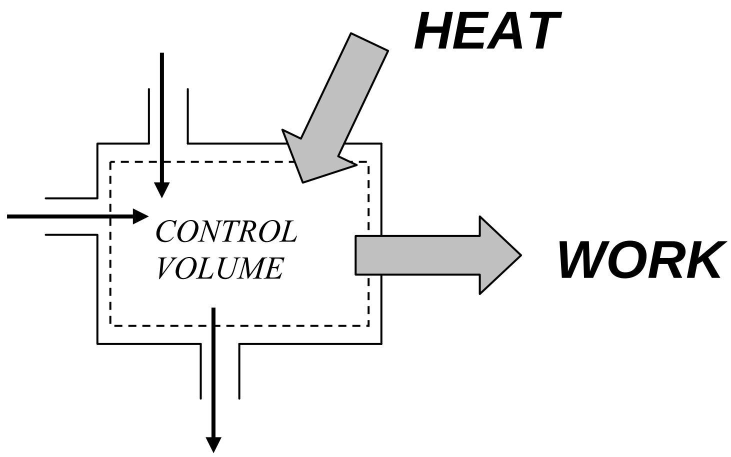
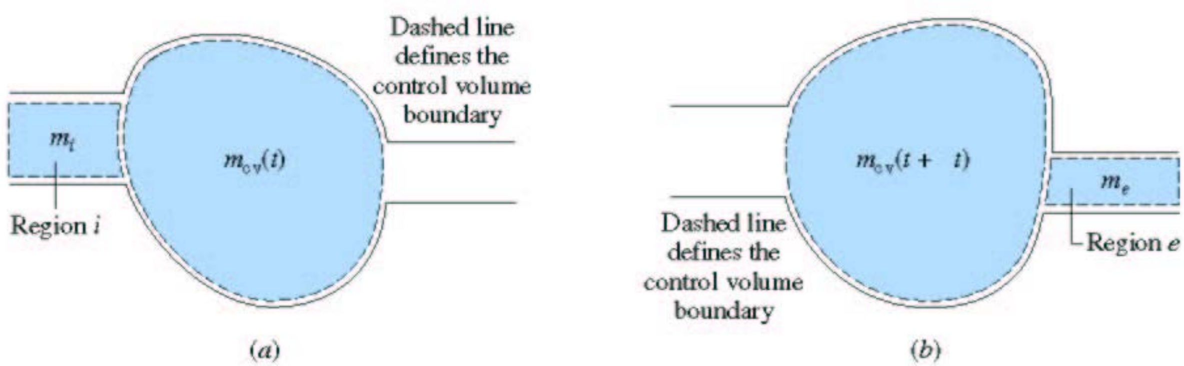
Control Volume Analysis

Used for flow through systems



Use closed system analysis (fixed system mass M) to derive expressions for conservation of mass and energy



Time t

Time t+Δt

$M_S(t) = M_{CV}(t) + m_i$

$M_S(t+\Delta t) = M_{CV}(t+\Delta t) + m_e$

Note: m_i doesn't have to be equal to m_e since $M_{CV}(t)$ doesn't have to be equal to $M_{CV}(t+\Delta t)$

For a closed system $M_S(t) = M_S(t+\Delta t)$, so

$$M_{CV}(t) + m_i = M_{CV}(t+\Delta t) + m_e$$

$$M_{CV}(t+\Delta t) - M_{CV}(t) = m_i - m_e$$

Divide through by Δt to get time rate quantities

$$\frac{M_{CV}(t+\Delta t) - M_{CV}(t)}{\Delta t} = \frac{m_i}{\Delta t} - \frac{m_e}{\Delta t}$$

Taking limit as Δt→0, closed system and control volume boundaries coincide

$$\lim_{\Delta t \to 0}\left[\frac{M_{CV}(t+\Delta t) - M_{CV}(t)}{\Delta t}\right] = \dot{m}_i - \dot{m}_e$$

Note: $\dot{m}$ is called the **mass flow rate** with units kg/s, subscript *i* is for inlet and *e* is for exits

$$\boxed{\frac{dM_{CV}}{dt} = \dot{m}_i - \dot{m}_e}$$

Time rate of change of mass contained within the control volume equals the net mass flow rate $\dot{m}$ into the control volume

