NUMBER THEORY AND STATISTCAL METHODS

QUESTION BANK

UNIT-I: FUNDAMENTAL SAMPLING DISTRIBUTIONS

Short answer questions

1. Explain the concept of sample with an example.
2. Classify the types of samples.
3. Suppose the data set is the following 1.7, 2.2, 3.9, 3.11,1 4.7. Calculate the sample mean and sample median.
4. Calculate the sample mean and the sample variance for the data 3, 4, 5, 6, 6, 7.
5. Find the value of the finite population correction factor if n = 10, N = 1000.
6. How many different samples of size two can be chosen, from a finite population of size 25?
7. In a random sample of 1000 packages shipped by air freight 13 had some damage. Find the standard error of proportions.
8. State the Central limit theorem.
9. The variance of a population is 2. The size of the sample collected from the population is 169. What is the standard error of mean?
10. What is the effect on standard error, if a sample is taken from an infinite population of sample size is increased from 400 to 900?

Long answer questions

1. A population consists of five numbers 2, 3, 6, 8, 11. Consider all possible samples of size two which can be drawn with replacement from this population. Find a) The mean of the population. b) The standard deviation of the population. c) The mean of the sampling distribution of means. d) The standard deviation of the sampling distribution of means.[(i.e.) the standard error of means].

2. A population consists of five numbers 2, 3, 6, 8, 11. Consider all possible

samples of size two which can be drawn without replacement from this population. Find a) The mean of the population. b) The standard deviation of the population. c) The mean of the sampling distribution of means. d) The standard deviation of the sampling distribution of means.[(i.e.) the standard error of means].

3. Let $u_1 = (3, 7, 8)$, $u_2 = (2, 4)$. Find a) μ_{u_1} b) μ_{u_2} c) Mean of the sampling distribution of the difference of means. d) σ_{u_1} e) σ_{u_2} f) The standard deviation of the sampling distribution of the difference of means $\sigma_{u_1-u_2}$

4. A random sample of size 64 is taken from an infinite population having the mean 45 and standard deviation 8. What is the probability that $\bar{x}$ will be between 46 and 47.5?

5. A normal population has a mean of 0.1 and standard deviation of 2.1. Find the probability that men of a sample size 900 will be negative?

6. Show that the sample variance S^2 is an unbiased estimator of the population variance σ^2

7. To illustrate that the mean of the random sample is an unbiased estimate of the mean of the population. Consider 5 slips of paper numbered 3, 6, 9, 15, 27. a) List all possible samples of size 3 that can be taken without replacement from this finite population. b) Calculate the mean of each of the samples listed in a) and assigning each sample a probability of each sample. Verify that the mean of these $\bar{x}$ equals 12, namely the mean of the population.

8. Find 95% confidence limits for the mean of the normally distributed population of which the following sample was taken from 15, 17, 10, 18, 16, 9, 7, 11, 13, 14.

9. A random sample of 400 items is found to have mean 82 and S.D. 18. Find the maximum error of estimation, and the confidence limits if the mean is $\bar{x} = 82$ at 95% confidence.

10. The mean weight loss of $n = 16$ grinding balls after a certain length of time in mill surry is 3.42 grams with S.D. of 0.68 gram. Construct 99% confidence interval for the mean weight loss of such grinding balls under the stated condition.

UNIT-II: TESTING OF HYPOTHESIS

Short answer questions

1. Briefly explain the term statistical hypothesis.
2. Classify the types of hypothesis.
3. Classify the types of errors involved in testing of hypothesis.

4. Tabulate the critical values for large sample tests(Two tailed, Right tailed, Left tailed) at 1%, 5%, 10% levels of significance?
5. A coin was tossed 400 times and returned heads 216 times. Test the hypothesis that the coin is unbiased.
6. A die is tossed 256 times and it turns up with an even digit 150 times. Is the die biased?
7. Among 900 people in a state 90 are found to be chapatti eaters. Construct 99% confidence interval for the true proportion.
8. A random sample of 500 apples was taken from a large consignment and 60 were found to be bad, obtain the 98% confidence limits for the percentage number of bad apples in the consignment.
9. In a random sample of 100 packages shipped by air freight 13 had some damage. Construct 95% confidence interval for the true proportion of damaged package.
10. A random sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Find the percentage of bad pineapples in the consignment.

Long answer questions

1. A sample of 900 members has a mean of 3.4 cm and S.D. 2.61 cm. Is this sample has been taken from a large population of mean 3.25 cm and S.D. 2.61 c,. If the population is normal and its mean is unknown, find the 95% fiducial limits of the true mean.

2. The means of two large samples of sizes 1000 and 2000 members are 67.5 inches and 68 inches respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 inches?

3. In a big city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?

4. A social worker believes that fewer than 25% of the couples in certain area have ever used any form of birth control. A random sample of 120 couples was contacted. Twenty of them said that they have used. Test the belief of the social worker at 0.05 level.

5. Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women in favour of the proposal are same, at 5% level.

6. On the basis of their total scores, 200 candidates of a civil service examination are divided into two groups, the upper 30% and the remaining 70%. Consider the first question of the examination. Among the first group, 40 had the correct answer, whereas among the second group, 80 had the correct answer. On the basis of these results, can one conclude that the first question is not good at discriminating ability of the type being examined here?

7. A sample of 26 bulbs gives a mean life of 990 hours with a S.D. of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours. Is the sample not up to the standard.

8. The life time of electric bulbs for a random sample of 10 from a large consignment gave the following data.

Item:	1	2	3	4	5	6	7	8	9	10
Life in 1000 hrs:	1.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6

Can we accept the hypothesis that the average life time of bulbs is 4000 hours.

9. Two horse A and B were tested according to the time (in seconds) to run a particular track with the following results.

Horse A	28	30	32	33	33	29	34
Horse B	29	30	30	24	27	29	

Test whether the two horse have the same running capacity.

10. I n one sample of 10 observations, the sum of the squares of the devitions of the sample values from sample mean was 120 and in the other sample of 12 observations it was 314. Test whether the difference is significant at 5% level?

UNIT-III: DIVISIBILITY THEORY AND CANONICAL DECOMPOSITIONS

Short answer questions

1. Explain Division algorithm.
2. Find the quotient and remainder when -23 is divided by 5.
3. State and prove the Pigeonhole principle.
4. Express 1776_{eight} in base ten.
5. Express $3AD_{sixteen}$ as a binary digit.
6. Find the value of the base b for $144_b = 49$.
7. Determine whether 1601 is a prime number.
8. Find the consecutive integers < 100 that are composite numbers.
9. Find the gcd of the pair a^2-b^2, a^4-b^4
10. Find [a,b] if ab = 156 and a and b are relatively prime.

Long answer questions

1. Find the number of positive integers in the range 1976 through 3766 that are

   a) Divisible by 13. b) Not - divisible by 19.

2. Show that the equation $n^3 + (n+1)^3 + (n+2)^3 = (n+3)^3$ has a unique solution.
3. Prove by induction principle $n^4 + 2n^3 + n^2$ is divisible by 4.
4. Using the formula for $\pi(n)$ find the number of primes $\leqslant 100$.
5. Find six consecutive integers that are composites.
6. Using recursion, evaluate $(a^2b^2, ab^3, a^2b^3, a^3b^4, a^4b^4)$.
7. Use the Euclidean algorithm to express 4076 and 1024 as a linear combination of

   4076 and 1024.

8. Find the gcd of 175, 192 using canonical decomposition.

9. Find the positive factors of 60 and the number of trailing zeros in the decimal value of

1010!

10. Using recursion, find the lcm of the given integers. [15, 18, 24, 30].

UNIT-IV: DIOPHANTINE EQUATIONS AND CONGRUENCES

Short answer questions

1. When do we say an LDE to be solvable?
2. Examine whether the LDE $12x + 16y = 18$ is solvable.
3. Determine if the LDE $8x + 10y + 16z = 25$ is solvable.
4. State the reflexive, symmetric and transitive properties of congruences.
5. Explain linear congruence.
6. Determine if the congruence $2x \equiv 3 \pmod 4$ is solvable?
7. Explain the divisibility test for 5.
8. State the Chinese remainder theorem.
9. When do we say the linear system $ax + by \equiv e \pmod m$, $cx + dy \equiv f \pmod m$ has a unique solution?
10. Show that $x \equiv 12 \pmod{13}$ and $y \equiv 2 \pmod{13}$ is a solution of the 2 x 2 linear system

    $2x + 3y \equiv 4 \pmod{13}$, $3x + 4y \equiv 5 \pmod{13}$

Long answer questions

1. Examine whether the LDE $12x + 13y = 14$ is solvable. Find the general solution if

solvable.

2. Find the general solution of $63x - 23y = -7$

3. Find the general solution of the LDE $6x + 8y + 12z = 10$.

4. If a cock is worth five coins, a hen three coins, and three chicks together one coin,

how many cocks, hens and chicks totaling 100, can be bought for 100 coins?

5. Compute the remainder when 3^{247} is divided by 25.

6. The seven digit number 21358ab is divisible by 99. Find a and b.

7. Using the CRT, solve Sun-Tsu's puzzle $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$ and

$x \equiv 3 \pmod 7$.

8. Find the smallest integer > 10000 such that $3|n$, $4|n+3$, $5|n+4$, $7|n+5$ and $11|n+7$

9. Solve the linear system of congruences $3x + 4y \equiv 5 \pmod 7$, $4x + 5y \equiv 6 \pmod 7$, by

using elimination method.

10. Solve the linear system $3x + 13y \equiv 8 \pmod{55}$, $5x + 21y \equiv 34 \pmod{55}$

UNIT-V: CLASSICAL THEOREMS AND MULTIPLICATIVE FUNCTIONS

Short answer questions

1. State Wilson's theorem.
2. Find the self-invertible least residues modulo each prime 7.
3. State Fermat's little theorem.
4. Let p be a prime and a be any positive integer. Then prove that, $a^p \equiv a \pmod{p}$
5. Verify $(12+15)^{17} \equiv 12^{17} + 15^{17} \pmod{17}$
6. Describe Euler's Phi function.
7. State Euler's theorem.
8. Find $\Phi(18)$.
9. Describe Tau and Sigma functions.
10. If $n = 2^{2^n}$, find $\tau(n)$, $\sigma(n)$.

Long answer questions

1. Solve the congruence $x^2 \equiv 1 \pmod{m}$ for each modulo m. a) 6 b)8.
2. Find the remainder when 24^{1947} is divided by 17.
3. Find the remainder when 30^{2020} is divided by 19.

4. Find the ones digit in the base − seven expansion of each decimal number for 5^{101} and 37^{3434}.

5. Find the remainder when 245^{1040} is divided by 18.

6. Find the remainder when 199^{2020} is divided by 28.

7. Using Euler' s theorem, find the ones digit in the decimal value of 23^{7777}

8. Solve the linear congruence $25x \equiv 13 \pmod{18}$.

9. Compute $\tau(n)$ for each n a)18 b)23 c)1560 d)2187 e)6120

10. Compute $\sigma(n)$ for each n a)12 b)28 c)36 d)2187 e)6120

NUMBER THEORY AND STATISTICAL METHODS

UNIT-1

FUNDAMENTAL SAMPLING DISTRIBUTIONS

SHORT ANSWER QUESTIONS :-

1) Explain the concept of sample with an example

Sol:- Sample : A finite subset of statistical individuals in a population is known as sample. The no. of objects or observations in the sample is called size of the sample. It is denoted by 'n'

Example :-

Total production of items in a month is a population and total production of items in a day is a sample.

2) Classify the types of samples :

Sol:- Samples are classified into two ways

i) large sample : If the size of the sample $n \geq 30$, the sample is said to be a large sample

ii) Small sample : If the size of the sample $n < 30$, the sample is said to be a small sample (or) an exact sample.

3) Suppose the data set is the following 1.7, 2.2, 3.9, 3.11, 14.7 calculate the sample mean and sample median.

Sol:- Sample mean $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$

$$= \frac{1.7 + 2.2 + 3.9 + 3.11 + 14.7}{5} = \frac{25.61}{5} = 5.122.$$

Sample mean $\bar{x} = 5.122$.

Sample median $\bar{x} = x_{(n+1)/2}$, if n is odd.

Given, $n = 5$

Sample median $\bar{x} = x_{6/2} = x_3$

Sample median $\bar{x} = 3.9$

4) Calculate the sample mean and the sample variance for the data 3, 4, 5, 6, 7

Sol:- Sample mean $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$

$$\frac{3+4+5+6+7}{5} = \frac{25}{5} = 5$$

sample mean $\bar{x} = 5$

Sample variance $s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (x_i - \bar{x})^2$

$$\frac{1}{(5-1)} \sum_{i=1}^{n} (x_i - 5)^2$$

$$\frac{1}{4}\left[(3-5)^2 + (4-5)^2 + (5-5)^2 + (6-5)^2 + (7-5)^2\right]$$

$$\frac{1}{4}\left[(-2)^2 + (-1)^2 + (0)^2 + (1)^2 + (2)^2\right]$$

$$\frac{1}{4}(4+1+1+4)$$

$$\frac{1}{4}(10) = \frac{10}{4} = 2.5$$

Sample variance $s^2 = 2.5$

5) find the value of the finite population correction factor if

$n=10, N=1000$

Sol:

Given $\left.\begin{array}{l} n=10 \\ N=1000 \end{array}\right\} \rightarrow (1)$

W.K.T finite population correction factor is $\left(\frac{N-n}{N-1}\right) \rightarrow (2)$

where N is the population size

n is the sample size

sub (1) in (2) we get, $\frac{1000-10}{1000-1} = \frac{990}{999} = 0.991$

finite population correction factor $= 0.991$

6) How many different samples of size two can be chosen, from a finite population of size 25?

