UNIT I : RANDOM VARIABLES

PART-A

1) Define Sample Space?

The set of all possible outcomes in random experiment is called sample space.

Ex :-

$S = \{H, T\}$ - 1 coin $= 2 = 2^1$

$S = \{HH, HT, TH, TT\} = 2^2 = 4$

In general $S = 2^n$ (for coin)

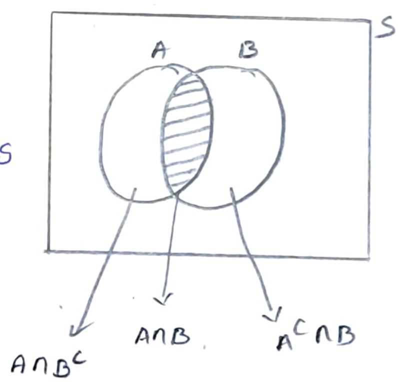
Ex2 :- For Dice

$S = \{1, 2, 3, 4, 5, 6\} = 6^1$

In general Sample Space $(S) = 6^n$

2) Define Addition theorem for two events

If A and B are two events in S then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$\Rightarrow$ Let A and B are any two events in S

Now, $(A \cup B) = (A \cup (A^c \cap B))$

Probability on both sides

$P(A \cup B) = P(A \cup (A^c \cap B))$

A & $A^c \cap B$ are MEE

$P(A \cup B) = P(A) + P(A^c \cap B)$

$= P(A) + P(B) - P(A \cap B)$ [$\because$ By 3rd Property]

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

3) Box A contains 5 red and 3 white marbles and Box B contains 2 red and 6 white marbles. If a marble is drawn from each box, what is the probability that they are both are same colour.

Given that

A	B
5R 3W	2R 6W

Probability of Box A $= 1/2$

Probability of Box B $= 1/2$

$\Rightarrow$ Both are same color

$$\frac{1}{2} \cdot \frac{5C_1}{8C_1} \cdot \frac{1}{2} \cdot \frac{2C_1}{8C_1} \quad (\text{or}) \quad \frac{1}{2} \cdot \frac{3C_1}{8C_1} \cdot \frac{1}{2} \cdot \frac{6C_1}{8C_1}$$

$$\frac{1}{4} \cdot \frac{5\times 2}{8\times 8} \quad (\text{or}) \quad \frac{1}{4} \cdot \frac{3\times 6}{8\times 8}$$

$$\frac{1}{4} \cdot \frac{10}{64} + \frac{1}{4} \cdot \frac{18}{64}$$

$$\Rightarrow 7/64.$$

4) Define Baye's Theorem?

If $E_1, E_2 \ldots\ldots E_n$ are Mutually Exclusive and Exhaustive Events such that $P(E_i > 0)$ $(i = 1, 2, \ldots\ldots n)$ in a sample space S and A is any other event in S intersecting with E_i such that $P(A) > 0$ then

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1} P(E_i) \cdot P(A/E_i)}$$

5) Define Discrete and Continuous Random Variables?

Discrete Random Variables:

A random variable X which can take only a finite no.of values in an interval of domain is called Discrete Random Variables.

Ex:-

$x = \{0, 1, 2, \ldots\ldots n\}$

Continuous Random Variables:

A Random variable X which can take values continuosly is called continuous Random Variables.

Ex:-

$X = \{\text{Height}, \text{Weight}, \text{Age}\}$

PART-B

1) State and Prove Baye's Theorem?

If $E_1, E_2, \ldots E_n$ are Mutually Exclusive and Exhaustive events such that $P(E_i > 0)$ $(i = 1, 2, \ldots n)$ in a sample space S and A is any other event in S intersecting with E_i such that $P(A) > 0$ then

$$P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1} P(E_i) \cdot P(A/E_i)}$$

