Unit-2, Electrostatics-II (Electric Fields in Material Space).

Introduction:- Just as Electric fields can exist in free space, they can exist in material media also. All the materials availa-ble in nature may be classified interms of their Conductivity, σ in mhos/permeter ($\mho$/m) or Siemens/meter (S/m) as Conductors and non conductors or technically as metals and insulators (or Dielectrics). A material with high Conductivity ($\sigma >> 1$) is referred to as a metal where as one with low Conductivity ($\sigma << 1$) is referred to as an insulator. The Conductivity of any metal means that the no. of charge Carriers available. Dielectric materials have few electrons available for conduction in Contrast to metals, which have more no. of electrons.

Convection and Conduction Currents:- The two fundamental quantities required for the study of Electric fields are Electric Voltage and Current. To see how Electric field behaves in a Conductor or dielectric, it is appropriate to Consider electric Current. Electric current is generally Caused by the motion of electric charges. The current in Amperes through a given area is the electric charge passing through the area per unit time.

That is $\vec{I} = \frac{dQ}{dt}$. Thus in a current of one ampere, charge is being transferred at a rate of one coulomb per second.

We now introduce the concept of current density, $\vec{J}$. If current ΔI flows through a surface ΔS, the current density is $\vec{J} = \frac{\Delta I}{\Delta S}$ or $\Delta I = \vec{J}\Delta S$. Then the total current through a surface, S is $\boxed{\vec{I} = \int_S \vec{J}\cdot d\vec{S}}$. Depending on how $\vec{I}$ is produced, there are three different kinds of current densities. They are Convection, Conduction and Displacement current densities. The displacement current exists in Magneto statics.

Convection current, as distinct from Conduction current, does not involve conductors and consequently doesnot satisfy Ohm's Law. It occurs in freespace or vaccum. The convection current density, $\vec{J}$ is the current through a unit area at that point. In freespace, $\vec{J}$ can be expressed as $\boxed{\vec{J} = \rho_v \vec{u}}$, where ρ_v is the charge density and $\vec{u}$ is the velocity of charge carriers. In similar way the current density, $\vec{J}$ in conductors is given by $\boxed{\vec{J} = \sigma \vec{E}}$.

Resistance and Capacitance:- All the materials having Conductivity, σ can also have the resistivity in contrast to conductivity. That is the resistance of any metal will the opposing nature for the flow of charges. From ohm's Law, we can express the resistance, R in ohms can as

$$\boxed{R = \frac{V}{I} = \frac{\int_L \vec{E} \cdot d\vec{l}}{\oint_S \vec{J} \cdot d\vec{s}} = \frac{\int_L \vec{E} \cdot d\vec{l}}{\oint_S \sigma \vec{E} \cdot d\vec{s}}}$$

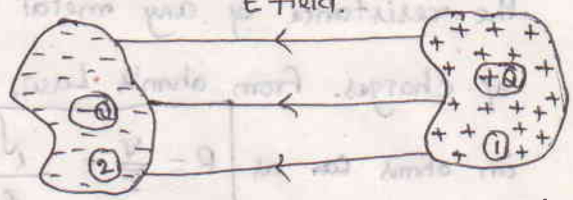
using this equation, the Resistance, R (or conductance, $G = \frac{1}{R}$) of a given conducting material can be found by following these steps:

1. Choose a suitable coordinate system.
2. Assume, V as the potential difference b/w conductor materials.
3. Determine $\vec{E}$ from $\vec{E} = -\nabla V$ and $\vec{I}$ from $\vec{I} = \oint_S \sigma \vec{E} \cdot d\vec{s}$.
4. Finally obtain R as $\frac{V}{I}$.

Alternatively, it is possible to assume current I, finding the corresponding potential difference V and determine R from $R = \frac{V}{I}$.

Basically, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges. The conductors are sometimes referred to as the plates of the capacitor. For example consider a two-conductor

Capacitor as shown in the figure. The Conductors are maintained at a potential difference, V. and $\vec{E}$ is the electric field existing between the conductors.