Sol:

we can take NC_n samples of size n from the finite population of size N $\rightarrow (1)$

Here $N = 25, n = 2 \rightarrow (2)$

Sub (2) in (1), we get, $25C_2 = 300$ $\left[\frac{25 \cdot \overset{12}{\cancel{24}}}{1 \cdot \cancel{2}} = 300\right]$

$\therefore$ we can take $25C_2 = 300$ samples of size 2 from a finite population of size 25.

7) In a random sample of 1000 packages shipped by air freight 13 had some damage. Find the standard error of proportions.

Sol:

sample size $n = 100$

proportion of damage packages, $P = \frac{13}{100}$

$\boxed{P = 0.13}$

W.K.T $Q = 1-P$

$1-0.13$

$\boxed{Q = 0.87} \rightarrow 2$

W.K.T standard error of sample proportion = S.E (P) =

$\sqrt{\frac{PQ}{n}} \rightarrow 3$

Sub 1, 2 in 3 we get $S.E\,(P) = \sqrt{\frac{(0.13)(0.87)}{1000}}$

$$\boxed{S.E\,(P) = 0.0106}$$

Standard error of sample proportion $S.E\,(P) = 0.0106$

8) State the central limit theorem.

Sol: If $\bar{x}$ be the mean of a random sample of size 'n' drawn from a population mean μ and finite variance σ^2 with standard deviation σ then the sampling distribution of sample mean $\bar{x}$ is approximately normal distribution with mean μ and standard deviation equal to the standard error $\bar{x} = \frac{\sigma}{\sqrt{n}}$ provided the sample size is large. (i.e.) $n \geq 30$

Theorem:

If $\bar{x}$ be the mean of a random sample of size n taken from a population mean μ and finite variance σ^2 and standard deviation then the limiting from of the distribution is the standardized sample mean then the $z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}}$ which is a random variable whose distribution function approaches that of standard normal distribution as $n \rightarrow \infty$

9) The variance of a population is 2. The size of the sample collected from the population is 169. what is the standard error of mean?

Sol:

Sample size $n = 169 \rightarrow (1)$

population standard deviation $\sigma = \sqrt{\text{variance}} = \sqrt{2} \rightarrow (2)$

W.K.T standard error (S.E) of sample mean, $\bar{x} = \frac{\sigma}{\sqrt{n}} \rightarrow (3)$

$= \frac{\sqrt{2}}{\sqrt{169}}$ [∵ Sub (1), (2) in (3)]

$= \frac{1.414}{13} = 0.109$

Standard error (S.E) of sample mean $\bar{x} = 0.109$

10) What is the effect on standard error, if a sample is taken from an infinite population of sample size is increased from 400 to 900?

Given: $n_1 = 400$

$n_2 = 900$

Sol:-

Sample size $= n$

Standard error (S.E) of sample mean, $\bar{x} = \frac{\sigma}{\sqrt{n}}$

$S.E_1 = \frac{\sigma}{\sqrt{n_1}} = \frac{\sigma}{\sqrt{400}} = \frac{\sigma}{20}$

$S.E_2 = \frac{\sigma}{\sqrt{n_2}} = \frac{\sigma}{\sqrt{900}} = \frac{\sigma}{30}$

$S.E_2 = \frac{\sigma}{30} = \frac{\cancel{2}}{30}\left(\frac{\sigma}{\cancel{20}_{10}}\right) = \frac{2}{3}(S.E_1)$

$S.E_2 = \frac{2}{3}(S.E_1)$

Thus if the sample size is increased from 400 to 900 the standard error (of sample mean $\bar{x}$) will be multiplied by $\frac{2}{3}$

LONG ANSWER QUESTIONS:-

1. A population consists of five numbers 2, 3, 6, 8, 11. consider all possible samples of size two which can be drawn with replacement from this population. Find
a) The mean of the population
b) The standard deviation of the population
c) The mean of the sampling distribution of means.
d) The standard deviation of the sampling distribution of means [(i.e) the standard error of means].

Sol:- Mean of the population, $\mu = \frac{2+3+6+8+11}{5}$

$\frac{30}{5} = 6$

a) mean of the population $\mu = 6 \rightarrow (1)$

b) variance of the population $\sigma^2 = \Sigma \frac{(x_i-\mu)^2}{n}$

$$= \frac{(2-6)^2+(3-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5}$$

$$= \frac{16+9+0+4+25}{5} = \frac{54}{5} = 10.8$$

variance of the population $\sigma^2 = 10.8$

Standard deviation of the population $\sigma = \sqrt{10.8} = 3.29$

Sampling with replacement (infinite population):

c) The total number of samples with replacement is.

$N^n = 5^2 = 25$

samples of size 2, where N = population size.

n = sample size

Standard deviation of the population $\sigma^2 = 10.8$

$\sigma = \sqrt{10.8} = 3.29$

Sampling with replacement (infinite population)

c) The total number of sample without replacement is

$NC_n = 5C_2 = \frac{5.\cancel{4}\,2}{1.\cancel{2}} = 10$ samples of size 2, where

N = population size

n = sample size

listing all possible samples of size 2 from population 2,3,6,8,11 without replacement, we get 10 samples. They are

$$\left\{\begin{matrix} (2,3) & (2,6) & (2,8) & (2,11) \\ (3,6) & (3,8) & (3,11) & \\ (6,8) & (6,11) & & \\ (8,11) & & & \end{matrix}\right\}$$

The corresponding sample means are $\left\{\begin{matrix} 2.5 & 4 & 5 & 6.5 \\ 4.5 & 5.5 & 7 & \\ 7 & 8.5 & & \\ 9.5 & & & \end{matrix}\right\} \rightarrow (J)$

The mean of the sampling distribution of mean,

$\mu\bar{x} = \frac{(J)}{10} = \frac{60}{10} = 6$

The mean of the sampling distribution of means,

$\mu\bar{x} = 6 \rightarrow (5)$

d) The variance of the sampling distribution of means is

$$\sigma_{\bar{x}}^2 = \frac{\left\{\begin{aligned}&(2.5-6)^2+(4-6)^2+(5-6)^2+(6.5-6)^2\\&+(4.5-6)^2+(5.5-6)^2+(7-6)^2\\&+(7-6)^2+(8.5-6)^2+(9.5-6)^2\end{aligned}\right\}}{10}$$

$$= \frac{40.5}{10} = 4.05$$

The variance of the sampling distribution of means

$$\sigma_{\bar{x}}^2 = 4.05$$

The standard deviation of the sampling distribution of means, $\sigma_{\bar{x}} = \sqrt{4.05} = 2.01$

Verification : for finite population involving sample without replacement,

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\left(\frac{\sqrt{N-n}}{\sqrt{N-1}}\right) = \left(\sqrt{\frac{N-n}{N-1}}\right)$$

$$2.32\left(\sqrt{\frac{5-2}{5-1}}\right) \quad [\because \text{from } 4]$$

$$2.32\left(\sqrt{\frac{3}{4}}\right)$$

$$2.32\left(\sqrt{0.75}\right)$$

$$\sigma_{\bar{x}} = 2.01 \rightarrow 7 \qquad \therefore 6 = 7$$

The standard deviation of the sampling distribution of means $\sigma_{\bar{x}} = 2.01$

The variance of the sampling distribution of mean $\sigma_{\bar{x}}^2 = 5.40$

The standard deviation of the sampling distribution of means,

$\sigma_{\bar{x}} = \sqrt{5.40} = 2.32$.

Verification: For in-finite population involving sampling with replacement,

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{3.29}{\sqrt{2}} = \frac{3.29}{1.414} = 2.32 \rightarrow (4) \qquad (3) = (4)$$

The standard deviation of the sampling distribution of mean $\sigma_{\bar{x}} = 2.32$.

2. A population consists of five numbers 2, 3, 6, 8, 11. consider all possible samples of size two which can be drawn without replacement from this population. Find

a) The mean of the population.

b) The standard deviation of the population.

c) The mean of the sampling distribution of means

d) The standard deviation of the sampling distribution of means [(i.e) the standard error of means]

Sol:- mean of the population $\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$

a) $\therefore$ mean of the population $\mu = 6 \rightarrow (1)$

b) variance of the population $\sigma^2 = \frac{\Sigma (x_i-\mu)^2}{n}$

$$\frac{(2-6)^2+(3-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5}$$

$$= \frac{54}{5} = 10.8$$

all possible samples of size 2 from infinite population 2, 3, 6, 8, 11 with replacement, we get 25 samples. They are,

$$\left\{\begin{matrix} (2,2) & (2,3) & (2,6) & (2,8) & (2,11) \\ (3,2) & (3,3) & (3,6) & (3,8) & (3,11) \\ (6,2) & (6,3) & (6,6) & (6,8) & (6,11) \\ (8,2) & (8,3) & (8,6) & (8,8) & (8,11) \\ (11,2) & (11,3) & (11,6) & (11,8) & (11,11) \end{matrix}\right\}$$

The corresponding sample mean are

$$\left\{\begin{matrix} 2 & 2.5 & 4 & 5 & 6.5 \\ 2.5 & 3 & 4.5 & 5.5 & 7 \\ 4 & 4.5 & 6 & 7 & 8.5 \\ 5 & 5.5 & 7 & 8 & 9.5 \\ 6.5 & 7 & 8.5 & 9.5 & 11 \end{matrix}\right\} \quad -(I)$$

The mean of the sampling distribution of mean $\mu_{\bar{x}} = \frac{(I)}{25}$

$$= \frac{150}{25} = 6$$

c) The mean of the sampling distribution of mean $\mu_{\bar{x}} = 6$

d) The variance of the sampling of mean is

$$\sigma_{\bar{x}}^2 = \left\{\begin{aligned} &(2-6)^2 + (2.5-6)^2 + (4-6)^2 + (5-6)^2 + (6.5-6)^2 \\ &+ (2.5-6)^2 + (3-6)^2 + (4.5-6)^2 + (5.5-6)^2 + (7-6)^2 \\ &+ (4-6)^2 + (4.5-6)^2 + (6-6)^2 + (7-6)^2 + (8.5-6)^2 \\ &+ (5-6)^2 + (5.5-6)^2 + (7-6)^2 + (8-6)^2 + (9.5-6)^2 \\ &+ (6.5-6)^2 + (7-6)^2 + (8.5-6)^2 + (9.5-6)^2 + (11-6)^2 \end{aligned}\right\} \quad [\because \text{from } ②]$$

$$\frac{135}{25} = 5.40$$

3. Let $u_1 = (3,7,8), u_2 = (2,4)$. find

a) μ_{u_1}

b) μ_{u_2}

c) Mean of the sampling distribution of the difference of means

d) σ_{u_1}

e) σ_{u_2}

f) The standard deviation of the sampling distribution of the difference of means $\sigma_{u_1-u_2}$

Sol:- Given $u_1 = (3,7,8)$ $u_2 = (2,4)$

$u_1 - u_2 = \{1,5,6,-1,3,4\}$

(a) $\mu_{u_1} = \frac{3+7+8}{3} = \frac{18}{3} = 6$

$\boxed{\mu_{u_1} = 6}$

(b) $\mu_{u_2} = \frac{2+4}{2} = \frac{6}{2} = 3$

$\boxed{\mu_{u_2} = 3}$

(c) $\mu_{u_1-u_2} = \frac{1+5+6-1+3+4}{6} = \frac{18}{6} = 3$

$\boxed{\mu_{u_1-u_2} = 3}$

(d) $\sigma_{u_1} = \sqrt{\frac{(3-6)^2+(7-6)^2+(8-6)^2}{3}}$

$= \sqrt{\frac{14}{3}} = \sqrt{4.67} = 2.16$

$\boxed{\sigma_{u_1} = 2.16}$

(e) $\sigma_{u_2} = \sqrt{\frac{(2-3)^2+(4-3)^2}{2}} = 1$

$\boxed{\sigma_{u_2} = 1}$

(f) $\sigma_{u_1-u_2} = \sqrt{\frac{(1-3)^2+(5-3)^2+(6-3)^2+(-1-3)^2+(3-3)^2+(4-3)^2}{6}} = \sqrt{\frac{34}{6}} \frac{17}{3}$

$= \sqrt{5.67} = 2.38$

$\boxed{\sigma_{u_1-u_2} = 2.38}$

4) A random sample of size 64 is taken from an infinite population having the mean 45 and standard deviation 8. what is the probability that $\bar{x}$ will be between 46 and 47.5 ?

Sol:-

population mean $\mu = 45$

population standard deviation $= \sigma = 8$

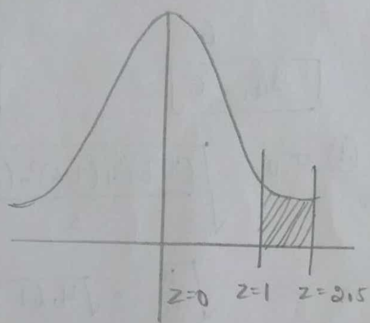
sample size $= n = 64$

sample means $\bar{x}_1 = \cancel{4.6}\ 46$

$\bar{x}_2 = 47.5$

W.K.T $z_1 = \dfrac{\bar{x}_1 - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{46-45}{\frac{8}{\sqrt{64}}} = \dfrac{1}{\frac{8}{8}} = 1$

$z_2 = \dfrac{\bar{x}_2 - \mu}{\frac{\sigma}{\sqrt{n}}} = \dfrac{47.5-45}{\frac{8}{\sqrt{64}}} = \dfrac{2.5}{8/8} = 2.5$



$\therefore P(46 \le \bar{x} \le 47.5) = P(1 < z < 2.5)$

$0.4938 - 0.3413$

$P(46 \le \bar{x} \le 47.5) = 0.1525$

The probality that $\bar{x}$ will be between 46 and 47.5 = 0.1525

5) A normal population has a mean of 0.1 and standard deviation of 2.1. Find the probability that mean of a sample size 900 will be negative ?

