**UNIT-IV**

**Partial Differential Equations**

**INTRODUCTION**

D.E.: An Equation involving a dependent variable and differential coefficient of the dependent variable with respect to one a more than one independent variables is called a D.E.

**Partial Differential Equations**

A. D.E. in which the diffentials involved are with reference to two or more than two independent variables is called partial differential equation.

Ex : 

**Linear & Non linear P.D.E**

If the partial derivatives as well as the dependent variable occur in first degree only and separately, Such a P.D.E is said to the linear P.D.E..

Otherwise it is a non –linear P.D.E.

**Homogeneous & Non Homogeneous P.D.E**

A P.D.E is said to the Homogeneous it each term of the equation entairs either the dependent variable as one of its derivations

Otherwise it is said to be Non - Homogeneous

**Notations**

If  = f(x,y), we employ the following rotations



**Formation & partial Differential Equations.**

Partial Differential equations can be formed by two methods

1. By the elimination of arbitrary constants
2. By the elimination of arbitrary functions
3. **Method of Elimination of arbitrary contants**

If the number of arbitrary constants to be eliminated is equal to the number of independent variables, we get a partial differential equation of first order.

If the number of arbitrary constants to be eliminated is grater than the number of independent variables we get a partial differential equation of higher order.

1. **From the partial differential equation by eliminating the arbitrary constants from.**

Z = ax+by+ab

Let Z = ax +by +ab ------------1

Diff (1). Partially w.r.t x and y

p = and q= 

=a = b

By eq 1 we have

Z = px+qy+pq

Which is the required D.E.

**Linear partial Differential Equations of First Order**

**Lagrange’s Linear Equation:**

A linear partial differential equation of the first order, commonly known as Lagrange’s linear equation is of the form

Pp + Qq = R

Where P,Q and R are functions of x,y,z

And p = and .

**Method of solving**

To solve the equation Pp+Qq = R

(i). form the auxiliary equations

dx/P =dy/Q= dz/R

(ii). Solve the auxiliary equations obtaining two independent solutions *u* =a and *v* =b

(iii). Then the solution is φ(u,v)=0 or u = F(*v*) or *v*=F(u).

**Solution of the subsidiary equations**

The subsidiary equations are

dx/P =dy/Q= dz/R

**I Method of Grouping:**

The equations are dx/P =dy/Q= dz/R by taking first two members dx/P =dy/Q and then

Integrating we get an equation, say u=a which gives one equation of the solution similarly, taking any other two members, and then integrating, we get another equation, say v= b which gives another equation of the solution.

II Method of Multiplies

If *l*,m,n are functions of x,y,z are constants, then

dx/P =dy/Q= dz/R = 

now if the multiplies *l*,m,n are chosen in such a way that *l*P+mQ+nR = 0 then *l*d*x*+mdy+ndz =0. Integrating we get u =a. This gives one equation of the solution.

Similarly, taking multiplies l1, m1, n1 in such a way that l1P+m1Q+n1R = 0 then *l1*d*x*+m1dy+n1dz =0. Integrating v =b. this gives another equation of solution.

The two solution u=a, v=b so obtained form the complete solution.

**Note:** We may get one solution u=a from the method of grouping and another solution *v*=b from the method of multiplies.

**Problems**

**1). Solve x(y-z)p+y(z-x)q=z(x-y).**

Sol:- The Auxiliary equations are



Taking 1,1,1 as multiplies

Each fraction = = 

This gives dx+dy+dz = 0

⇒ x+y+z =a (u=a)

Again taking 1/x, 1/y, 1/z as multiplies

Each fraction = 

= 

This gives =0

⇒log x+log y+log z = log b

⇒ xyz =b (v=b)

The general integral is

φ(x+y+z, xyz) =0.

**2). Solve y2zp+x2zq = xy2**

Sol: Auxiliary equations are



From 

⇒ x2dx = y2dy

⇒ x3/3 = y3/3+a

⇒ x3-y3 = a1 where a1= 3a (u=a)

Again ⇒ xdx = zdz

⇒x2/2 = z2/2+b

⇒ x2-z2 = b1 where b1 = 2b (*v*=b1)

The general integral is

φ(x3-y3, x2-z2) =0.