We define the Capacitance, C of the capacitor as the ratio of magnitude of the charge on one of the conductors to the potential difference between them, that is

$$\boxed{C = \frac{Q}{V} = \frac{\oint_S \vec{D}.d\vec{s}}{-\int_l \vec{E}.d\vec{l}} = \frac{\varepsilon \oint_S \vec{E}.d\vec{s}}{-\int_l \vec{E}.d\vec{l}}}$$ (in Farads).

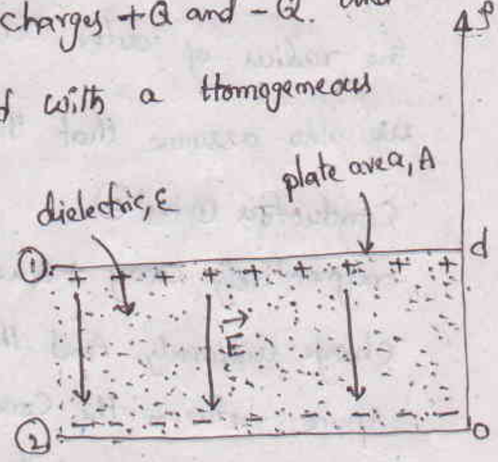
The capacitance of two conductors can be found by following these steps: ① Choose a suitable co-ordinate system.

②. Let the two conducting plates carry charges +Q and −Q respectively.

③ Determine $\vec{E}$ from either Coulomb's Law or Gauss's Law and find V from $V = -\int_l \vec{E}.d\vec{l}$.

④ Finally obtain C from $C = Q/V$.

parallel-plate Capacitor :- Consider the arrangement of parallel plate Capacitor as shown in figure. Assume that each of the plates has, A and they are seperated by a distance, d. and also plates ① and ②, respectively Carrying charges $+Q$ and $-Q$. and the space between the plates is filled with a Homogeneous dielectric with permittivity, ε.



For this arrangement of two Conductors or parallel plates, we can write

$$Q = \oint_S \vec{D}\cdot d\vec{s} = \oint_S \epsilon \vec{E}\cdot d\vec{s}$$

$$= \varepsilon \oint_S \vec{E}\cdot d\vec{s}$$

$$= \varepsilon E_\rho \oint_S d\vec{s}$$

$$= \varepsilon E_\rho A.$$

$$E_\rho = \frac{Q}{\varepsilon A} \text{ or in general } \vec{E} = \frac{Q}{\varepsilon A}\vec{a_\rho} \text{ and}$$

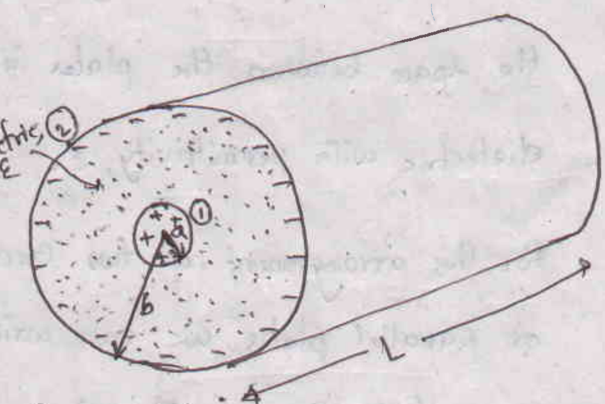
$$V = -\int_1^2 \vec{E}\cdot d\vec{l} = -\int_d^0 \frac{Q}{\varepsilon A}\vec{a_\rho}\cdot d\rho\,\vec{a_\rho} = \frac{-Q}{\varepsilon A}\int_d^0 d\rho = \frac{-Q}{\varepsilon A}\left[\rho\right]_d^0$$

$$V = 0 - \left(\frac{-Qd}{\varepsilon A}\right) = \frac{Qd}{\varepsilon A}.$$

we know that, Capacitance $\boxed{C = \frac{Q}{V} = \frac{\varepsilon A}{d}}$

Coaxial or Cylindrical Capacitor :- This is essentially a coaxial Cable or Coaxial Cylindrical Capacitor. Consider the length of the Conductors or cylinder is L. The radius of inner conductor is a and the radius of outer Conductor is b $(b>a)$ as shown in the figure.