Sol:-

population mean $\mu = 0.1$

population standard deviation $\sigma = 2.1$

sample size $= n = 900$

W.K.T The standard normal variate is

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{x}-0.1}{\frac{2.1}{\sqrt{900}}} = \frac{\bar{x}-0.1}{\frac{2.1}{300}} = \frac{\bar{x}-0.1}{0.07}$$

$0.072 = \bar{x} - 0.1$

$\bar{x} = 0.072 + 0.1$ where $Z \sim N(0,1)$

$\therefore P(\bar{x} < 0) = P(0.072 + 0.1 < 0)$

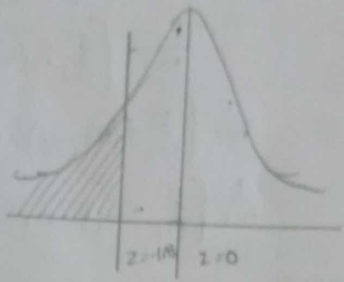
$P(0.072 < -0.1)$

$P\left(Z < \frac{-0.1}{0.07}\right)$

$P(Z < -1.43)$

$0.5 - 0.4236$

$P(\bar{x} < 0) = 0.0764$



The probability that mean of a sample size 900 will be negative $= 0.0764$

6) Show that the sample variance s^2 is an unbiased estimator of the population variance σ^2

Sol:- To prove:- $E(s^2) = \sigma^2$

W.K.T Sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$$\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\sum_{i=1}^{n}\left[(x_i - \mu) - (\bar{x} - \mu)\right]^2$$

$$\sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu)\sum_{i=1}^{n}(x_i - \mu) + n(\bar{x} - \mu)^2$$

$$\sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu)\left[(x_1 - \mu) + (x_2 - \mu) + \dots + (x_n - \mu)\right] + n(\bar{x} - \mu)^2$$

$$\sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu)\left[\frac{n(x_1 + x_2 + \dots + x_n)}{n} - n\mu\right] + n(\bar{x} - \mu)^2$$

$$\sum_{i=1}^{n} (x_i - \mu)^2 - 2(\bar{x}-\mu)\, n(\bar{x}-\mu) + n(\bar{x}-\mu)^2$$

$$\sum_{i=1}^{n} (x_i - \mu)^2 - 2n(\bar{x}-\mu)^2 + n(\bar{x}-\mu)^2$$

$$\boxed{\sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{i=1}^{n} (x_i-\mu)^2 - n(\bar{x}-\mu)^2} \rightarrow ①$$

Now. $E(s^2) = E\left[\dfrac{\sum_{i=1}^{n}}{n-1} (x_i - \bar{x})^2\right]$

$$\frac{1}{n-1}\left[\sum_{i=1}^{n} E(x_i-\bar{x})^2\right]$$

$$\frac{1}{n-1}\left[\sum_{i=1}^{n} E(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right]$$

$$\frac{1}{n-1}\left[\sum_{i=1}^{n} \sigma_{x_i}{}^2 - n\sigma_{\bar{x}}{}^2\right] \rightarrow ②$$

However $\sigma_{x_i}{}^2 = \sigma^2$ for $i = 1, 2, \ldots n \rightarrow ③$

and $\sigma_{\bar{x}}{}^2 = \dfrac{\sigma^2}{n}$

Sub ③ in ②, we get

$$E(s^2) = \frac{1}{n-1}\left[n\sigma^2 - \cancel{n}\frac{\sigma^2}{\cancel{n}}\right]$$

$$= \frac{1}{\cancel{(n-1)}} \cancel{(n-1)}\sigma^2$$

$$\boxed{E(s^2) = \sigma^2}$$

Hence the sample variance s^2 is an unbiased estimation of the population varience σ^2

7) To illustrate that the mean of the random sample is an unbiased estimate of the mean of the population. consider 5 slips of paper numbered 3, 6, 9, 15, 27

a) list all possible samples of size 3 that can be taken without replacement from this finite population.

b) calculate the mean of each of the samples listed in a) and assigning each sample a probability of each sample. verify that the mean of these $\bar{x}$ equal 12, namely the mean of the population.

Sol:- The total no. of samples without replacement is $NC_n = 5C_3$

$\frac{5 \cdot \cancel{4}^{2} \cdot \cancel{3}}{1 \cdot \cancel{2} \cdot \cancel{3}} = 10$ samples of size 3, where N = population size, n = sample size

listing all possible samples of size 3 from finite population 3, 6, 9, 15, 27 without replacement, we get 10 samples. They are,

$$\left\{ \begin{array}{l} (3,6,9), (3,6,15), (3,6,27), (3,9,15), (3,9,27), (3,15,27) \\ (6,9,15), (6,9,27), (6,15,27), (9,15,27) \end{array} \right\}$$

b) population mean $\mu = \frac{3+6+9+15+27}{5} = \frac{60}{5} = 12$

$\boxed{\mu = 12} \rightarrow (1)$

The mean of each of the samples $\bar{x}$ are 6, 8, 12, 9, 13, 15, 10, 14, 16, 17

$\bar{x}$	6	8	12	9	13	15	10	14	16	17
$P(\bar{x})$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

w.k.T $E(\bar{x}) = \sum_{i=1}^{n} \bar{x}_i p_i$

$6\left(\frac{1}{10}\right) + 8\left(\frac{1}{10}\right) + 12\left(\frac{1}{10}\right) + 9\left(\frac{1}{10}\right) + 13\left(\frac{1}{10}\right) + 15\left(\frac{1}{10}\right) + 10\left(\frac{1}{10}\right)$

$+14\left(\frac{1}{10}\right)+16\left(\frac{1}{10}\right)+17\left(\frac{1}{10}\right)$

$= \frac{1}{10}\left[6+8+12+9+13+15+10+14+16+17\right]$

$\frac{12\cancel{0}}{1\cancel{0}}$

$\boxed{E(\bar{x}) = 12} \rightarrow (2)$

$\therefore$ from (1), (2) $E(\bar{x}) = \mu = 12$.

Hence the mean of a random sample $\bar{x}$, is an unbiased estimate of the population mean μ

Hence proved.

8) Find 95% confidence limits for the mean of the normally distributed population of which the following sample taken from 15, 17, 10, 18, 16, 9, 7, 11, 13, 14

Sol:- Sample size $\boxed{n = 10} \rightarrow (1)$

Sample mean, $\bar{x} = \frac{15+17+10+18+16+9+7+11+13+14}{10}$

$\boxed{\bar{x} = 13} \rightarrow (2)$

sample variance $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

$\frac{1}{10-1}\left[\begin{array}{l}(15-13)^2+(17-13)^2+(10-13)^2+(18-13)^2+(10-13)^2 \\ +(9-13)^2+(7-13)^2+(11-13)^2+(13-13)^2+(14-13)^2\end{array}\right]$

$s^2 = \frac{40}{3}$

Sample standard deviation $\boxed{s = \sqrt{\frac{40}{3}}} \rightarrow (3)$

W.K.T maximum error estimate, E

$Z_{\alpha/2}\frac{s}{\sqrt{n}}$

$1.96 \times \frac{\sqrt{40}}{\sqrt{3}\sqrt{10}}$ [$\because$ from ①, ③]

$\boxed{E = 2.26} \rightarrow$ ④

95% confidence limits are given by $\bar{x} \pm E$

(13 ± 2.26) [$\because$ from ②, ④]

95% confidence limits = $(10.74, 15.26)$

9) A random sample of 400 items is found to have mean 82 and S.D 18. Find the maximum error of estimation, and the confidence limits if the mean is $\bar{x} = 82$ at 95% confidence.

Sol:-

Sample size $\boxed{n = 400} \rightarrow$ ①

Sample mean $\boxed{\bar{x} = 82} \rightarrow$ ②

sample S.D $\boxed{s = 18} \rightarrow$ ③

W.K.T maximum error estimate $E = Z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$

$1.96 \times \frac{18}{\sqrt{400}}$ [$\because$ from ①, ②]

$\boxed{E = 1.764} \rightarrow$ ④

95% confidence limits are given by $\bar{x} \pm E$

(82 ± 1.764) [$\because$ from ②, ④]

95% confidence limits = $(80.236, 83.764)$

10) The mean weight loss of $n = 16$ grinding balls after a certain length of time in mill surry is 3.42 grams with S.D of 0.68 grams. Construct 99% confidence interval for the mean weight loss of such grinding balls under the stated condition.

Sol:-

sample size $\boxed{n = 16} \rightarrow$ ①

Sample mean $\boxed{\bar{x} = 3.42} \rightarrow$ ②

Sample S.D $\boxed{S = 0.68} \rightarrow$ ③

maximum error estimate $E = t_{\alpha/2} \frac{S}{\sqrt{n}}$

$2.947 \times \frac{0.68}{\sqrt{16}}$ [∵ from ①, ③]

$\boxed{E = 0.4998} \rightarrow$ ④

99% confidence limits are given by $\bar{x} \pm E$

(3.42 ± 0.4998) [∵ from ②, ④]

99% confidence limits = $(2.9202, 3.9198)$

TESTING OF HYPOTHESIS

Short Answer Questions

1. Briefly explain the term statistical hypothesis.

<u>Solution</u>: In many circumstances, to arrive at decisions about the population on the basis of sample information, we make assumptions about the population parameters involved. such an assumption (or) statement is called a 'statistical hypothesis'.

2. Classify the types of hypothesis.

<u>Solution</u> There are two types of hypothesis. They are (i) Null hypothesis. (ii) Alternative hypothesis.

<u>Null hypothesis</u> (H_0):

The Null Hypothesis (H_0) is the negation of the claim. The Null Hypothesis is in the form $H_0 : \mu = \mu_0$ where μ_0 is the value which is assumed or claimed for the population of characteristic.

<u>Alternative hypothesis</u> (H_1):

The Alternative Hypothesis (H_1) is the claim we wish to establish. Any hypothesis which contradicts the Null Hypothesis (H_0) is called an Alternative Hypothesis (H_1). The Alternative Hypothesis is one of the following forms.

$H_1 : \mu \neq \mu_0$ (Two tailed test) (or) $H_1 : \mu > \mu_0$ (One tailed right test)

(or) $H_1 : \mu < \mu_0$ (One tailed left test)

3. Classify the types of errors involved in testing of hypothesis.

Solution:

Type I error: (Rejection of H_0 when it is true)

When a Null hypothesis is true, but the difference of means is significant and the hypothesis is rejected, then a type I error is made. The probability of making a type I error is denoted by 'α', the level of significance. The probability of making a correct decision is $(1-\alpha)$.

Type 2 error: (Acceptance of H_0 when it is false)

If the Null hypothesis is false, but it is accepted by test, then error committed is called a type 2 error. The probability of making a type 2 error is denoted by 'β', the level of significance. The probability of making a correct decision is then $(1-\beta)$.

4. Tabulate the critical values for large sample tests (two tailed, right tailed, left tailed) at 1%, 5%, 10% levels of significance?

Solution:

Test types	Level of Significance		
	1% (0.01)	5% (0.05)	10% (0.1)
two tailed test	$\lvert Z_\alpha \rvert = 2.58$	$\lvert Z_\alpha \rvert = 1.96$	$\lvert Z_\alpha \rvert = 1.645$
right tailed test	$Z_\alpha = 2.33$	$Z_\alpha = 1.645$	$Z_\alpha = 1.28$
left tailed test	$Z_\alpha = -2.33$	$Z_\alpha = -1.645$	$Z_\alpha = -1.28$

5. A coin was tossed 400 times and returned heads 216 times. Test the hypothesis that the coin is unbiased.

Solution

Given: Sample size, $n = 400$

probability of getting a head, $p = \frac{1}{2}$

probability of getting a tail, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Number of successes, $\boxed{x = 216}$

W.K.T for binomial distribution,

population mean $\mu = np$

$= 400 \times 1/2$

$\boxed{\mu = 200}$

population standard deviation,

$\sigma = \sqrt{npq}$

$= \sqrt{400 \times 1/4} = 10$

$\boxed{\sigma = 10}$

Null Hypothesis (H_0): The coin is unbiased.

Alternative Hypothesis (H_1): The coin is biased.

Level of significance: $\alpha = 0.05$

Test statistic: $Z = \frac{x - \mu}{\sigma} = \frac{216 - 200}{10} = 1.6$

$|Z|_{cal} = 1.6 \quad |Z|_{tab} = 1.96$

$|Z|_{cal} < |Z|_{tab}$

∴ We accept the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that the coin is unbiased.

6. A die is tossed 256 times and it turns up with an even ~~gi~~ digit 150 times. Is the die is biased?

Solution:

Given: sample size, $n = 256$

probability of getting an even digit (2 or 4 or 6), $p = 3/6 = 1/2$

probability of not getting an even digit, $q = 1 - p = 1/2$

W.K.T for binomial distribution,

population mean, $\mu = np$

$= 256 \times 1/2$

$\boxed{\mu = 128}$

population standard deviation, $\sigma = \sqrt{npq}$

$= \sqrt{256 \times 1/2 \times \frac{1}{2}}$

$\boxed{\sigma = 8}$

Number of successes, $\boxed{x = 150}$

Null Hypothesis (H_0): The die is unbiased.