For multiple inlets and outlets

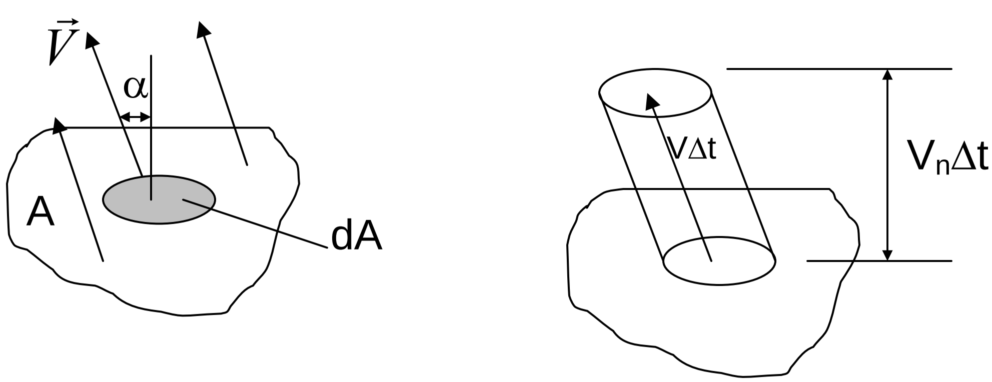
$$\frac{dM_{CV}}{dt} = \sum_i \dot{m}_i - \sum_e \dot{m}_e$$

For steady-state ($dM_{CV}/dt=0$)

$$\sum_i \dot{m}_i = \sum_e \dot{m}_e$$

Mass flow rate in terms of local properties

Consider a small quantity of matter flowing with velocity $\vec{V}$ across an incremental area dA, in a time interval Δt



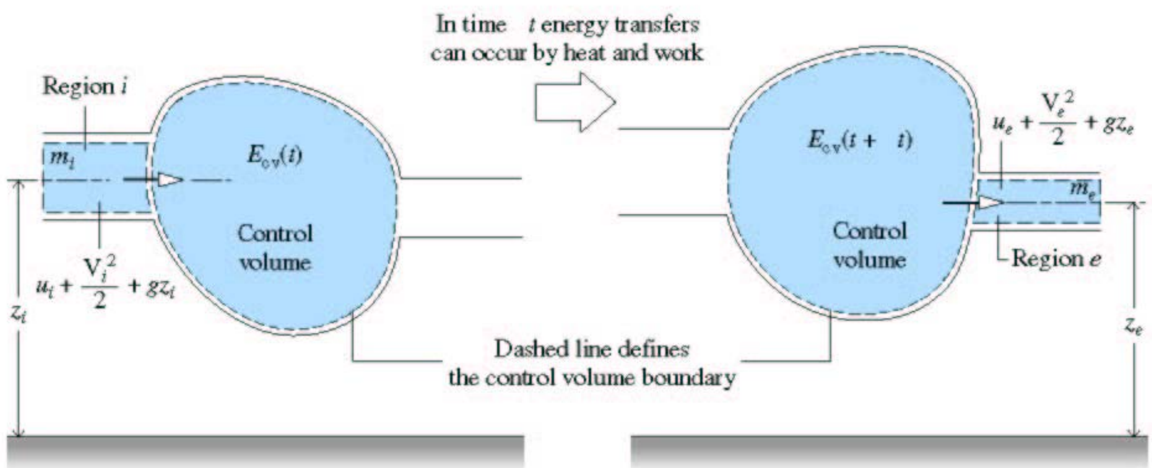
V_n is the normal component of the velocity vector

The volume of matter in the cylinder is $V_n \Delta t dA$

Mass of matter crossing dA in Δt is $\Delta m = \rho V_n \Delta t dA$

Conservation of Energy for a Control Volume

Perform closed system analysis where the system at time t includes the fluid in the CV and inlet region i



Time t   Time t+Δt

At time t, the energy of the system is:

$$E_S(t) = E_{CV}(t) + m_i(u_i + V_i^2/2 + gZ_i)$$

At time t+Δt, the energy of the system is:

$$E_S(t+\Delta t) = E_{CV}(t+\Delta t) + m_e(u_e + V_e^2/2 + gZ_e)$$

First law applied to the closed system gives:

$$E_S(t+\Delta t) - E_S(t) = Q_S - W_S$$

$$E_{CV}(t+\Delta t) + m_e(u_e + V_e^2/2 + gZ_e) - [E_{CV}(t) + m_i(u_i + V_i^2/2 + gZ_i)] = Q_S - W_S$$

$$E_{CV}(t+\Delta t) - E_{CV}(t) = Q_S - W_S + m_i(u_i + V_i^2/2 + gZ_i) - m_e(u_e + V_e^2/2 + gZ_e)$$

The change in energy in the CV is equal to the energy transferred to the system by heat and work plus the energy entering into the CV by m_i less the energy leaving the CV by m_e