Proof :-

Given that

$E_1, E_2 \ldots E_n$ are MEE and Exhaustive Events

i.e $E_1 \cup E_2 \ldots E_n = S$

Let a is any event in S

$A = A \cap S$

$A = A \cap [E_1 \cup E_2 \cup \ldots\ldots E_n]$

$A = (A \cup E_1) \cup (A \cap E_2) \cup \ldots\ldots (A \cap E_n)$

$A \cap E_1$; $A \cap E_2$; ...... $A \cap E_n$ are MEE

$A = (A \cap E_1) + (A \cap E_2) + \ldots\ldots (A \cap E_n)$

Taking Probability on both sides

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) \text{ --- } (1)$$

eq (1) by the defination of conditional property

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P\left(\frac{E_i}{A}\right) = \frac{P(A \cap E_i)}{P(A)}$$

$$= \frac{P(A \cap E_i)}{\sum_{i=1} P(A \cap E_i)} \quad \text{(By Multiplication theorem)}$$

$$\Rightarrow P\left(\frac{E_i}{A}\right) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}$$ //

2) Three machines I, II, III produce 40%, 30%, 30% of the total no. of items in the factory. The percentage of defective items of these machines are 4%, 2%, 3%. if a item is selected at random find the probability that the item is defective.

$$P(A) = 40\% = \frac{40}{100} = 2/5$$

$$P(B) = 30\% = \frac{30}{100} = 3/10$$

$$P(C) = 30\% = \frac{30}{100} = 3/10$$

Let D be defective items.

$P(D/A) = 4\% = \frac{4}{100} = \frac{2}{50} = \frac{1}{25}$;

(3)

$P(D/B) = 2\% = \frac{2}{100} = \frac{1}{50}$;

$P(D/C) = 3\% = \frac{3}{100}$;

To find $P(D) = ?$

Here A, B, C be the events of M-I, M-II, M-III

$$P(D) = P(A \cap D) + P(B \cap D) + P(C \cap D)$$

$$= P(A) \cdot P\left(\frac{D}{A}\right) + P(B) \cdot P\left(\frac{D}{B}\right) + P(C) \cdot P(D/C)$$

$$= \frac{2}{5} \cdot \frac{1}{25} + \frac{3}{10} \cdot \frac{1}{50} + \frac{3}{10} \cdot \frac{3}{100}$$

$$= \frac{31}{1000}$$

3) A Random Variable X has the following Probability function.

X	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Determine k

ii) Evaluate $P(X<6)$, $P(X \geq 6)$, $P(0<x<5)$ & $P(0 \leq x \leq 4)$

iii) If $P(x \leq k) > 0.5$, find minimum value of k

iv) Determine distribution function of x

v) Mean

vi) Variance

i) $\sum_{x=0}^{7} P(x) = 1$

$P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)+P(7) = 1$

$0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$10k^2+9k-1=0$

$10k(k+1)-(k+1)=0$

$(10k-1)(k+1)=0$

$10k-1=0$	$k+1=0$
$10k=1$	$k=-1$
$k=1/10$	$k \neq -1$
$k=0.1$	

$\boxed{\therefore k=0.1}$

ii) $P(X<6)$

$= 1-P(X \geq 6)$

$= 1-[P(x=6)+P(x=7)]$

$= 1-[2k^2+7k^2+k]$

$= 1-[9k^2+k]$

$= 1-[9(0.1)^2+(0.1)]$

$= 0.8$

$P(X \geq 6)$

$= P(X=6)+P(X=7)$

$= 2k^2+7k^2+k$

$= 9k^2+k$

$= 9(0.1)^2+0.1 \Rightarrow 0.19$

$P(0<x<5)$

$= P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= k+2k+2k+3k$

$= 8k$

$\Rightarrow 8(0.1)$

$\Rightarrow 0.8$

$P(0 \leq x \leq 4)$

$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$= 0+k+2k+2k+3k$

$\Rightarrow 8k$

$= 8(0.1)$

$= 0.8$

iii) If $P(X \leq k) > 1/2$

If $k=0 \Rightarrow P(X \leq 0) > 1/2$

$P(0) > 1/2$

$0 > 0.5$ X

Put $k=1 \Rightarrow P(X \leq 1) > 1/2$

$P(X=0)+P(X=1) > 1/2$

$0+k > 1/2$

$0.1 > 1/2$ X

Put $k=4 \Rightarrow P(X \leq 4) > 1/2$

$P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4) > 1/2$

$0+k+2k+2k+3k > 1/2$

$8k > 1/2$

$8(0.1) > 1/2$

$0.8 > 0.5$ ✓

The min value of k is 4

iv)