Alternative Hypothesis (H_1): The die is biased.

Level of significance : $\alpha = 0.05$

Test statistic : $Z = \frac{x - \mu}{\sigma} = \frac{150 - 128}{8} = 2.75$

$|Z|_{cal} = 2.75 \quad |Z|_{tab} = 1.96$

$|Z|_{cal} > |Z|_{tab}$

∴ We reject the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that the die is biased.

7. Among 900 people in a state 90 are found to be chapatti eaters. Construct 99% confidence interval for the true proportion.

Solution :

Given : sample size, $\boxed{n = 900}$

population proportion, $P = \frac{90}{900} = 0.1 \quad Q = 1 - P = 0.9$

W.K.T 99% confidence interval is given by,

$$P \pm Z_{\alpha/2} \sqrt{\frac{PQ}{n}}$$

$$= 0.1 \pm 2.58 \sqrt{\frac{0.1 \times 0.9}{900}}$$

$$= (0.07, 0.12)$$

8. A random sample of 500 apples was taken from a large consignment and 60 were found to be bad, obtain the 98% confidence limits for the percentage number of bad apples in the consignment.

Solution : Given : sample size, $\boxed{n = 500}$

population proportion, $P = \frac{60}{500} = 0.12 \quad Q = 1 - P = 0.88$

W.K.T 98% confidence interval is given by,

$$P + Z_{\alpha/2} \sqrt{\frac{PQ}{n}}$$

$$= 0.12 \pm 2.33 \sqrt{\frac{0.12 \times 0.88}{500}}$$

$$= (0.087 \times 100, 0.153 \times 100)$$

$$= (8.7, 15.3)$$

9. In a random sample of 100 packages shipped by air freight 13 had some damage. Construct 95% confidence interval for the true proportion of damaged package.

Solution: Given : sample size, $n = 100$

population proportion, $P = \frac{13}{100} = 0.13$ $Q = 1 - P = 0.87$

W.K.T 95% confidence interval is given by,

$$P \pm Z_{\alpha/2} \sqrt{\frac{PQ}{n}}$$

$$= 0.13 \pm 1.96 \sqrt{\frac{0.13 \times 0.87}{100}}$$

$$= (0.06, 0.19)$$

10. A random sample of 500 pineapples was taken from a large consignment and 65 were found to be bad. Find the percentage of bad pineapples in the consignment.

Solution:

Given: sample size, $\boxed{n = 500}$

Sample proportion, $p = \frac{65}{500} = 0.13$, $q = 0.87$

W.K.T Limits for population proportion are given by, $p \pm 3\sqrt{\frac{pq}{n}}$

$$= 0.13 \pm 3 \sqrt{\frac{0.13 \times 0.87}{500}}$$

$$= (0.085 \times 100, 0.175 \times 100)$$

$$= (8.5, 17.5)$$

$\therefore$ The percentage of bad pineapples in the consignment lies between 8.5 and 17.5.

Long Answer Questions

1. A sample of 900 members has a mean of 3.4 cm and S.D 2.61 cm. Is this sample has been taken from a large population of mean 3.25 cm and S.D 2.61 cm. If the population is normal and its mean is unknown, find the 95% fiducial limits of the true mean.

Solution:

Given:

sample size, $\boxed{n = 900}$

population mean, $\boxed{\mu = 3.25}$

sample mean, $\boxed{\bar{x} = 3.4}$

sample S.D, $\boxed{S = 2.61}$

population S.D, $\boxed{\sigma = 2.61}$

Null Hypothesis (H_0) : $\mu = 3.25$

Alternative Hypothesis (H_1) : $\mu \neq 3.25$ (two tailed test)

Level of significance : $\alpha = 0.05$

Test statistic : $Z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{3.4 - 3.25}{2.61/\sqrt{900}} = 1.724$

$|Z|_{cal} = 1.724 \quad |Z|_{tab} = 1.96$

$|Z|_{cal} < |Z|_{tab}$

∴ we accept the Null Hypothesis (H_0) at 5% level of significance

∴ we conclude that the sample has been drawn from the population mean, $\mu = 3.25$

95% confidence or fiducial limits is given by

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$

$$= 3.4 \pm 1.96 \frac{(2.61)}{\sqrt{900}}$$

$$= (3.23, 3.57)$$

2. The means of two large samples of sizes of 1000 and 2000 members are 67.5 inches and 68 inches respectively. Can the samples be regarded as drawn from the same population of S.D 2.5 inches?

Solution:

Given sample sizes, $\boxed{n_1 = 1000, n_2 = 2000}$

sample means, $\boxed{\bar{x}_1 = 67.5, \bar{x}_2 = 68}$

population S.D, $\boxed{\sigma = 2.5}$

Null Hypothesis (H_0) : $\mu_1 = \mu_2$

Alternative Hypothesis (H_1) : $\mu_1 \neq \mu_2$ (two tailed test)

Level of significance : $\alpha = 0.05$

Test statistic: $Z = \dfrac{(\bar{x}_1 - \bar{x}_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \dfrac{67.5 - 68}{2.5\sqrt{\frac{1}{1000} + \frac{1}{2000}}} = -5.16$

$|Z|_{cal} = 5.16 \quad |Z|_{tab} = 1.96$

$|Z|_{cal} > |Z|_{tab}$

$\therefore$ We reject the Null Hypothesis (H_0) at 5% level of significance.

$\therefore$ We conclude that the samples are not drawn from the same population of S.D 2.5 inches ($\mu_1 \neq \mu_2$)

3. In a big city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers?

Solution:

Given : sample size, $\boxed{n = 600}$

sample proportion of smokers, $p = \dfrac{325}{600} \quad p = 0.54$

population proportion of smokers, $P = 1/2$

population proportion of nonsmokers, $Q = 1 - P = 1/2$

Null Hypothesis (H_0) : The number of smokers and non-smokers are equal in the city ($P = 0.5$)

Alternative Hypothesis (H_1) $P > 0.5$ (one tailed right test)

Level of significance : $\alpha = 0.05$

Test statistic : $Z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.54-0.5}{\sqrt{\frac{0.5 \times 0.5}{600}}} = 1.95$

$$Z_{cal} = 1.95 > Z_{tab} = 1.64$$

∴ We reject the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that majority of men in this city are smokers ($P > 0.5$)

4. A social worker believes that fewer than 25% of the couples in certain area have ever used any form of birth control. A random sample of 120 couples was contacted. Twenty of them said that they have used. Test the belief of the social worker at 0.05 level.

Solution:

Given : sample size, $\boxed{n = 120}$

sample proportion, $p = \frac{20}{120} = 0.166$

population proportion, $P = 0.25$

$Q = 1 - P = 0.75$

Null Hypothesis (H_0) : $P = 0.25$

Alternative Hypothesis (H_1) : $P < 0.25$ (one tailed left test)

Level of significance : $\alpha = 0.05$

Test statistic : $Z = \frac{p-P}{\sqrt{PQ/n}} = \frac{0.166-0.25}{\sqrt{\frac{0.25 \times 0.75}{120}}} = -2.1$

$$Z_{cal} = -2.1 < Z_{tab} = -1.645$$

∴ We accept the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that the belief of the social worker is true.

5. Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence. 200 men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and wowen in favour of the proposal are same, at 5% level.

Solution:

Given sample sizes, $\boxed{n_1 = 400 \quad n_2 = 600}$

sample proportion of men, $P_1 = \frac{200}{400} \Rightarrow \boxed{P_1 = 0.5}$

sample proportion of women, $P_2 = \frac{325}{600} \Rightarrow \boxed{P_2 = 0.5416}$

Null Hypothesis (H_0): Assume that there is no significant difference between the opinion of men and women as far as proposal of flyover is concerned. $P_1 = P_2$

Alternative Hypothesis (H_1): $P_1 \neq P_2$ (two tailed test)

Level of significance: $\alpha = 0.05$

Test statistic: $z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ where $p = \frac{n_1 P_1 + n_2 P_2}{n_1 + n_2}$

$p = \frac{400(0.5) + 600(0.5416)}{400 + 600}$

$\boxed{p = 0.525}$

$\boxed{q = 0.475}$

$z = \frac{0.5 - 0.5416}{\sqrt{(0.525)(0.475)\left(\frac{1}{400} + \frac{1}{600}\right)}} = -1.3$

$|z|_{cal} = 1.3 \quad |z|_{tab} = 1.96$

$|z|_{cal} < |z|_{tab}$

$\therefore$ We accept the Null Hypothesis (H_0) at 5% level of significance.

$\therefore$ We conclude that there is no difference of opinion between men and women as far as proposal of flyover is concerned.

6. On the basis of their total scores, 200 candidates of a civil service examination are divided into two groups, the upper 30% and the remaining 70%. Consider the first question of the examination. Among the first group, 40 had the correct answer, where as among the second group, 80 had the correct answer. On the basis of these results, can one conclude that the first question is not good at discriminating ability of the type being examined here?

Solution:

Given: sample sizes, $\boxed{\begin{array}{l} n_1 = 60 \\ n_2 = 140 \end{array}}$ $\boxed{\begin{array}{l} x_1 = 40 \\ x_2 = 80 \end{array}}$

$p_1 = \frac{x_1}{n_1} = \frac{40}{60}$

$\boxed{p_1 = 0.667}$

$\boxed{q_1 = 0.333}$

$p_2 = \frac{x_2}{n_2} = \frac{80}{140}$

$\boxed{p_2 = 0.571}$

$\boxed{q_2 = 0.429}$

Null Hypothesis(H_0): $P_1 = P_2$

Alternative Hypothesis(H_1): $P_1 \neq P_2$

(two tailed test)

Level of significance: $\alpha = 0.05$

Test statistic: $Z = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$

where $p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$

$p = \frac{60(0.667) + 140(0.571)}{60 + 140}$

$\boxed{p = 0.6}$ $q = 1 - p$ $\boxed{q = 0.4}$

$$Z = \frac{0.667 - 0.571}{\sqrt{(0.6)(0.4)\left(\frac{1}{60} + \frac{1}{140}\right)}}$$

$|Z|_{cal} = 1.26$

$|Z|_{tab} = 1.96$

$|Z|_{cal} < |Z|_{tab}$

∴ We accept the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that the first question is good enough in discriminating ability of the 57 students of both groups.

7. A sample of 26 bulbs gives a mean life of 990 hours with a S.D of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours. Is the sample not up to the standard,

Solution:

Given: sample size, $\boxed{n=26} < 30$, degrees of freedom, $\nu = n-1 = 26-1$ $\boxed{\nu = 25}$

∴ The sample is a small sample.

sample mean, $\boxed{\bar{x} = 990}$

sample S.D, $\boxed{S = 20}$

population mean, $\boxed{\mu = 1000}$

Here we know $\bar{x}, S, n, \mu$ so we use students t-test.

Null Hypothesis (H_0): The sample is upto the standard. $\mu = 1000$

Alternative Hypothesis (H_1): $\mu < 1000$. The sample is below the standard. (one tailed left test)

Level of significance: $\alpha = 0.05$

Test statistic: $t = \dfrac{\bar{x} - \mu}{S/\sqrt{n-1}} = \dfrac{990 - 1000}{20/\sqrt{25}} = -2.5$

$$|t|_{cal} = 2.5 \quad |t|_{tab} = 1.708$$

$$|t|_{cal} > |t|_{tab}$$

∴ We reject the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that the sample is not upto the standard (below the standard) $\mu < 1000$.

---

8. The lifetime of electric bulbs for a random sample of 10 from a large consignment gave the following data.

Item	1	2	3	4	5	6	7	8	9	10
Life in 1000 hours:	1.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6

Can we accept the hypothesis that the average life time of bulbs is 4000 hours.

Solution:

Given: Sample size, $\boxed{n=10} < 30$ Degrees of freedom, $\nu = n-1$
$= 10-1$
$\boxed{\nu = 9}$

$\therefore$ The sample is a small sample.

sample mean, $\bar{x}$ = Average life time of bulbs

$$= \frac{1.2+4.6+3.9+4.1+5.2+3.8+3.9+4.3+4.4+5.6}{10}$$

$$= 41/10 \Rightarrow \boxed{\bar{x} = 4.1}$$

sample variance, $s^2 = \sum_{i=1}^{10} \frac{1}{n-1}(x_i - \bar{x})^2$

$$= \frac{1}{9}\left\{\begin{matrix}(1.2-4.1)^2 + (4.6-4.1)^2 + (3.9-4.1)^2 + (4.1-4.1)^2 + (5.2-4.1)^2 + \\ (3.8-4.1)^2 + (3.9-4.1)^2 + (4.3-4.1)^2 + (4.4-4.1)^2 + (5.6-4.1)^2\end{matrix}\right\}$$

$$= \frac{1}{9}\{8.41+0.25+0.04+1.21+0.09+0.04+0.03+0.09+2.25\} = 12.42/9 = 1.38$$

Sample S.D, $s = \sqrt{1.38} \Rightarrow \boxed{s = 1.175}$

Null Hypothesis (H_0) : $\mu = 4000$

Alternative Hypothesis (H_1) : $\mu \neq 4000$

Level of significance : $\alpha = 0.05$

Test statistic : $t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{4100-4000}{1.175/\sqrt{10}} = \frac{4100-4000}{\sqrt{1380}/\sqrt{10}}$

$|t|_{cal} = 8.512$

$|t|_{tab} = 2.262$

$|t|_{cal} > |t|_{tab}$

$\therefore$ We reject the Null Hypothesis (H_0) at 5% level of significance.