We also assume that the Conductors ① and ② respectively carry $+Q$ and $-Q$ Charge uniformly. And the space between the Conductors is filled with a dielectric medium of pirmittivity, ε.



Applying Gauss's Law to an arbitary Gaussian cylindrical surface of radius ρ $(a<\rho<b)$, we obtain

$$Q = \varepsilon \oint_s \vec{E} \cdot d\vec{s} = \varepsilon E_\rho \oint_s d\vec{s} = \varepsilon E_\rho \, 2\pi \rho L \quad \text{or} \quad E_\rho = \frac{Q}{2\pi \varepsilon \rho L}$$

or in General $\vec{E} = \frac{Q}{2\pi \varepsilon \rho L} \vec{a_\rho}$ and

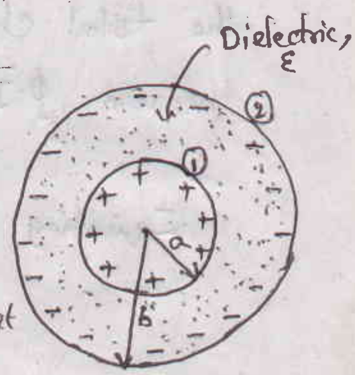
$$V = -\int_2^1 \vec{E} \cdot d\vec{l} = -\int_b^a \frac{Q}{2\pi \varepsilon \rho L} \cdot d\rho \, \vec{a_\rho} = -\frac{Q}{2\pi \varepsilon L} \int_b^a \frac{1}{\rho} d\rho \, \vec{a_\rho}$$

$$V = \frac{-Q}{2\pi \varepsilon L} \left[\log \rho\right]_b^a = \frac{-Q}{2\pi \varepsilon L} \left[\log a - \log b\right]$$

$$V = \frac{Q}{2\pi \varepsilon L} (\log b - \log a) = \frac{Q}{2\pi \varepsilon L} \log\left(\frac{b}{a}\right).$$

we Know that $\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\log\left(\frac{b}{a}\right)}}$

Spherical Capacitor:- This is the case of two Concentric spherical Conductors. Consider the inner radius, a and outer radius b. (b>a). Seperated by dielectric medium with permittivity, ε as shown in the figure. We assume charges +Q and -Q on the inner and outer spheres respectively.



Applying Gauss's Law to an arbitary Gaussian spherical surface of radius ρ $(a<\rho<b)$, we get

$$Q = \varepsilon \oint_s \vec{E}.\vec{ds} = \varepsilon E_\rho \oint_s \vec{ds} = \varepsilon E_\rho (4\pi\rho^2)$$

or $E_\rho = \frac{Q}{4\pi\varepsilon\rho^2}$ or in General $\vec{E} = \frac{Q}{4\pi\varepsilon\rho^2}\vec{a_\rho}$ and

$$V = -\int_2^1 \vec{E}.\vec{dl} = -\int_b^a \frac{Q}{4\pi\varepsilon\rho^2}\vec{a_\rho}.\, d\rho\,\vec{a_\rho} = \frac{-Q}{4\pi\varepsilon}\int_b^a \frac{1}{\rho^2}d\rho.$$

$$V = \frac{-Q}{4\pi\varepsilon}\left[\frac{-1}{\rho}\right]_b^a = \frac{+Q}{4\pi\varepsilon a} - \frac{+Q}{4\pi\varepsilon b} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

we Know that, Capacitance, $\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$

Continuity of Current equation & Relaxation Time :-

Due to the principle of conservation of charge, the time rate of decrease of charge with in a given volume must be equal to the net outward current flow through the closed surface & volume. That is $I_{out} = \oint_S \vec{J}\cdot d\vec{S} = \frac{-dQ_{in}}{dt}$. where Q_{in} is the total charge enclosed by the surface. Applying Divergence theorem $\oint_S \vec{J}\cdot d\vec{S} = \int_V \nabla\cdot\vec{J}\, dv$. But $\frac{-dQ_{in}}{dt} = \frac{-d}{dt}\int_V \rho_v dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$

Equating the two volume integrals $\int_V \nabla\cdot\vec{J}\, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$.

or $\boxed{\nabla\cdot\vec{J} = \frac{-\partial \rho_v}{\partial t}}$ which is called the Continuity of Current equation.