X	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$
k=0.1	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17
Dist of X $P(X \le x)$	0	0.1	0.3	0.5	0.8	0.81	0.83	1

v) Mean

$$= \mu = \sum_{x=0}^{7} x \cdot P(x)$$

$$= (0) \cdot P(10) + (1) \cdot P(11) + 2 \cdot P(12) + 3\,P(8) + 4(P_4) + 5(P_5) + 6 \cdot (P_6) + 7 \cdot P_7$$

$$\Rightarrow 0 + 1(0.1) + 2(0.2) + 3(0.2) + 4(0.3) + 5(0.01) + 6(0.02) + 7(0.17)$$

$$\Rightarrow 0.1 + 0.4 + 0.6 + (1.2) + 0.05 + 0.12 + 1.19$$

$$\Rightarrow 3.66$$

vi) Variance

$$\sigma^2 = \sum_{x=0}^{7} x^2 \cdot P(x) - \mu^2$$

$$\sigma^2 = (0)^2 \cdot P(x) + (1)^2 \cdot P(1) + (2)^2 \cdot P(2) + 3^2 + P(3) + 4^2 \cdot P(4) + 5^2 \cdot P(5) + 6^2(P(6)) + 7^2 \cdot P(7) - (3.66)^2$$

$$\Rightarrow 0 + 0.1 + 0.8 + 1.8 + 4.8 + 0.25 + 0.72 + 8.33 - (3.66)^2$$

$$\Rightarrow 16.80 - (3.66)^2$$

$$\sigma^2 = 3.40$$

4) A continuous random variable has a probability density function

$$f(x) = \begin{cases} kxe^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0 & \text{, otherwise} \end{cases}$$

Determine i) k ii) Mean iii) Variance

Given

$$f(x) = \begin{cases} kxe^{-\lambda x} & \text{for } x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

i) WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{\infty} kxe^{-\lambda x}\,dx = 1$$

$$k \int_{0}^{\infty} \underset{u}{x} \cdot \underset{v}{e^{-\lambda x}}\,dx = 1$$

$$k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{(-\lambda)^2}\right)\right]_0^{\infty} = 1$$

$$k\left[0 - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$$

$$k\left(\frac{1}{\lambda^2}\right) = 1$$

$$k = \lambda^2$$

ii) Mean

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$$

$$\mu = \int_{-\infty}^{0} x \cdot f(x)\,dx + \int_{0}^{\infty} x \cdot f(x)\,dx$$

$$\mu = 0 + \int_0^\infty x \cdot kx e^{-\lambda x} \, dx$$

$$\mu = k \int_0^\infty \underset{u}{x^2} \, \underset{v}{e^{-\lambda x}} \, dx$$

$$\mu = \lambda^2 \left[x^2 \left[\frac{e^{-\lambda x}}{-\lambda} \right] - 2x \left[\frac{-e^{-\lambda x}}{\lambda^2} \right] + 2 \left[\frac{e^{-\lambda x}}{-\lambda^3} \right] \right]_0^\infty$$

$$\mu = \lambda^2 \left[0 - \left[0 - 0 - 2 \times \frac{1}{\lambda^3} \right] \right.$$

$$\mu = \lambda^2 \times \frac{2}{\cancel{\lambda^3}}$$

$$\boxed{\mu = \frac{2}{\lambda}}$$

iii) Variance

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx - \mu^2$$

$$\sigma^2 = \int_{-\infty}^{0} x^2 \cdot f(x) dx + \int_0^\infty x^2 \cdot f(x) \, dx - \left(\frac{2}{\lambda} \right)^2$$

$$\sigma^2 = 0 + \int_0^\infty x^2 \cdot kx e^{-\lambda x} \, dx - \frac{4}{\lambda^2}$$

$$\sigma^2 = k \int_0^\infty \underset{u}{x^3} \, \underset{v}{e^{-\lambda x}} \, dx - \frac{4}{\lambda^2}$$

$$\sigma^2 = \lambda^2 \left[x^3 \left(\frac{e^{\lambda x}}{-\lambda} \right) - 3x^2 \left(\frac{e^{-\lambda x}}{(-\lambda)^2} \right) + 6x \left(\frac{e^{-\lambda x}}{(-\lambda)^3} \right) - 6 \left(\frac{e^{-\lambda x}}{(-\lambda)^4} \right) \right]_0^\infty - \frac{4}{\lambda^2}$$