$\therefore$ We conclude that the average life time of bulbs is not equal to 4000 hours.

---

9. Two horse A and B were tested according to the time (in seconds) to run a particular track with the following results.

Horse A	28	30	32	33	33	29	34
Horse B	29	30	30	24	27	29	

Test whether the two horse have the same running capacity.

Solution:

Sample sizes, $\boxed{n_1 = 7}$ $\boxed{n_2 = 6}$

sample means, $\bar{x} = \frac{\Sigma x}{n}$

$= \frac{28+30+32+33+33+29+34}{7} = \frac{219}{7}$

$\boxed{\bar{x} = 31.286}$

x	$(x-\bar{x})$	$(x-\bar{x})^2$	y	$(y-\bar{y})$	$(y-\bar{y})^2$
28	-3.286	10.8	29	0.84	0.7056
30	-1.286	1.6538	30	1.84	3.3856
32	0.714	0.51	30	1.84	3.3856
33	1.714	2.94	24	-4.16	17.3056
33	1.714	2.94	27	-1.16	1.3456
29	-2.286	5.226	29	0.84	0.7056
34	2.714	7.366			
$\Sigma x = 219$		$\Sigma(x-\bar{x})^2 = 31.4358$	$\Sigma y = 169$		$\Sigma(y-\bar{y})^2 = 26.8326$

Now sample variance, $S^2 = \frac{1}{n_1+n_2-2}\left[\Sigma(x_i-\bar{x})^2 + \Sigma(y_i-\bar{y})^2\right]$

$= \frac{1}{11}[31.4358 + 26.8336] = \frac{1}{11}[58.2694]$

$S^2 = 5.23 \Rightarrow \boxed{S = 2.3}$

Null Hypothesis (H_0) : $\mu_1 = \mu_2$

Alternative Hypothesis (H_1) : $\mu_1 \neq \mu_2$ (two tailed test)

Level of significance : $\alpha = 0.05$ Degrees of freedom $= \nu = n_1+n_2-2$

$\boxed{\nu = 11}$

Test statistic : $t = \frac{\bar{x}-\bar{y}}{S\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{31.286-28.16}{2.3\sqrt{\frac{1}{7}+\frac{1}{6}}} = 2.443$

$|t|_{cal} = 2.443$ $|t|_{tab} = 2.201$

$|t|_{cal} > |t|_{tab}$

$\therefore$ We reject the Null Hypothesis (H_0) at 5% level of significance.

$\therefore$ We conclude that both horses A and B do not have the same running capacity.

10. In one sample of 10 observations, the sum of the squares of the deviations of the sample values from sample mean was 120 and in the other sample of 12 observation it was 314. Test whether the difference is significant at 5% level?

Solution:

Given: sample sizes, $n_1 = 10$, $n_2 = 12$

sum of squares of deviation,

$\Sigma (x_i - \bar{x})^2 = 120$

$\Sigma (y_i - \bar{y})^2 = 314$

W.K.T Sample S.D's

$$S_1^2 = \frac{\Sigma (x_i - \bar{x})^2}{n_1 - 1} = \frac{120}{10-1} = 13.33$$

Level of significance; $\alpha = 0.05$

$$S_2^2 = \frac{\Sigma (y_i - \bar{y})^2}{n_2 - 1} = \frac{314}{12-1} = 28.54$$

Degrees of freedom
$= (\nu_1, \nu_2)$
$= (n_1 - 1, n_2 - 1)$
$= (9, 11)$

Null Hypothesis (H_0): $\sigma_1^2 = \sigma_2^2$

Alternative Hypothesis (H_1): $\sigma_1^2 \neq \sigma_2^2$

Test statistic: $F = \frac{S_2^2}{S_1^2} = \frac{28.54}{13.33}$

$F_{cal} = 2.14$ $F_{tab} = 2.9$

$F_{cal} < F_{tab}$

∴ We accept the Null Hypothesis (H_0) at 5% level of significance.

∴ We conclude that the variances of two populations are same.

UNIT-III : DIVISIBILITY THEORY AND CANONICAL DECOMPOSITIONS

Short answer questions

1. Explain Division algorithm.

Ans: Division algorithm: Suppose that an integer "a" is divided by a positive integer "b" they we get a unique quotient "q" and remainder "r" where the remainder satisfy, the condition $0 \leq r \leq b$ "a" is divident and "b" is the divisor.

2. Find the quotient and remainder when -23 is divided by 5.

Sol: $a = -23$

$b = 5$

$$\begin{array}{r} -5 \\ 5\,\overline{)\,-23} \\ \oplus\ 25 \\ \hline 2 \end{array}$$

quotient $= -5$

Remainder $= 2$

[NOTE: The remainder r can never be negative i.e $r \geq 0$]

3. state and prove the pigeonhole principle.

Sol: Statement: If "m" pigeons are assinged to "n" pigeon hole where "m>n" then atleast two pigeons must occupy the same pigeon hole.

Proof:

We proof by contradaction

Suppose the given statement false is No two pigeons occupy the same pigeon hole. Then every pigeon must occupy the distinct pigeon hole. So $n \geq m$. which the contradation. Hence two are more pigeon Must occupy same pigeon holes.

T.P.O

4. Express 1776_{eight} in base ten.

Sol: $(1776)_8 = 1\times8^3+7\times8^2+7\times8^1+6\times8^0$

$= 1\times512+7\times64+56+6$

$= 512+448+56+6$

$= 1022$

5. Express $3AD_{sixteen}$ as a binary digit

Sol: $3 = 0(2^3)+0(2^2)+1(2^1)+1(2^0) = 0011$

$A = 10 = 1(2^3)+0(2^2)+1(2^1)+0(2^0) = 1010$

$D = 13 = 1(2^3)+1(2^2)+0(2^1)+1(2^0) = 1101$

$(3AD)_{16} = (001110101101)_2$

$= (1110101101)_2$

6. Find the value of the base b for $144_b = 49$.

Sol: $1(b)^2+4(b^1)+4(b^0) = 49$

$b^2+4b+4 = 49$

$b^2+4b-45 = 0$

$(b+9)(b-5) = 0$

$b+9=0$ | $b-5=0$

$b=-9$ | $b=5$

(Not admissible).

7. Determine whether 1601 is a prime number.

T.P.O

Sol: First we list all primes $\leq [\sqrt{1601}] = 40$

The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 and 37.

Since none of them is a factor of 1601.

Hence 1601 is a prime.

8. Find the consecutive integers <100 that are composite numbers.

Sol: So we know that $5! = 120 > 100$

So choose $4!, 4!+1, 4!+2, 4!+3, 4!+4$

$= 24, 25, 26, 27, 28$

are the 5 consecutive composite number <100.

9. Find the gcd of the pair a^2-b^2, a^4-b^4

Sol: $(a^2-b^2, a^4-b^4) = a^2-b^2$

$a^4-b^4 = (a^2)^2 - (b^2)^2$

$= (a^2-b^2)(a^2+b^2)$

10. Find $[a,b]$ if $ab=156$ and a and b are relatively prime.

Sol: $[a,b] = \frac{ab}{(a,b)} = \frac{ab}{1} = 156$

Since $(a,b) = 1$.

T.P.O

Long answer questions

1. Find the number of positive integers in the range 1976 through 3766 that are

a] Divisible by 13

b] Not - divisible by 19.

Sol: a] $\lfloor 3766/13 \rfloor - \lfloor 1976/13 \rfloor$

$= 289 - 152$

$= 137$

b] $3766 - \lfloor 3766/19 \rfloor$

$3766 - 198$

$= 3568$

$1976 - \lfloor 1976/19 \rfloor$

$1976 - 104$

$= 1872$

Required answer $= 3568 - 1872$

$= 1698.$

2. Show that the equation $n^3 + (n+1)^3 + (n+2)^3 = (n+3)^3$ has a unique solution.

Sol: We know that the sum of the cubes of three consecutive integers is a cube "k^3", then $3/k$."

$\Rightarrow 3/n$

$n = 3m$, where m is an integer.

T.P.O

$n^3 + (n+1)^3 + (n+2)^3 = (n+2)^3$

$(3m)^3 + (3m+1)^3 + (3m+2)^3 = (3m+2)^3$

$27m^3 + (27m^2 + 27m^2 + 9m + 1) + (27m^3 + 54m^2 + 36m + 8)$

$= 27(m^3 + 3m^2 + 3m + 1)$

$54m^3 - 36m - 18 = 0$

$\Rightarrow 3m^3 - 2m - 1 = 0$ $\quad \because \langle m^2 + 3m + 1 \neq 0 \rangle$

$(m-1)(3m^2 + 3m + 1) = 0$ [Since no real roots we get].

$m - 1 = 0$

$m = 1$

$n = 3m$

$n = 3$

So the solution (1) is unique

3. Prove by induction principle $n^4 + 2n^3 + n^2$ is divisible by 4.

Sol: Let $p(n) = n^4 + 2n^3 + n^2$

$n=1 \Rightarrow P(1) = 1 + 2 + 1 = 4$ is divisible by 4

So $P(1)$ is true.

Assume, $n = k \Rightarrow P(K) = K^4 + 2K^3 + K^2 = 4m$ for some $m \in z^+$ is true

To prove $p(K+1)$ is true

$(K+1)^4 + 2(K+1)^3 + (K+1)^2$

$= (K+1)^2 [(1K+1)^2 + 2(K+1) + 1]$

$= (K+1)^2 [K^2 + 2K + 1 + 2K + 2 + 1]$

$= (K+1)^2 [K^2 + 4K + 4]$

T.P.O

$= (K^4 + 2K^3 + K^2) + 4(K^3 + 3K^2 + 2K + 1)$

$= 4m + 4(K^3 + 3K^2 + 2K + 1)$ which is divisible by 4.

$\Rightarrow$ $P(K+1)$ is true

Hence by induction principle the resolt is true for every integer "$n \geq 0$."

4. Using the formula for $\pi(n)$ find the number of primes ≤ 100.

Sol: Here. $n = 100$

Then $\pi\sqrt{n} = \pi(\sqrt{100}) = \pi(10) = 4$

The four primes ≤ 10 are 2, 3, 5 and 7

let $P_1 = 2$

$P_2 = 3$

$P_3 = 5$

$P_4 = 7$

Then $\pi(100) = 100 - 1 + 4 - \left(\left[\frac{100}{2}\right] + \left[\frac{100}{3}\right] + \left[\frac{100}{5}\right] + \left[\frac{100}{7}\right]\right) +$

$\left(\left[\frac{100}{2.3}\right] + \left[\frac{100}{2.5}\right] + \left[\frac{100}{2.7}\right] + \left[\frac{100}{3.5}\right] + \left[\frac{100}{3.7}\right] + \left[\frac{100}{5.7}\right]\right) - \left(\left[\frac{100}{2.3.5}\right]\right.$

$\left. + \left[\frac{100}{2.3.7}\right] + \left[\frac{100}{2.5.7}\right] + \left[\frac{100}{3.5.7}\right]\right) + \left[\frac{100}{2.3.5.7}\right]$

$= 103 - (50 + 33 + 20 + 14) + (16 + 10 + 7 + 6 + 4 + 2) -$
$(3 + 2 + 1 + 0) + 0$

$= 25$

$\therefore$ The number of primes ≤ 100 is "25."

T.P.O

5. Find six consecutive integers that are composites

Sol: Here $n=6$

They the six consecutive composite are

$(n+1)!+2, (n+1)!+3, (n+1)!+4, (n+1)!+5, (n+1)!+6, (n+1)!+7$

put $n=6$

$(7)!+2, (7)!+3, (7)!+4, 7!+5, 7!+6, 7!+7$

$= 5040+2, 5040+3, 5040+4, 5040+5, 5040+6, 5040+7$

$= 5042, 5043, 5044, 5045, 5046, 5047.$

The six consecutive composites are
5042, 5043, 5044, 5045, 5046 & 5047

$(n+1)!+2 = (6+1)!+2 = 7!+2 = 5040+2$
$= 5042$

6. Using recursion, evaluate $(a^2b^2, ab^3, a^2b^3, a^3b^4, a^4b^4)$

Sol: $(a^2b^2, ab^3, a^2b^3, a^3b^4, a^4b^4)$

$= ((a^2b^2, ab^3, a^2b^3, a^3b^4), a^4b^4)$

$= (((a^2b^2, ab^3, a^2b^3)\ a^3b^4)\ a^4b^4)$

$= ((((a^2b^2, ab^3)\ a^2b^3)\ a^3b^4)\ a^4b^4)$

$= (ab^2, a^2b^3)\ a^3b^4)\ a^4b^4)$

$= (ab^2, a^3b^4)\ a^4b^4)$

$= (ab^2, a^4b^4)$

$= ab^2$

T.P.O

7. Use the Euclidean algorithm to express 4076 and 1024 as a linear combination of 4076 and 1024.

Sol: Here $4076 > 1024$

By apply division algorithm successfully we get

$$4076 = 3(1024) + 1004$$
$$1024 = 1(1004) + 20$$
$$1004 = 50(20) + 4$$
$$20 = 5(4) + 0$$

The last non-zero remainder is 4

$(4076, 1024)$ gcd $= 4$

Using the above equation in reverse order

$$(4076, 1024) = 4$$
$$= 1004 - 50(20)$$
$$= 1004 - 50[1024 - 1(1004)]$$
$$= 51[(1004)] - 50(1024)$$
$$= 51[4076 - 3(1024)] - 50(1024)$$
$$= 51[4076) - 203(1024)$$
$$= 51(4076) + (-203)(1024)$$

$\therefore$ gcd is a linear combination of the number 1024 and 4076.