Relaxation time :- It is the time taken by the charges placed inside the material to drop or reduce by an amount $e^{-1} = 36.8\%$ of its initial value. The relaxation time, T_r for any material can be equivalent to the ratio of the medium permittivity to the conductivity i.e.

$$\boxed{T_r = \frac{\varepsilon}{\sigma}}.$$

Dielectric Constant and Strength :- We can write for any medium of permittivity, ε, $\vec{D} = \varepsilon \vec{E}$ where $\varepsilon = \varepsilon_0 \varepsilon_r$ or $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$. ε_0 is the permittivity of free space, and ε_r is called the Dielectric Constant or relative permittivity. Therefore the Dielectric Constant (or relative permittivity) ε_r is the ratio of the permittivity of the Dielectric to that of free space or Vaccum. The dielectric strength is the maximum electric field that a dielectric can withstand.

Linear, Isotropic and Homogeneous Dielectrics :- A material is said to be linear if $\vec{D}$ varies linearly with $\vec{E}$ and non linear otherwise. Materials for which $\vec{D}$ and $\vec{E}$ are in the same direction are said to be isotropic. non isotropic otherwise. Materials for which ε (or σ) does not vary in the region being Considered and is therefore the same at all points are said to be Homogeneous. otherwise inhomogeneous or non-Homogeneous.

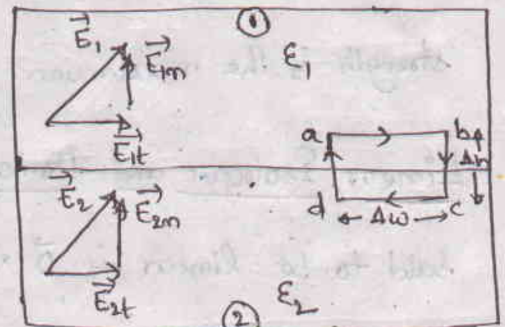
Boundary Conditions :- If the field exists in a region Consists of two different media, the Conditions that the field must satisfy at the interface seperating the media are called Boundary Conditions

These Conditions are helpful in determining the field on side of the boundary if the field on the other side is Known.

① Dielectric-Dielectric Boundary Conditions :- Consider the $\vec{E}$ field existing in a region consisting of two different dielectrics characterized by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$ as shown in the figure. $\vec{E}_1$ and $\vec{E}_2$ in media ① and ② respectively. Can be decomposed as

$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n}$$



We Know that the potential of any closed path is zero. That is $\oint_L \vec{E}\cdot d\vec{l} = 0$. we apply this equation to the closed path, abcda of figure. we obtain.

$$0 = \vec{E}_{1t}\Delta w - \vec{E}_{1n}\frac{\Delta h}{2} - \vec{E}_{2n}\frac{\Delta h}{2} - \vec{E}_{2t}\Delta w + \vec{E}_{2n}\frac{\Delta h}{2} + \vec{E}_{1n}\frac{\Delta h}{2}$$

if $\Delta h \to 0$, then.

$$0 = \vec{E}_{1t}\Delta w - \vec{E}_{2t}\Delta w \quad \text{or}$$

$$\boxed{\vec{E}_{1t} = \vec{E}_{2t}} \quad -①$$

Thus the tangential Component of $\vec{E}$ is Continous or same on the two sides of the boundary.

Since $\vec{D} = \varepsilon\vec{E} = \vec{D}_t + \vec{D}_n$ we can write.

$$\frac{\vec{D}_{1t}}{\varepsilon_1} = \vec{E}_{1t} = \vec{E}_{2t} = \frac{\vec{D}_{2t}}{\varepsilon_2} \quad \text{or}$$

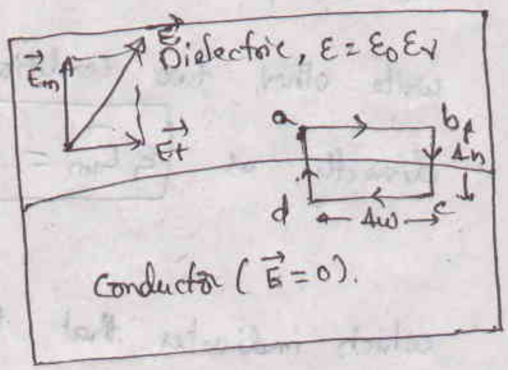
$$\boxed{\frac{\vec{D}_{1t}}{\varepsilon_1} = \frac{\vec{D}_{2t}}{\varepsilon_2}} \quad -(2)$$