**3). Solve (x2-y2-z2)p+2xyq = 2xz**

Sol: Auxiliary equations are



from 



⇒ log y = log z+log a

⇒ y/z =a (u=a)

Again using x,y,z as the multieplies



= 

Now = dz/2xz

⇒= dz/z

Integrating

Log(x2+y2+z2) = log z+ log b

 (*v* =b)

The general integral is φ(y/z, )=0

**Non- Linear Equations of First order**

A partial differential equation of first order but of degree more than one is called a non-linear partial differential equation.

**Standard Form I:**

Equations involving only p,q and not x,y,z.

i.e f(p,q) = 0 --------(1)

an integral of (1) is given by

z=ax+by+c-----------(2)

where a and b are connected by the relation

f(a,b)=0----(3)

since from (2) p =and 

which when substituted in (3) yields (1)

i.e (2) satisfies the given equation

now solving (3) for b, let b =F(a). putting this value of b in (2), the complete integral is given by

z= ax+y F(a)+c -----------(4)

The singular integral is obtained by eliminating a and c between the complete integral (4) and the equations obtained by differentiating (4) w.r.t ‘a’ and c.

**Standard Form IV:**

Z=px+qy+f(a,b)

Clairaut’s Type:

Equations of this type have form

Z=px+qy+f(p,q)--------(1)

We can easily verify that a solution 1 is

Z=ax+by+f(a,b)-------------(2)

Where a, b are arbitrary constants, therefore it is the complete integral.

Partially differentiating (2) w.r.t a and b in turn and equating to zero the results derived, we have the equations.

0= x+of/oa--------(3)

And 0= y+of/ob----------------(4)

Eliminating a and b from the equations (2), (3) and (4) we get singular solution.

To obtain the general integral, we put b = φ(a) in (2), where φ is an arbitrary function.

Then z= ax+y φ(a)+f[a, φ(a)] --------------(5)

Partially differentiating (5) w.r.t a and equating it to zero we get

0= x+y φ1(a)+f1(a) --------------------(6)

The elimination of a between the equations (5) and (6) is the general integral.

**Standard Form II:**

Equation does not involve x and y

i.e f(z,p,q) = 0 ------------------(1)

we take q= ap -------------(2)

where a is an orbitary constant.

Solve (1) and (2) for p in terms of z say, we obtain

P=φ(z)----------------(3)

dz = pdx+qdy

= pdx+a pdy

=p(ax+ady)

dx+ady = dz/φ(z) ------------------(4)

integrating (4),

x+ay = ----------(5)

which is the complete integral of (1) working rule of solve f(p,q,z)= 0;

1. Let us assume u = x+ay and using p= dz/du and q = adz/du in the given equation

f(z,p,q) = 0 and which transform into f(z,dz/du, adz/du) = 0.

1. Solve the resulting ordinary differential equation

f(z,dz/du, adz/du) =0

1. Substituting x+ay in place of u.

**STANDARD FORM III. VARIABLES SEPARABLE**

Equation of the form f1 (x,p) = f2 (y,q) i.e. equations not involving z and the terms containing x and p can be separated from those containing y and q.

As a trail solution, we assume each side equal to an arbitrary constant a, solve for p and q from the resulting equation.

f1(x,p) = a and f2(x,p)=a

Solving for p and q, we obtain

P = F1(x,a) and q= F2 (y,a)

Since z is a function of x and y, we have





Integrating z = 

Which is the required complete solution containing two arbitrary constants a and b.

**Example : Solve p –q = x2+y2**

**Solution:** Seperating p and x from q and y, the given equation can be written as p-x2=q+y2=a, (say)

gives p = a+x2 and q=y2=a gives q=a-y2

Putting the values of p and q and dz = pdx + qdy, we get

dz= (a=x2) dx+ (a-y2) dy

Integrating z = ax+ 

Which is the desired solution.

**Example : Solve p2+q2 = x2+y2**

**Solution:**Given equation can be written as

p2- x2= y2- q2= a, say



and 

Substituting these values of p and q in dz= pdx + qdy, we get



Integrating,we get







Which is the required solution

**ONE DIMENSIONAL WAVE EQUATION**

Let OA be a stretched string of length l with fixed ends O and A. Let us take x-axis along OA and y-axis along OB perpendicular to OA, with O as origin. Let us assume that the tension T in the string is constant and large when compares with the string so that the effects of gravity are negligeable. Let us pluck the string in the BOA plane and allow it to vibrate. Let p be any point of the string at time t. Let there be no external forces acting on the string. Let each point of the string make small vibrations at right angles to OA in the plane of BOA. Draw  perpendicular to OA. Let and. Then y is a function of x and t. Under the assumptions, using Newton’s Second Law of motion, it can be proved that  is governed by the equation,



With T = tension in the string at any point and m is mass per unit length of the string.