8. Find the gcd of 175, 192 using conoical decomposition.

Sol:

$$\begin{array}{r|l} 5 & 175 \\ \hline 5 & 35 \\ \hline & 7 \end{array} \qquad \begin{array}{r|l} 2 & 192 \\ \hline 2 & 96 \\ \hline 2 & 48 \\ \hline 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

$175 = 5^2 \cdot 7 \cdot 1$

$192 = 2^6 \cdot 3 \cdot 1$

$\therefore$ gcd $(175, 192) = 1$

T.P.O

9. Find the positive factors of trailing zeros in the decimal value of 1010!

Sol: a] First notice that $60 = 2^2 . 3 . 5$

By the fundamental theorem of airthematic every factor of 60 is of the form $2^a . 3^b . 5^c$ where $0 \leq a \leq 2$ and $0 \leq b, c \leq 1$ Thus, the various factors are

$2^0 3^0 5^0 = 1$	$2^0 3^0 5^1 = 5$	$2^0 3^1 5^0 = 3$	$2^0 3^1 5^1 = 15$
$2^1 3^0 5^0 = 2$	$2^1 3^0 5^1 = 10$	$2^1 3^1 5^0 = 6$	$2^1 3^1 3^1 = 30$
$2^2 3^0 5^0 = 4$	$2^2 3^0 5^1 = 20$	$2^2 3^1 5^0 = 12$	$2^2 3^1 5^1 = 60$

Hence 60 has 12 factors.

b] $\left\lfloor \frac{1010}{5} \right\rfloor + \left\lfloor \frac{1010}{25} \right\rfloor + \left\lfloor \frac{1010}{125} \right\rfloor + \left\lfloor \frac{1010}{625} \right\rfloor$

$= 202 + 40 + 8 + 1$

$= 251.$

10. Using recursion, find the lcm of the given integers $\{15, 18, 24, 30\}$.

Sol: $\{15, 18, 24, 30\} = \{\{15, 18, 24\} 30\}$

$= \{\{\{15, 18\} 24\} 30\}$

$= \{\{90, 40\}, 30\}$

$= \{360, 30\}$

$3 \underline{| 15, 8}$

$5, 6$

Lcm $= 3 \times 5 \times 6 = 90$

$3 \underline{| 90, 24}$

$2 \underline{| 30, 8}$

$15, 4$

Lcm $= 3 \times 2 \times 15 \times 4 = 360$

T.P.O

$$\begin{array}{r|l} 2 & 360, 30 \\ \hline 5 & 180, 15 \\ \hline 3 & 36, 3 \\ \hline & 12, 1 \end{array}$$

Lcm $= 2 \times 5 \times 3 \times 12 = 360.$

UNIT-IV
DIOPHANTINE EQUATIONS AND CONGRUENCES

Short answer Questions:-

1. when do we say an LDE to be solvable?

Sol A linear diophantine equation in two variables x and y is a diophantine equation of the form $ax+by=c$.

→ The LDE $ax+by=c$ is solvable. if and only if $d|c$, where $d=(a,b)$. If x_0, y_0 is a particular solution of the LDE, then all its solutions are given by

$$x=x_0+\left(\frac{b}{d}\right)t \text{ and } y=y_0-\left(\frac{a}{d}\right)t$$

where t is an arbitary integer.

2. Examine whether the LDE $12X+16Y=18$ is solvable.

Sol Given: $12x+16y=18$

Compare with $ax+by=c$

Here, $a=12, b=16, c=18$

$(a,b)=d$

$\Rightarrow (12,16)=4$

so, $d=4$

Here $d \nmid c$ i.e., $4 \nmid 18$

Hence, the given LDE is not solvable.

3. Determine if the LDE $8x+10y+16z=25$ is solvable.

Sol: Given: $8x+10y+16z=25$

Here, $(8,10,16)=2$

But $2 \nmid 25$ so the LDE is not solvable.

4. State the reflexive, symmetric and transitive properties of congruences.

Sol: Reflexive property:-

$a \equiv a \pmod m$

Symmetric property:-

If $a \equiv b \pmod m$ then $b \equiv a \pmod m$

Transitive property:-

If $a \equiv b \pmod m$ and $b \equiv c \pmod m$, then $a \equiv c \pmod m$

5. Explain linear congruence.

Sol: Linear Congruence:

A congruence is of the form $ax \equiv b \pmod m$, where m is a positive integer and a, b are integers and x is a variable, is called a linear congruence.

-> The linear congruence $ax \equiv b \pmod m$ is solvable if and only if $d \mid b$, where $d=(a,m)$. If $d \mid b$, then it has d incongruent solutions.

6. Determine if the congruence $2x \equiv 3 \pmod 4$ is solvable.

Sol: Given, $2x \equiv 3 \pmod 4$

$(2,4) = 2$, but $2 \nmid 3$, so the congruence $2x \equiv 3 \pmod 4$ is not solvable and has no solutions.

7. Explain the divisibility test for 5.

Sol: Divisibility test for 5:-

We know that $n \equiv n_0 \pmod{10}$, n is divisible by 5 if and only if n_0 is divisible by 5.

But the only single-digit numbers divisible by 5 are 0 and 5, so an integer is divisible by 5 if and only if it ends in a 0 or 5.

8. State the chinese remainder theorem.

Sol: The chinese Remainder Theorem:-

The linear system of congruences $x \equiv a_i \pmod{m_i}$, where the modules are pairwise relatively prime and $1 \le i \le k$, has a unique solution modulo $m_1 m_2 \cdots m_k$

9. When do we say the linear system $ax+by \equiv e \pmod m$, $cx+dy \equiv f \pmod m$ has a unique solution?

Sol: The linear system,

$$ax+by \equiv e \pmod m$$
$$cx+dy \equiv f \pmod m$$

has a unique solution if and only if $(\Delta, m) = 1$, where $\Delta \equiv ad-bc \pmod m$

10. show that $x \equiv 12 \pmod{13}$ and $y \equiv 2 \pmod{13}$ is a solution of the 2×2 linear system $2x+3y \equiv 4 \pmod{13}$, $3x+4y \equiv 5 \pmod{13}$.

Sol:

Given linear system,

$$2x+3y \equiv 4 \pmod{13} \longrightarrow (1)$$

$$3x+4y \equiv 5 \pmod{13} \longrightarrow (2)$$

when $x \equiv 12 \pmod{13}$ and $y \equiv 2 \pmod{13}$ is a solution of (1) & (2), we get

$13 \,\underline{|\, 30}$
$2 - 4$

$(1) \Rightarrow 2x+3y \equiv 2(12)+3(2) \equiv 4 \pmod{13}$

$13 \,\underline{|\, 44}$
$3 - 5$

$(2) \Rightarrow 3x+4y \equiv 3(12)+4(2) \equiv 5 \pmod{13}$

$\therefore$ Every pair $x \equiv 12 \pmod{13}$, $y \equiv 2 \pmod{13}$ is a solution of the system.

The general solution of the system is $x = 12+13t$, $y = 2+13t$, with t being an arbitary integer.

Long answer questions:-

1. Examine whether the LDE $12X+13Y=14$ is solvable. Find the general solution if solvable.

Sol: Given: $12X+13Y=14 \longrightarrow (1)$

Compare with $ax+by=c$

Here $a=12,\ b=13,\ c=14$

$$(a,b)=d \Rightarrow (12,13)=1$$

So, $d=1$ Here, $d|c$

Hence, the given LDE is ~~not~~ solvable.

$(1) \Rightarrow 13Y=14-12X$

$$Y=\frac{14-12X}{13}$$

X	-1	0	....
Y	2	$\frac{14}{13}$	....

$x_0=-1,\ y_0=2$ is a particular solution.

So the general solution is

$x = x_0+\left(\frac{b}{d}\right)t$ | $Y = Y_0-\left(\frac{a}{d}\right)t$

$x = -1+\left(\frac{13}{1}\right)t$ | $Y = 2-\left(\frac{12}{1}\right)t$

i.e., $X=-1+13t$ | $Y=2-12t$

$\therefore X=-1+13t,\ t\in Z$

$Y=2-12t,\ t\in Z$

2. Find the general solution of $63x-23y=-7$

Sol: Given: $63x-23y=-7 \longrightarrow (1)$

$(63,23)=1$

$\therefore$ the LDE has a solution.

To find a particular solution x_0, y_0.

We apply the Euclidean algorithm.

$63=2.23+17 \quad \ldots (1)$

$23=1.17+6 \quad \ldots (2)$

$17=2.6+5 \quad \ldots (3)$

$6=1.5+1 \quad \ldots (4)$

$5=5.1+0 \quad \ldots (5)$

Now, use the first four equations in reverse order.

$1=6-1.5$
$=6-1(17-2.6)$
$=3.6-1.17$
$=3(23-1.17)-1.17$
$=3.23-4.17$
$=3.23-4(63-2.23)$
$1=(-4).63+11.23$

Multiply both sides of the equation by -7

$-7=(-7)(-4).63+(-7).11.23$
$=63.28-23.77$

$\Rightarrow$ a particular solution of the LDE is $x_0=28, y_0=77$

$\Rightarrow$ the general solution is $x=x_0+bt=28-23t$

$y=y_0-at=77-63t$

where $t \in z$

3. Find the general solution of the LDE $6x+8y+12z=10$

Sol: Given: $6x+8y+12z=10 \rightarrow (1)$

Here $(6,8,12)=2$

$2|10$, so the given LDE is solvable and has infinitely many solutions.

Since $8y+12z$ is a linear combination of 8 and 12, it must be a multiple of $(8,12)=4$

So, we get $8y+12z=4u \rightarrow (2)$

$(1) \Rightarrow 6x+4u=10 \rightarrow (3)$

In equation (3)
$a=6; b=4;$
$c=10; d=2$

Here $(6,4)=2$

$2|10$, so the LDE (3) is solvable.

We find $x_0=5, u_0=-5$ is a particular solution of (3)

$[\because 6(5)+4(-5)=10]$

$\therefore$ the general solution of (3) is

$x=x_0+\frac{b}{d}t \Rightarrow x=5+\frac{4}{2}t$

$\Rightarrow x=5+2t$

$u=u_0-\frac{a}{d}t \Rightarrow u=-5-\frac{6}{2}t$

$u=-5-3t$

$(2) \Rightarrow 8y+12z=4(-5-3t)$

Since $(8,12)=4$ and $4=2.8+(-1).12$

Multiply by $-5-3t$, we get

$$(4)(-5-3t) = 2(-5-3t)\cdot 8 + (-1)(-5-3t)\cdot 12$$

$$= (-10-6t)\cdot 8 + (5+3t)\cdot 12$$

$\therefore$ a solution of (2) is

$$y_0 = -10-6t, \; z_0 = 5+3t$$

So the general solution of (2) is

$$y = y_0 + \frac{b}{d}t' \text{ and } z = z_0 - \frac{a}{d}t', \; t' \in Z$$

$$y = (-10-6t) + \left(\frac{12}{4}\right)t' \text{ and } z = 5+3t - \frac{8}{4}t'$$

$$y = -10-6t+3t' \text{ and } z = 5+3t-2t'$$

Hence, the general solution of (1) is

$$x = 5+2t, \; y = -10-6t+3t', \; z = 5+3t-2t'$$

where t and t' are arbitrary integers.

4. If a cock is worth five coins, a hen three coins and three chicks together one coin, how many cocks, hens and chicks, totaling 100, can be bought for 100 coins?

Sol: Let $x \to$ number of cocks

$y \to$ number of hens

$z \to$ number of chicks

clearly $x, y, z \geq 0$, then the given data has two LDEs

$$x+y+z = 100 \longrightarrow (1)$$

$$5x+3y+\frac{1}{3}z = 100 \longrightarrow (2)$$

(1) $\Rightarrow z = 100 - x - y$

(2) $\Rightarrow 5x + 3y + \frac{1}{3}[100 - x - y] = 100$

i.e, $5x + 3y + \frac{100}{3} - \frac{1}{3}x - \frac{1}{3}y = 100$

$$(5 - \tfrac{1}{3})x + (3 - \tfrac{1}{3})y = 100 - \frac{100}{3}$$

$$\frac{14}{3}x + \frac{8}{3}y = \frac{200}{3}$$

$$7x + 4y = 100 \longrightarrow (3)$$

(3) $\Rightarrow y = \frac{100 - 7x}{4} = 25 - \frac{7}{4}x \longrightarrow (4)$

y be an integer $\Rightarrow -\frac{7}{4}x$ must be an integer.

but $4 \nmid 7$ so, $x = 4t, t \in z$

Then, $y = 25 - \frac{7}{4}x$

$\Rightarrow y = 25 - 7t$

and $z = 100 - x - y$

$\Rightarrow z = 100 - 4t - (25 - 7t)$

$= 75 + 3t$

Hence, every solution to the puzzle is of the form $x = 4t$, $y = 25 - 7t$, $z = 75 + 3t$, where t is an arbitary integer.

Now, to find the possible actual solutions.

Since $x \geq 0$, $t \geq 0$. Since $y \geq 0$, $25 - 7t \geq 0$; i.e., $t \leq 25/7$, so $t \leq 3$

Since $z \geq 0$ $75 + 3t \geq 0$; i.e., $t \geq -25$, but this does not give us any additional information, so $0 \leq t \leq 3$.