That is $\vec{D}_t$ undergoes some change across the interface. Hence $\vec{D}_t$ is said to be discontinous across the interface. From the above equations (1) and (2), we can write other two conditions at boundary for normal components directly as $\boxed{\varepsilon_1 \vec{E}_{1n} = \varepsilon_2 \vec{E}_{2n}}$ - (3) and $\boxed{D_{1n} = D_{2n}}$ - (4).

which indicates that the normal component of $\vec{E}$ is not same or discontinous at the boundary and the Normal Component of $\vec{D}$ is same or Continous at the Boundary. Therefore equations (1) to (4) gives the conditions to be satisfied at the boundary. Hence these four equations gives all the Boundary conditions of a Dielectric - Dielectric mediums.

② Conductor - Dielectric Boundary Conditions :-

In this case, the conductor is assumed to be perfect i.e $\vec{E}=0$ inside the conductor as shown in the figure. To determine the Boundary conditions for a conductor - dielectric interface, we follow the same procedure for Dielectric - Dielectric interface except that we incorporate the fact that $\vec{E}=0$ inside the conductor. For the closed path abcda in the figure we can write



$$0 = 0.\Delta w + 0.\frac{\Delta h}{2} + \vec{E}_n \frac{\Delta h}{2} - \vec{E}_t \Delta w - \vec{E}_n \frac{\Delta h}{2} - 0.\frac{\Delta h}{2}.$$

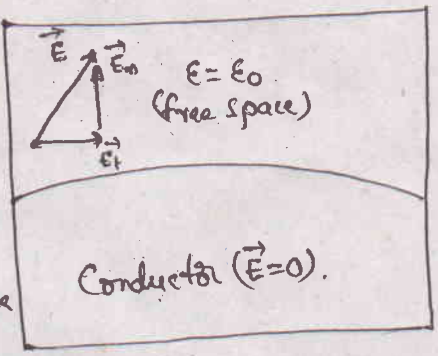
As $\Delta h \to 0$ $\boxed{\vec{E}_t = 0}$ —① Similarly the remaining boundary conditions can also be obtained as

$$\boxed{\frac{\vec{D}_t}{\varepsilon} = 0} - ②, \quad \boxed{\varepsilon \vec{E}_n = 0} - ③ \quad \text{and} \quad \boxed{\vec{D}_n = 0} - ④.$$

Therefore equations ① to ④ gives the conditions to be satisfied at the boundary for Conductor - Dielectric medium in which the Conductor was assumed as a perfect Conductor that is inside the Conductor, $\vec{E}=0$.

③ Conductor- Freespace Boundary Conditions :-

This is a special case of Conductor- Dielectric Conditions and is shown in the figure. the Boundary conditions at the interface are given directly by substituting $\vec{E}=0$ in Conductor and $\epsilon = \epsilon_0$ $(\epsilon_r=1)$ in freespace as.



$$\boxed{\vec{E}_t = 0} \quad -①$$

$$\boxed{\frac{\vec{D}_t}{\epsilon_0} = 0} \quad -②$$

$$\boxed{\epsilon_0 \vec{E}_n = 0} \quad -③ \quad \& \quad \boxed{\vec{D}_n = 0} \quad -④$$

The above equations ① to ④ Gives the required Conditions to be satisfied at the interface.

Poisson's and Laplace's Equations :- These equations are derived from Gauss's Law. for a linear material medium. from Gauss's Law, we know that $\nabla . \vec{D} = \nabla . \epsilon \vec{E} = \rho_v$ and w.k.t. $\vec{E} = -\nabla V$. Therefore $\nabla . (-\epsilon \nabla V) = \rho_v$ or

$$\boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}}$$ This is known as poisson's equation.

A Special Case of this equation occurs when $\rho_v = 0$ then $\boxed{\nabla^2 V = 0}$ is called the Laplace's equation.