Since the points O and A are not disturbed from their original positions for any time t we get 



These are referred to as the end conditions or boundary conditions. Further it is possible that, we describe the initial position of the string as well as the initial velocity at any point of the string at time  through the conditions



Where  and  are functions such that ;and . Thus to study the subsequent motion of any point of the string we have to solve following :

Determine  such that 

Subject to the condition





The equation (1) is called one dimensional wave equation

Solution of equation (1) to (5)

Consider the equation 

Let us use the method of separation of variables. Here . Let us take 

As solution of (1). Then



Using these in (1) we get



Since the left hand side is function of x and right hand side is a function of t the equality is possible if and only if each side is equal to the same constant (say).

Hence we shall take



Let us take  to be real. Then three cases are possible 

**Case 1:-** let , then 

Then 

Hence 



Hence in this case, a typical solution is like



Where  are arbitary constants

**Case 2:-** let  then







Where  are arbitary constants

**Case 3:-** Let . Then we can write  where  then





Hence a typical solution in this case is



Thus the possible solution forms of equation (1) are



Consider  (I.e.,)



Using conditions (2) 





Using condition (3), 



Solving 

And 

We get 

Thus 

This implies that there is no displacement for any x and for any t. this is impossible. Thus  is not an appropriate solution

Consider :



Using (2), 

Hence 

Using (3), 



 since A = 0

Here  Hence B = 0

Thus here again 

Thus as before, this solution also is not valid

Hence  is also not appropriate for the present problem

Consider 

 ( using condition 2)



Using condition 3





 and this is invalid

Hence 

 where n = 1, 2, 3………….

Thus 

Thus a typical solution of (1) satisfying conditions (2) & (3) is



Since different solutions correspond to different positive integer n.

**An Important observation here :**

If  are functions satisfying (1) as well as conditions (2) and (3). As the equation (1) is linear. The most general solution of (1) here is 

Thus the most general solution of (1) satisfying (2) & (3) is



Where  are constants to be determined using (3) and (4)

Let us use condition 4: 

Thus putting t = 0 in (6)



Hence  n = 1, 2, ……

Thus  are all determined

Let us consider condition (5):



Hence 

Thus  are all determined

Hence the displacement  at any point  and at any subsequent time t is given by



Where 



**TWO DIMENSIONAL WAVE EQUATION:-**

Two dimensional wave equation is given by



Where , for the unknown displacement  of a point  of the vibrating membrane from rest  at time t.s

The boundary conditions (membrane fixed along the boundary in the xy- plane for all times , are  on the boundary ----(2)

And the initial conditions are



Now we have to find a solution of the partial differential equation (1) satisfying the conditions (2) and (3) . we shall do this in 3 steps, as follows:

**Working rule to solve two – dimensional wave equation :-**

**Step1:** By the “method of separating variables” setting  and later  we obtain from (1) an ordinary differential equation for G and one partial differential equation for F, two ordinary differential equations for H & Q.

**Step 2:** We determine solutions of these equations that satisfy the boundary conditions (2). Step(2) to obtain a solution of (1) satisfying both (2) and (3). That is the solution of the regular membrane as follows.

The double Fourier series for  is given by



Hence  and  are called Fourier co-efficients of  and are given by



1. **Find the solution of the wave equation  corresponding to the triangular initial deflection**

 and initial velocity is equal to 0.

Ans. To find  we have to solve



Where





Equation (1) can be in the form



The three solutions of (1) are



The appropriate solution is S.3

Hence 

Using (2) & (3)



 The most general solution of (1) satisfying (2) & (3) is



Using (4)



Now we can expand the given function  in a half range fourier sine series for 



Comparing (7) & (8) we get 



The required solution of (1) is of the form



Using (2) & (3), we have



 General solution of (1) satisfying (2) & (3) is



Now using condition (4)  we get



The most general solution of (1) is





From (5) & above result



If  ( an even number ) 

If 

Thus all  are determined

Using



Hence, 

1. **Solve the boundary value problem**



Ans.  is the solution of the wave equation



Given conditions are



Comparing the coefficients of like terms,



Hence, satisfying the values in (9)