Hence, the riddle has four possible solutions,

If $t=0$, then $x=0$, $y=25$, $z=75$

If $t=1$, then $x=4$, $y=18$, $z=78$

If $t=2$, then $x=8$, $y=11$, $z=81$

If $t=3$, then $x=12$, $y=4$, $z=84$

5. Compute the remainder when 3^{247} is divided by 25.

Sol:

$3^2 \equiv 9 \pmod{25}$ $\quad$ $25\,|\,81$ $\quad 3-6$

$(3^2)^2 \equiv 9^2 \pmod{25} \Rightarrow 3^4 \equiv 6 \pmod{25}$ $\quad$ $25\,|\,36$ $\quad 1-11$

$(3^4)^2 \equiv 6^2 \pmod{25} \Rightarrow 3^8 \equiv 11 \pmod{25}$ $\quad$ $25\,|\,121$ $\quad 4-21$

$(3^8)^2 \equiv 11^2 \pmod{25} \Rightarrow 3^{16} \equiv 21 \pmod{25}$ $\quad$ $25\,|\,441$ $\quad 17-16$

$(3^{16})^2 \equiv 21^2 \pmod{25} \Rightarrow 3^{32} \equiv 16 \pmod{25}$ $\quad$ $25\,|\,256$ $\quad 10-6$

$(3^{32})^2 \equiv 16^2 \pmod{25} \Rightarrow 3^{64} \equiv 6 \pmod{25}$

$(3^{64})^2 \equiv 6^2 \pmod{25} \Rightarrow 3^{128} \equiv 11 \pmod{25}$

(128 is the largest power of 2 contained in 247)

$$
\begin{aligned}
3^{247} &= 3^{128+64+32+16+4+2+1} \\
&= 3^{128}\,3^{64}\,3^{32}\,3^{16}\,3^{4}\,3^{2}\,3 \\
&\equiv (11)(6)(16)(21)(6)(9)(3) \pmod{25} \\
&\equiv (11)(6.16)(21)(6.9)(3) \pmod{25} \\
&\equiv [(11(-4)][(-4)(-4)](3) \pmod{25} \\
&\equiv (6)(9)(3) \pmod{25} \\
&\equiv (4)(3) \pmod{25} \\
&\equiv 12 \pmod{25}
\end{aligned}
$$

Hence, 12 is the desired remainder

6. The seven digit number 21358ab is divisible by 99. Find a and b.

Sol: Since $21358ab \equiv 0 \pmod{99}$, ~~21358ab)~~ $21358ab \equiv 0 \pmod{9}$ and $21358ab \equiv 0 \pmod{11}$.

Then $2+1+3+5+8+a+b \equiv 0 \pmod{8}$ and

$(2+3+8+b)-(1+5+a) \equiv 0 \pmod{11}$

i.e, $a+b \equiv 8 \pmod{9}$ and $b \equiv a+4 \pmod{11}$

The pairs of values ~~st~~ satisfying $a+b \equiv 8 \pmod{9}$ are given by:

a	0	1	2	3	4	5	6	7	8
b	8	7	6	5	4	3	2	1	0

and those satisfying the congruence $b \equiv a+4 \pmod{11}$ are given by

a	5	4	3	2	1	0
b	9	8	7	6	5	4

There is just one pair $(2,6)$ common to the pairs of values satisfying both congruences.

Hence $a=2$ and $b=6$

7. Using the CRT, solve Sun-Tsu's Puzzle $x \equiv 1 \pmod{3}$, $x \equiv 2 \pmod{5}$ and $x \equiv 3 \pmod{7}$

Sol: Since the moduli $m_1=3$, $m_2=5$ and $m_3=7$ are pairwise relatively prime, by CRT, the linear system has a unique solution.

To find M_1, M_2, M_3

$$M_1 = \frac{M}{m_1} = \frac{3.5.7}{3} = 35,$$

$$M_2 = \frac{M}{m_2} = \frac{3.5.7}{5} = 21,$$

$$M_3 = \frac{M}{m_3} = \frac{3.5.7}{7} = 15$$

To find y_1, y_2 and y_3

y_1 is the solution of the congruence $M_1 y_1 \equiv 1 \pmod{m_1}$;

i.e., $35 y_1 \equiv 1 \pmod 3 \Rightarrow (-1) y_1 \equiv 1 \pmod 3 \Rightarrow y_1 \equiv 2 \pmod 3$

$|||^{ly}$ $M_2 y_2 \equiv 1 \pmod{m_2}$

$\Rightarrow 21 y_2 = 1 \pmod 5$

$\Rightarrow y_2 = 1 \pmod 5$

Finally, $M_3 y_3 \equiv 1 \pmod{m_3}$ yields

$\Rightarrow 15 y_3 \equiv 1 \pmod 7$

$\Rightarrow y_3 \equiv 1 \pmod 7$

Hence by the CRT,

$$x \equiv \sum_{i=1}^{3} a_i M_i y_i \pmod M$$

$$\equiv 1.35.2 + 2.21.1 + 3.15.1 \pmod{105}$$

$$\equiv 52 \pmod{105}$$

$\therefore$ 52 is the unique solution of the linear system modulo 105.

Hence the general solution is

$$x = 52 + 105t$$

8. Find the smallest integer >10000 such that $3|n$, $4|n+3$, $5|n+4$, $7|n+5$ and $11|n+7$

Sol: Given: $n \equiv 0 \pmod{3}$

$n \equiv 1 \pmod{4}$

$n \equiv 1 \pmod{5}$

$n \equiv 2 \pmod{7}$

$n \equiv 4 \pmod{11}$

Since the moduli are pairwise relatively prime,

$$M_2 = \frac{M}{m_2} = \frac{3.4.5.7.11}{4} = 1155$$

$$M_3 = \frac{M}{m_3} = \frac{3.4.5.7.11}{5} = 924$$

$$M_4 = \frac{M}{m_4} = \frac{3.4.5.7.11}{7} = 660$$

$$M_5 = \frac{M}{m_5} = \frac{3.4.5.7.11}{11} = 420$$

$M_2 Y_2 \equiv 1 \pmod{m_2}$

$\Rightarrow 1155 Y_2 \equiv 1 \pmod{4}$

$\Rightarrow Y_2 \equiv 3 \pmod{4}$

$M_3 Y_3 \equiv 1 \pmod{m_3}$

$\Rightarrow 924 Y_3 \equiv 1 \pmod{5}$

$\Rightarrow Y_3 \equiv 4 \pmod{5}$

$M_4 Y_4 \equiv 1 \pmod{m_4}$

$\Rightarrow 660 Y_4 \equiv 1 \pmod{7}$

$\Rightarrow Y_4 = 4 \pmod{7}$

$M_5 Y_5 \equiv 1 \pmod{m_5}$

$\Rightarrow 420 Y_5 \equiv 1 \pmod{11}$

$\Rightarrow Y_5 \equiv 6 \pmod{11}$

Hence by CRT,

$$x = (0)(M_1)(Y_1) + (1)(1155)(3) + (1)(924)(4) + (2)(660)(4) + (4)(420)(6)$$

$$x = 4041 \pmod{4620}$$

So, the desired integer greater than $10,000 = 4041+4620(2)$

$= 4041 + (2)(4620)$

$= 13281$

9. solve the linear system of congruences $3x+4y\equiv 5 \pmod 7$, $4x+5y\equiv 6 \pmod 7$ by using elimination method.

sol: Given: $3x+4y\equiv 5 \pmod 7 \rightarrow (1)$

$4x+5y\equiv 6 \pmod 7 \rightarrow (2)$

$(1)\times 5 \Rightarrow 15x+20y\equiv 25 \pmod 7 \rightarrow (3)$

$(2)\times 4 \Rightarrow 16x+20y\equiv 24 \pmod 7 \rightarrow (4)$

$(3)-(4) \Rightarrow -x = 1 \pmod 7$

$\Rightarrow x\equiv -1 \pmod 7$

$\Rightarrow x\equiv 6 \pmod 7$

$(1) \Rightarrow (3)(6)+4y\equiv 5 \pmod 7$

$18+4y\equiv 5 \pmod 7$

$\Rightarrow 4y\equiv -13 \pmod 7 \rightarrow (\cancel{8})$

$\Rightarrow 4y\equiv 1 \pmod 7 \rightarrow (5)$

$\Rightarrow 8y\equiv 2 \pmod 7$

$\Rightarrow y\equiv 2 \pmod 7$

Hence, $x\equiv 6 \pmod 7$, $y\equiv 2 \pmod 7$

10. solve the linear system $3x+13y\equiv 8 \pmod{55}$, $5x+21y\equiv 34 \pmod{55}$

sol: Given: $3x+13y\equiv 8 \pmod{55} \rightarrow (1)$

$5x+21y\equiv 34 \pmod{55} \rightarrow (2)$

Here $a=3$, $b=13$, $c=5$, $d=21$

$e=8$, $f=34$, $m=55$

$\Delta\equiv (ad-bc) \pmod m$

$\equiv [(3)(21)-(13)(15)] \pmod{55}$

$\equiv 53 \pmod{55}$

$(\Delta, m) = (53, 55) = 1$

$\therefore$ the system has unique solution modulo 55

$\Delta\Delta^{-1} \equiv 1 \pmod{55}$

$\Rightarrow 53\,\Delta^{-1} \equiv 1 \pmod{55}$

$\Rightarrow -2\,\Delta^{-1} \equiv 1 \pmod{55}$

If $\Delta^{-1} = -28$ then $(-2).(-28) = 56 \equiv 1 \pmod{55}$

$-28 \equiv 27 \pmod{55}$

$\Rightarrow \Delta^{-1} \equiv 27 \pmod{55}$

$x_0 \equiv \Delta^{-1}(de - bf) \pmod{55}$

$\equiv (27)[(21)(8) - (13)(34)] \pmod{55}$

$\equiv 27 \pmod{55}$

$\boxed{-274 \pmod{55} \equiv 1 \pmod{55}}$

$y_0 \equiv \Delta^{-1}(af - ce) \pmod{55}$

$\equiv 27[(3)(34) - (5)(8)] \pmod{55}$

$\equiv 24 \pmod{55}$

$\boxed{62 \pmod{55} \equiv 7 \pmod{55}}$

Hence, $x \equiv 27 \pmod{55}$

$y \equiv 24 \pmod{55}$

is the unique solution to the given system.

UNIT-V

CLASSICAL THEOREMS AND MULTIPLICATIVE FUNCTIONS

Short answer questions

(1) State Wilson's theorem.

Sol. According to Wilson's theorem

If p is a prime, then $(p-1)! \equiv -1 \pmod{p}$

(2) Find the self-invertible least residues modulo each prime 7.

Sol. Given $p=7$

Then $(p-1)! = 6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6$

The least residues modulo 7 that are self-invertible are 1 & 6.

(3) State Fermat's little theorem.

Sol. The Fermat's little theorem states,

If; Let p be a prime and a any integer such that $p \nmid a$. Then the least residues of the integers $a, 2a, 3a, \ldots, (p-1)a$ modulo p are a permutation of the integers $1, 2, 3, \ldots, (p-1)$.

(4) Let p be a prime and a be any positive integer. Then prove that $a^p \equiv a \pmod{p}$

Sol. Case 1: Suppose $p \nmid a$. Then, by Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$, so $a^p \equiv a \pmod{p}$

Case 2: Suppose $p \mid a$. Then $p \equiv a \equiv 0 \pmod{p}$, so $a^p \equiv 0 \pmod{p}$

$\therefore a^p \equiv a \pmod{p}$

Hence, in both cases, $a^p \equiv a \pmod{p}$

(5) Verify $(12+15)^{17} \equiv 12^{17} + 15^{17} \pmod{17}$

Sol. $(12+15)^{17} = (27)^{17} \equiv 27 \pmod{17}$

$=(12+15) \pmod{17}$

$= 12^{17}+15^{17} \pmod{17}$

(6) Describe Euler's phi function

Sol· Let m be a positive integer. Then Euler's phi function $\phi(m)$ denotes the number of positive integer $\leq m$ and relatively prime to m

(7) State Euler's theorem.

Sol· Euler's theorem states:

Let m be a positive integer and a any integer with $(a,m)=1$

Then $a^{\phi(m)} \equiv 1 \pmod{m}$

(8) Find $\phi 18$

Sol·

$$18 = (2)(3^2)$$

$$\begin{array}{r|l} 2 & 18 \\ \hline 3 & 9 \\ \hline & 3 \end{array}$$

$$\phi(18) = \phi(2)\,\phi(3^2)$$

$$= \phi(2)\, 3^2\left(1-\frac{1}{3}\right)$$

$$=(1)(9)\left(\frac{2}{3}\right) \quad [\because \phi(2) = 2-1 \text{ by Theorem 1}]$$

$$= 6 \quad \left[\phi(p^e) = p^e\left(1-\frac{1}{p}\right) \text{ Theorem 5}\right]$$

(9) Describe Tau and Sigma functions.