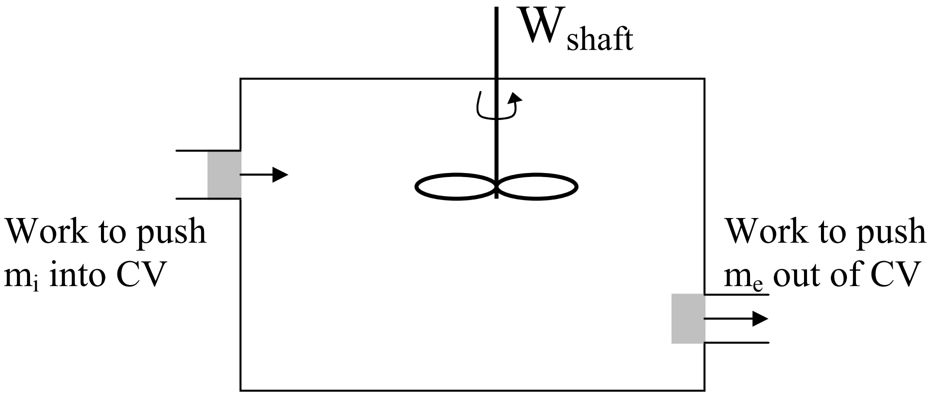
Divide through by Δt, take limit as Δt→0, closed system and control volume boundaries coincide, note
$\dot{Q}_s = \dot{Q}_{CV}, \dot{W}_s = \dot{W}_{CV}$

$$\frac{dE_{CV}(t)}{dt} = \dot{Q} - \dot{W} + \dot{m}_i(u_i + V_i^2/2 + gZ_i) - \dot{m}_e(u_e + V_e^2/2 + gZ_e)$$

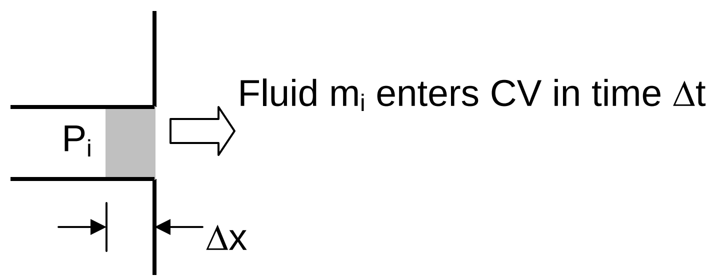
where u_i, V_i, Z_i are evaluated at the inlet
u_e, V_e, Z_e are evaluated at the exit

Consider the work term

Work consists of **shaft type work** and **flow work**



Consider the inlet of cross-sectional area A



Flow work is the energy used to get fluid into CV in time Δt

$$\Delta W = F\Delta x = (P_i A_i)\Delta x$$

Dividing through by Δt and taking limit as $\Delta t \rightarrow 0$

$$\lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = (P_i A_i) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

This yield the rate of work done, note $\dot{m}_i = \rho_i A_i V_i$

$$\dot{W}_i = (P_i A_i) V_i = P_i (A_i V_i) = P_i (\dot{m}_i / \rho_i)$$

Flow work at the inlet is $\boxed{\dot{W}_i = \dot{m}_i (P_i v_i)}$

Similarly, flow work at the exit $\boxed{\dot{W}_e = -\dot{m}_e (P_e v_e)}$

Substituting into the energy conservation equation

$$\frac{dE_{CV}(t)}{dt} = \dot{Q} - \dot{W}_{shaft} + \dot{m}_i P_i v_i - \dot{m}_e P_e v_e + \dot{m}_i (u_i + V_i^2/2 + gZ_i) - \dot{m}_e (u_e + V_e^2/2 + gZ_e)$$

Generalizing for multiple inlets and exits and substituting for $h = u + pv$

$$\boxed{\frac{dE_{CV}(t)}{dt} = \dot{Q} - \dot{W}_{shaft} + \sum_i \dot{m}_i (h_i + V_i^2/2 + gZ_i) - \sum_e \dot{m}_e (h_e + V_e^2/2 + gZ_e)}$$

Common engineering applications involve one inlet and one exit and the flow is steady, $dE/dt = 0, \dot{m}_i = \dot{m}_e = \dot{m}$

$$\frac{\cancel{dE/dt}}{\dot{m}} = \frac{\dot{Q}}{\dot{m}} - \frac{\dot{W}_{shaft}}{\dot{m}} + (h_i + V_i^2/2 + gZ_i) - (h_e + V_e^2/2 + gZ_e)$$

$$0 = \dot{q} - \dot{w}_s + (h_i + V_i^2/2 + gZ_i) - (h_e + V_e^2/2 + gZ_e)$$

where $\dot{q} = \dfrac{\dot{Q}}{\dot{m}}$, $\dot{w} = \dfrac{\dot{W}_{shaft}}{\dot{m}}$ are heat added and shaft work done *per unit mass,* units are J/kg

Note $\dot{q}$, $\dot{w}$ are *not* rates, i.e., J/s