Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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1-5momem.tex,

1.5 Law of Momentum Conservation

Let us apply Newton's second law to a volume containing the same fluid :

$$\frac{D}{Dt}\iiint_V \rho\vec{q}dV = \iint_S \vec{\Sigma}\,dS \,+ \iiint_V \rho\vec{f}dV \tag{1.5.1}$$

where the successive terms represent, from left to right, the rate of change of fluid momentum in the volume, the total surface force acting on the fluid, and the total body force on the fluid.

Using the kinematic transport theorem, the left-hand side becomes

$$\frac{D}{Dt}\iiint_V \rho\vec{q}dV = \iiint_V \rho\frac{D\vec{q}}{Dt}dV,$$

In index notation, (1.5.1) can be written

$$\begin{aligned}\iiint_V \rho\frac{Dq_i}{Dt}dV &= \iint \Sigma_i dS + \iiint \rho f_i dV \\ &= \iint_S \sigma_{ij}n_j dS + \iiint_V \rho f_i dV.\end{aligned} \tag{1.5.2}$$

after using Cauchy's formula.

For a fixed i, σ_{ij} are three components of a vector. By Gauss' theorem of divergence, the surface integral can be turned to a volume integral:

$$\iint_S \sigma_{ij}n_j dS = \iiint_V \frac{\partial\sigma_{ij}}{\partial x_j}dV.$$

Now (1.5.2) can be written

$$\iiint_V dV\left(\rho\frac{Dq_i}{Dt} - \frac{\partial\sigma_{ij}}{\partial x_j} - \rho f_i\right) = 0.$$

Because V is an arbitrary material volume, the integrand must be zero everywhere

$$\rho\frac{Dq_i}{Dt} = \frac{\partial\sigma_{ij}}{\partial x_j} + \rho f_i \qquad i = 1, 2, 3. \tag{1.5.3}$$

We now have 10 unknowns : one of ρ, three of q_i and six of σ_{ij}. Equations (1.2.3) and (1.5.3) constitute only four scalar equations; six more equations are needed.

In the first part of this course we shall restrict to isothermal incompressible fluids only. We then have the *equation of state*[1]

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{q} \cdot \nabla \rho = 0 \tag{1.5.4}$$

It follows from mass conservation that

$$\nabla \cdot \vec{q} = 0. \tag{1.5.5}$$

Four more conditions are still needed.

[1]If thermal effects are important, the equation of state is relation among ρ, the fluid pressure, and the fluid temperature which would be an additional dynamical quantity. This will discussed later.