Sol·

The Tau fuction: Let n be a positive integer. Then $\tau(n)$ denotes the number of positive factors of n,

i.e $\tau(n) = \sum_{d|n} 1$

The Sigma fuctions: Let n be a positive integer. Then $\sigma(n)$ denotes the sum of the positive factors of n,

i.e $\sigma(n) = \sum_{d|n} d$

(10) If $n=2^{2^e}$, find $\tau(n)$ and $\sigma(n)$

Sol·

Given $n = 2^{2^e}$

$$\tau(n) = \tau(2^{2^e}) = 2^e + 1$$

$\tau(n) = \sigma(2^{2^e}) = 2^{2^{e+1}} - 1$

Long answer questions

(1) Solve the congruence $x^2 \equiv 1 \pmod m$ for each modulo m (a) 6 (b) 8

Sol. (a) 6

Given: $m = 6$

$\therefore x^2 \equiv 1 \pmod 6$

$x = 1 \Rightarrow 1^2 \equiv 1 \pmod 6$

$x = 2 \Rightarrow 4 \not\equiv 1 \pmod 6$

$x = 3 \Rightarrow 9 \not\equiv 1 \pmod 6$

$x = 4 \Rightarrow 16 \not\equiv 1 \pmod 6$

$x = 5 \Rightarrow 25 \equiv 1 \pmod 6$

$\therefore x = 1, 5$

(b) 8

Given: $m = 8$

$\therefore x^2 \equiv 1 \pmod 8$

$x = 1 \Rightarrow 1 \equiv 1 \pmod 8$

$x = 2 \Rightarrow 4 \not\equiv 1 \pmod 8$

$x = 3 \Rightarrow 9 \equiv 1 \pmod 8$

$x = 4 \Rightarrow 16 \not\equiv 1 \pmod 8$

$x = 5 \Rightarrow 25 \equiv 1 \pmod 8$

$x = 6 \Rightarrow 36 \not\equiv 1 \pmod 8$

$x = 7 \Rightarrow 49 \equiv 1 \pmod 8$

$\therefore x = 1, 3, 5, 7$

(2) Find the remainder when 24^{1947} is divided by 17

Sol.

$24 \equiv 7 \pmod{17}$

$\Rightarrow 24^{1947} \equiv 7^{1947} \pmod{17}$ ---- ①

$\boxed{\text{Fermat's little theorem is } a^{p-1} \equiv 1 (\text{mod } p), \text{ p prime, any integer } p \nmid a}$

Here, $p = 17$, $p-1 = 16$, $a = 7$, $17 \nmid 7$

$\therefore 7^{16} = 1 (\text{mod } 17)$ ---- ②

$7^{1947} \equiv 7^{(16)(121)+11} = (7^{16})^{121} \, 7^{11}$

$16 \overline{)1947}$
$1936 - 11$

$\therefore 7^{1947} \equiv 1^{121} \, 7^{11} (\text{mod } 17)$

$7^{1947} \equiv 7^{11} (\text{mod } 17)$ ---- ③

$7^2 = -2 (\text{mod } 17)$

$7^{11} \equiv (7^2)^5 \, 7 \, (\text{mod } 17)$

$\equiv (-2)^5 \, 7 (\text{mod } 17)$

$\equiv (-32)(7) (\text{mod } 17)$

$\equiv (2)(7) \text{ mod } 17$

$\equiv 14 \text{ mod } 17$ ---- ④

$\Rightarrow 24^{1947} \equiv 14 \text{ mod}(17)$ by ②, ③, ④

So, the remainder is 14

(3) Find the remainder when 30^{2020} is divided by 19

Sol. $30 \equiv (-8) (\text{mod } 19)$

$30^{2020} \equiv (-8)^{2020} (\text{mod } 19)$

$\equiv 8^{2020} (\text{mod } 19)$ ---- ①

$\boxed{\text{Fermat's little theorem is } a^{p-1} \equiv 1 (\text{mod } p), \text{ p prime, any integers } p \nmid a}$

$8^{18} \equiv 1 (\text{mod } 19)$

$18 \overline{)2020}$
$112 - 4$

$8^{2020} \equiv (8^{18})^{112} \, 8^4 (\text{mod } 19)$

$\equiv (1)\ 8^4 \pmod{19}$ ---- (2)

$8^4 = 4096$

$30^{2020} \equiv 11 \pmod{19}$

$19 \,|\, 4096$
$215 - 11$

(4) Find the ones digits in the base - seven expansion of each decimal number (a) 5^{101} (b) 37^{3434}

Sol. (a) 5^{101}

$5 \equiv -2 \pmod 7$

$5^{101} \equiv (-2)^{101} \pmod 7$ ---- (1)

$\boxed{\text{Fermat's little theorem is } a^{p-1} \equiv 1 \pmod p, p \text{ prime, any integer } p \nmid a}$

$\therefore (-2)^6 \equiv 1 \pmod 7$

$6 \,|\, 101$
$16 - 5$

$(-2)^{101} \equiv (-2)^{(6 \times 16)+5} \pmod 7$

$\equiv (-2)^{6(16)} (-2)^5 \pmod 7$

$7 \,|\, 32$
$4 - 4$

$\equiv (1)\ (-2)^5 \pmod 7$

$= -4 \pmod 7$

$5^{101} \equiv 3 \pmod 7$

(b) 37^{3434}

$37 \equiv 2 \pmod 7$

$(37)^{3434} \equiv 2^{3434} \pmod 7$

$\boxed{\text{Fermat's little theorem is } a^{p-1} \equiv 1 \pmod p, p \text{ prime, any integer } p \nmid a}$

$2^6 \equiv 1 \pmod 7$

$2^{3434} \equiv (2^6)^{572}\ 2^2 \pmod 7$

$6 \,|\, 3434$
$572 - 2$

$\equiv (1)^{572}\ 2^2 \pmod 7$

$= 4 \pmod 7$

(5) Find the remainder when 245^{1040} is divided by 18.

Sol. $245 \equiv 11 \pmod{18}$

$245^{1040} \equiv 11^{1040} \pmod{18}$ ---- (1)

$\phi 18 = \phi(3^2 \times 3)$

$= \phi(3^2)\,\phi(2)$

$= 3^2\left(1-\frac{1}{3}\right)(1)$

$= 9\left(\frac{2}{3}\right)(1)$

$= 6$

$11^{\phi(18)} \equiv 11^6 \equiv 1 \pmod{18}$ by Euler

(1) $\Rightarrow 11^{1040} = (11^6)^{173} \cdot 11^2 \equiv 1^{173}(121) \bmod 18$

$\equiv 13 \bmod 18$

$6 \,\underline{|\, 1040}$
$173 - 2$

Hence, the remainder is 13

(6) Find the remainder when 199^{2020} is divided by 28

Sol.

$199 \equiv 3 \pmod{28}$

$199^{2020} \equiv 3^{2020} \pmod{28}$ ---- (1)

$2 \,\underline{|\, 28}$
$2 \,\underline{|\, 14}$
7

$\phi(28) = \phi(2^2)\,\phi(7)$

$= 2^2\left(1-\frac{1}{2}\right)(6)$

$= (4)\left(\frac{1}{2}\right)(6) = 12$

$3^{\phi(28)} \equiv 3^{12} \equiv 1 \pmod{28}$

$12 \,\underline{|\, 2020}$
$168 - 4$

(1) $\Rightarrow 199^{2020} = 3^{2020} \pmod{28}$

$\equiv (3^{12})^{168}\, 3^4 \pmod{28}$

$= 1^{168}(81) \pmod{28}$

$= 25 \pmod{28}$

Hence, the remainder is 25

(7) Using Euler's theorem, find the ones digits in the decimal value of 23^{7777}

Sol. $23 \equiv 3 \bmod (10)$

$23^{7777} \equiv 3^{7777} \pmod{10}$ ----(1)

$\phi(10) = \phi(2)\,\phi(5)$

$= (1)(4) = 4$

$3^{\phi(10)} \equiv 3^4 \equiv 1 \pmod{10}$ by Euler.

$4 \overline{)7777}$
$1944 \; -1$

(1) $\Rightarrow 23^{7777} \equiv 3^{7777} \pmod{10}$

$\equiv (3^4)^{1944}\, 3 \pmod{10}$

$\equiv 1^{1944}\, 3 \pmod{10}$

$\equiv 3 \pmod{10}$

So the ones digit is 3

(8) Solve the linear congruence $25x \equiv 13 \pmod{18}$

Sol. $25x \equiv 13 \pmod{18}$

$\Rightarrow 7x \equiv 13 \pmod{18}$ ---- (1)

$\phi(18) = \phi(2)\,\phi(3^2)$

$= (2-1)\, 3^2 \left[1 - \frac{1}{3}\right]$

$= (1)(9)\left(\frac{2}{3}\right)$

$= 6$

(1) $\Rightarrow x = 7^{\phi(18)-1}\, 13 \pmod{18}$

$x \equiv 7^{6-1}\, 13 \pmod{18}$

$\equiv 7^{6}(13) \pmod{18}$ ---- (2)

$7^{6} \equiv 13 \pmod{18}$

$\therefore (2) \Rightarrow x \equiv (13)(13) \pmod{18}$

$\equiv 7 \pmod{18}$

(9) Compute $\tau(n)$ for each n (a) 18 (b) 23 (c) 1560 (d) 2187 (e) 6120

Sol. (a) 18

$18 = (2^1)(3^2)$

$\tau(18) = \tau(2^1)\,\tau(3^2)$

$= (1+1)(2+1)$ $\quad [\because \tau = (p^e) = e+1]$

$= (2)(3) = 6$

2	18
3	9
	3

NOTE:- The number of positive divisors of 18 are 1, 2, 3, 6, 9 and 18

(b) 23

23, being a prime

23 has exactly two positive divisors

So $\tau(23) = 2$

NOTE:- The number of positive divisors of 23 are 1, 23

(c) 1560

$1560 = (2^3)(3)(5)(13)$

$\tau(1560) = \tau(2^3)\,\tau(3^1)\,\tau(5^1)\,\tau(13)^1$

$= (3+1)(1+1)(1+1)(1+1)$

$= (4)(2)(2)(2)$

$= 32$

2	1560
2	780
2	390
3	195
5	65
	13

(d) 2187

$2187 = 3^7$

$\tau(2187) = \tau(3^7)$

$= (7+1) = 8$

3	2187
3	729
3	243
3	81
3	27
3	9
	3

e. 6120

$6120 = (2^3)(5)(3^2)(17)$

$$\begin{array}{r|l} 2 & 6120 \\ 2 & 3060 \\ 2 & 1530 \\ 5 & 765 \\ 3 & 153 \\ 3 & 51 \\ & 17 \end{array}$$

$\tau(6120) = \tau(2^3)\,\tau(5)\,\tau(3^2)\,\tau(17)$

$= (3+1)(1+1)(2+1)(1+1)$

$= (4)(2)(3)(2) = 48$

(10) Compute $\sigma(n)$ for each n (a) 12 (b) 28 (c) 36 (d) 2187 (e) 6120

Sol: a. 12

Let p be any prime and e any prime, then

$$\sigma(p^e) = \frac{p^{e+1} - 1}{p-1}$$

$12 = (2^2)(3)$

$$\begin{array}{r|l} 2 & 12 \\ 2 & 6 \\ & 3 \end{array}$$

$\sigma(12) = \sigma(2^2).\sigma(3^1)$

$$= \left[\frac{2^{2+1} - 1}{2-1}\right]\left[\frac{3^{1+1} - 1}{3-1}\right]$$

$$= \left[\frac{8-1}{1}\right]\left[\frac{9-1}{2}\right]$$

$= (7)(4)$

$= 28$

b. 28

$28 = (2^2)(7)$

$$\begin{array}{r|l} 2 & 28 \\ 2 & 14 \\ & 7 \end{array}$$

$\sigma(28) = \sigma(2^2)\,\sigma(7^1)$

$$= \left[\frac{2^{2+1} - 1}{2-1}\right]\left[\frac{7^{1+1} - 1}{7-1}\right]$$

$$= \left[\frac{8-1}{1}\right]\left[\frac{49-1}{6}\right] \Rightarrow [7]\,[8]$$

$= 56$

c. 36

$$36 = (2^2)(3^2)$$

$$\begin{array}{r|l} 2 & 36 \\ 2 & 18 \\ 3 & 9 \\ & 3 \end{array}$$

$$\sigma(36) = \sigma(2^2)\,\sigma(3^2)$$

$$= \left[\frac{2^{2+1}-1}{2-1}\right]\left[\frac{3^{2+1}-1}{3-1}\right]$$

$$= \left[\frac{8-1}{1}\right]\left[\frac{27-1}{2}\right]$$

$$= (7)(13) \Rightarrow 91$$

(d) 2187 =

$$2187 = 3^7$$

$$\begin{array}{r|l} 3 & 2187 \\ 3 & 729 \\ 3 & 243 \\ 3 & 81 \\ 3 & 27 \\ 3 & 9 \\ & 3 \end{array}$$

$$\sigma(2187) = \sigma(3^7)$$

$$= \frac{3^{7+1}-1}{3-1} = \frac{6560}{2}$$

$$= 3280$$

(e) 6120

$$6120 = (2^3)(3^2)(5)(17)$$

$$\begin{array}{r|l} 2 & 6120 \\ 2 & 3060 \\ 2 & 1530 \\ 3 & 765 \\ 3 & 255 \\ 5 & 85 \\ & 17 \end{array}$$

$$\sigma(6120) = \sigma(2^3)\,\sigma(3^2)\,\sigma(5)\,\sigma(17)$$

$$= \left[\frac{2^{3+1}}{2-1}\right]\left[\frac{3^{2+1}-1}{3-1}\right]\left[\frac{5^{1+1}-1}{5-1}\right]\left[\frac{17^{1+1}-1}{17-1}\right]$$

$$= \left[\frac{15}{1}\right]\left[\frac{26}{2}\right]\left[\frac{24}{4}\right]\left[\frac{288}{16}\right]$$

$$= (15)(13)(6)(18)$$

$$= 21060$$