1. **If a string of length l is initially at rest in equillibrium position and each of its points is given the velocity , find the displacement **

Ans. with the explained notation, the displacement  is given by



The most general solution of (1) satisfying (2) & (3) is



Using (4) we get  which implies  for all n

Now, using (5), we get



Hence 

Hence 

**Fill in the blanks:**

1. If the number of arbitary constants to be eliminated is equal to the number of independent variables then we get a partial differential equation of \_\_\_\_\_\_\_\_\_\_\_\_ order
2. If the number of arbitary constants to be eliminated is greater than the number of independent variables then we get a partial differential equation of \_\_\_\_\_\_\_\_\_\_\_\_ order
3. The partial differential equation by eliminating the arbitary constants from  is \_\_\_\_\_\_\_\_\_\_
4. The partial differential equation by eliminating the arbitary constants from  is
5. The partial differential equation by eliminating the arbitary constant from

 is\_\_\_\_\_\_\_\_\_\_\_\_\_

1. The partial differential equation by eliminating the arbitary constants from

 is\_\_\_\_\_\_\_\_\_\_\_

1. The partial differential equation by eliminating the arbitary constants from

 is\_\_\_\_\_\_\_\_\_\_

1. The partial differential equation of all spheres whose centers lie on the z-axis is \_\_\_\_\_\_\_\_
2. The partial differential equation by eliminating arbitary function from  is\_\_\_\_\_\_\_\_\_\_\_
3. The partial differential equation by eliminating arbitary function from  is\_\_\_\_\_\_\_\_\_\_
4. The partial differential equation by eliminating arbitary function from  is\_\_\_\_\_\_\_\_\_\_
5. The partial differential equation by eliminating arbitary function from  is\_\_\_\_\_\_\_\_\_\_
6. The partial differential equation by eliminating the arbitary function from the relation  is \_\_\_\_\_\_\_\_\_\_
7. The partial differential equation by eliminating the arbitary function from the relation  is \_\_\_\_\_\_\_\_\_\_
8. The general solution of  is \_\_\_\_\_\_\_\_\_
9. The general solution of  is \_\_\_\_\_\_\_\_
10. The general solution of  is \_\_\_\_\_\_\_\_
11. The general solution of  is \_\_\_\_\_\_\_\_\_
12. The general solution of  is \_\_\_\_\_\_\_\_\_
13. The general solution of  is \_\_\_\_\_\_\_\_\_\_
14. The general solution of  is \_\_\_\_\_\_\_\_\_\_
15. The general solution of  is \_\_\_\_\_\_\_\_\_\_
16. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_
17. General form of Clairauti equation is\_\_\_\_\_\_\_\_\_\_
18. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_
19. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_\_
20. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_\_\_
21. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_\_\_
22. The general solution of  is \_\_\_\_\_\_\_\_\_\_
23. The general solution of  is \_\_\_\_\_\_\_\_\_\_
24. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_\_
25. The general solution of  is \_\_\_\_\_\_\_\_\_\_
26. The general solution of  is \_\_\_\_\_\_\_\_\_\_
27. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_
28. By eliminating a & b from , the partial differential equation is \_\_\_\_\_\_\_\_\_\_
29. By eliminating a & b from , the partial differential equation formed is \_\_\_\_\_\_\_\_\_
30. By eliminating a & b from , the partial differential equation formed is \_\_\_\_\_\_\_\_\_\_\_
31. By eliminating a & b from , the partial differential equation formed is \_\_\_\_\_\_\_\_\_\_\_
32. By eliminating a & b from , the partial differential equation formed is \_\_\_\_\_\_\_\_\_\_\_\_
33. The general solution of  is \_\_\_\_\_\_\_\_\_\_
34. The general solution of  is \_\_\_\_\_\_\_\_\_\_
35. The general solution of  is \_\_\_\_\_\_\_
36. The general solution of  is \_\_\_\_\_\_\_\_\_\_
37. The general solution of  is \_\_\_\_\_\_\_\_\_
38. The general solution of  is \_\_\_\_\_\_
39. The general solution of  is \_\_\_\_\_\_\_\_\_\_\_\_\_
40. The general solution of  is \_\_\_\_\_\_\_\_\_

48.The general solution of  is \_\_\_\_\_\_\_\_\_