$$\sigma^2 = \lambda^2 \left[0 - \left(0 + 0 + 0 - 6 \left(\frac{1}{\lambda^4} \right) \right) \right] - \frac{4}{\lambda^2}$$

$$\sigma^2 = \cancel{\lambda^2} \left[\frac{6}{\lambda^{\cancel{4}2}} \right] - \frac{4}{\lambda^2}$$

$$\sigma^2 = \frac{6}{\lambda^2} - \frac{4}{\lambda^2}$$

$$\therefore \text{variance}, \ \sigma^2 = \frac{2}{\lambda^2}$$

i) If x is a continuous random variable whose density function is $f(x) = \begin{cases} x, & \text{if } 0<x<1 \\ 2-x, & \text{if } 1\le x<2 \\ 0, & \text{elsewhere} \end{cases}$

find $E[25x^2+30x-5]$

Given

$E[25x^2+30x-5]$

$= 25E(x^2)+30E(x)-5$ —— ①

Find mean for both $E(x^2)$ & $E(x)$

$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx$

$= \int_{-\infty}^{0} x^2 \cdot f(x)dx + \int_{0}^{1} x^2 \cdot f(x)dx + \int_{1}^{2} x^2 \cdot f(x)dx + \int_{2}^{\infty} x^2 \cdot f(x)dx$

$= 0 + \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)dx - 0$

$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)dx$

$E(x^2) = \left(\frac{x^4}{4}\right)_0^1 + \left[2\cdot\frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$

$= \left[\frac{1^4}{4}\right] - \left[\frac{0^4}{4}\right] + \left[2\cdot\frac{2^3}{3} - \frac{2^4}{4}\right] - \left[2\cdot\frac{1^3}{3} - \frac{1^4}{4}\right]$

$= \frac{1}{4} - 0 + 2\left[\frac{8}{3} - \frac{16}{4}\right] - 2\left[\frac{1}{3} - \frac{1}{4}\right]$

$= \frac{1}{4} + 2\left[\frac{8}{3} - \frac{16}{4}\right] - 2/3 - 1/4$

$= \frac{1}{4} + \left[\frac{16}{3} - \frac{16}{4}\right] - \left[\frac{2}{3} - 1/4\right]$

$= 1/4 + \left[\left(\frac{16-2}{3}\right) - \left(\frac{8-3}{12}\right)\right]$

$= 1/4 + \left[\frac{4}{3} - \frac{5}{12}\right]$

$$\Rightarrow \frac{1}{4} + \left(\frac{16-5}{12}\right)$$

$$= \frac{1}{4} + \frac{11}{12}$$

$$E(x^2) = \frac{14}{12} = \frac{7}{6} \text{ ——— } (2)$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{0} x \cdot f(x) dx + \int_{0}^{1} x \cdot f(x) dx + \int_{1}^{2} x \cdot f(x) dx + \int_{2}^{\infty} x \cdot f(x) dx$$

$$= 0 + \int_{0}^{1} x \cdot x\, dx + \int_{1}^{2} x(2-x)\, dx + 0$$

$$= \int_{0}^{1} x^2 dx + \int_{1}^{2} (2x - x^2)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2$$

$$= \left(\frac{1-0}{3}\right) + \left[\left(2^2 - \frac{2^3}{3}\right) - \left(1 - 1/3\right)\right]$$

$$= \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \frac{2}{3}\right]$$

$$= \frac{1}{3} + \left[\left(\frac{12-8}{3}\right) - \frac{2}{3}\right]$$

$$E(x) = 1/3 + \left[\frac{4}{3} - \frac{2}{3}\right] = \frac{1}{3} + \left[\frac{2}{3}\right] = 1 \longrightarrow (3)$$

substitute (2) & (3) in 1

$$= 25\, E(x^2) + 30\, E(x) - 5$$

$$= 25\left(\frac{7}{6}\right) + 30(1) - 5$$

$$= 25\left(\frac{7}{6}\right) + 30 - 5$$

$$= \frac{175}{6} + 30 - 5$$

$$= \frac{325}{6}$$

$$= 54.16$$