1.204 Lecture 17

Branch and bound:
Method
Warehouse location problem

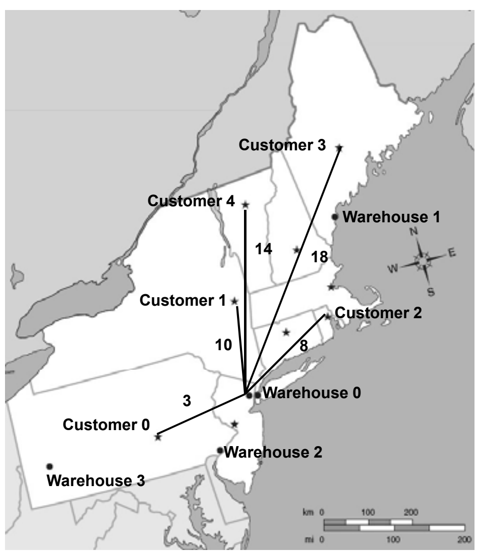
Breadth first search

- Breadth first search manages E-nodes in the branch and bound tree
  - An E node is the node currently being explored
  - In breadth first search, E-node stays live until all its children have been generated
  - The children are placed on a queue, stack or heap
- Typical strategies to select E-nodes
  - Choose node with largest upper bound (in a maximization problem), using a heap
  - Choose node likely to be optimal, even if we can't prove it's optimal immediately
    - Use problem specific, heuristic rule
    - It can be a previous optimal result to similar problem
  - Choose 'quick improvement' node based on a gradient estimate from the upper and lower bounds on a node

Branching on nodes

- Several strategies are used to decide which branch (0 or 1) to take:
  - User specified rules. Again, heuristics are used
  - Set a group of 0-1 variables to given values, not just one
    - This seems to perform better in many problems
    - Our code in this lecture does not do this
- Other strategies to improve performance:
  - If dual can be computed, it provides a lower bound
  - Bound tightening, such as truncation (like we used in the last lecture on the knapsack problem)
  - Adding linear constraints (0<=x<=1) in subproblems in hopes that integer answers are obtained
  - Greedy heuristics, including dual descent and others…

Facility location problem



e.g., Amazon
Intel
Tropicana

Facility location example

Warehouse	Fixed cost	Cost to ship to customer j				
k	f[k]	0	1	2	3	4
0	4	3	10	8	18	14
1	6	9	4	6	5	5
2	6	12	6	10	4	8
3	8	8	6	5	12	9

- **Set of 4 possible warehouses (0-3) to serve 5 possible customers (0-4)**
- **Table gives annual capital (fixed) cost of warehouse if it is built, and the annual cost of shipping to each customer via that warehouse**
- **Decision is whether to build (x_i= 1) or not build (x_i=0) each warehouse**
- **Objective is to minimize fixed plus shipping costs**

Computing bounds

- **Lower (optimistic) bound at each node is sum of:**
  - **Minimum transport cost over all built or unknown warehouses**
  - **Fixed cost of built warehouses**
- **Upper (pessimistic) bound at each node is sum of**
  - **Minimum transport cost over all built warehouses**
  - **Fixed cost of built warehouses**
- **Pruning rules**
  - **If minimum (pessimistic) savings from building a warehouse are greater than its fixed cost, we build it**
  - **If maximum (optimistic) savings from building a warehouse are less than its fixed cost, we don't build it**
- **All combinations are feasible in this problem, so there is no reduction in the size of the tree from feasibility constraints**
  - **We can introduce capital budget constraints in some cases**

Pruning rules from Akinc, Khumwala

Computational strategy

- Start at root node
  - Apply upper and lower bound at root
  - Try to lock in or lock out some warehouses
- Then create tree node with arbitrary warehouse locked in or out
  - Apply upper and lower bound at this node
  - Try to lock in or lock out additional warehouses
  - Generate children if bounds don't prune them
    - Use stack, queue or heap to hold children
- Continue until all E-nodes have been explored
  - Output optimal solution
  - Difference between lower and upper bound decreases as algorithm continues
    - We can stop when the difference is small enough, even without an exact optimal solution

Computational example: root node

- Root node
  - All warehouses x_i are unknown
  - Upper bound at root is "infinity", by convention
  - Lower bound at root is sum of:
    - Cost of built warehouses (none) plus
    - Minimum transport cost over all built or unknown warehouses, which is all of them. Lower bound= 21
  - Use convention:
    - x= 1 is built warehouse
    - x= 0 is unknown warehouse
    - x= -1 is warehouse not built
  - Thus, root node solution is {0, 0, 0, 0}

Pruning rules at root

- **Minimum savings at all warehouses (if warehouse is cheapest, compare it with next cheapest):**

Warehouse k	Fixed cost f[k]	Cost to ship to customer j: 0	1	2	3	4	Minimum savings	Pruning decision
0	4	3	10	8	18	14	5	x0=1
1	6	9	4	6	5	5	5	None
2	6	12	6	10	4	8	1	None
3	8	8	6	5	12	9	1	None

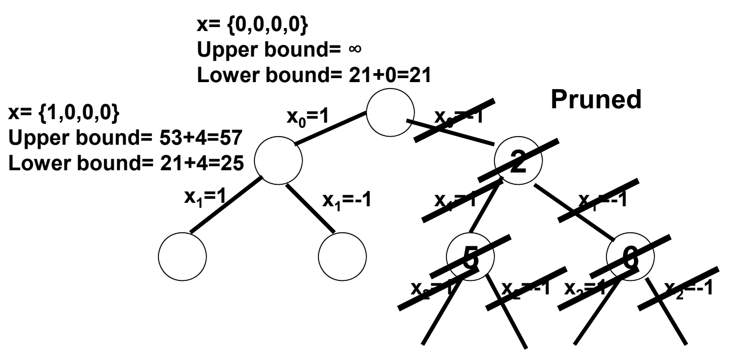
- **Maximum savings at all warehouses (compare it with most expensive):**

Warehouse k	Fixed cost f[k]	Cost to ship to customer j: 0	1	2	3	4	Maximum savings	Pruning decision
0	4	3	10	8	18	14	11	None
1	6	9	4	6	5	5	35	None
2	6	12	6	10	4	8	24	None
3	8	8	6	5	12	9	24	None

- **Thus we are able to prune the x_0 branch of the tree**

Orange is cheapest warehouse to serve customer; gray is next cheapest

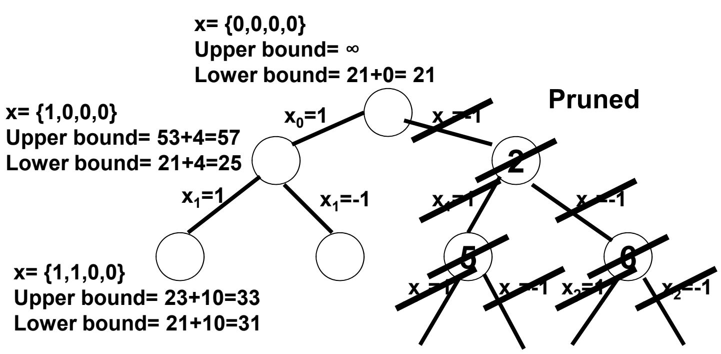
Tree from root



Generate E-node

- **Generate E-node to left of root:**
  - Warehouse 0 is built (x_0= 1 in root solution)
- **Compute upper and lower bounds at E-node**
  - All customers served from warehouse 0
  - Upper bound= 4 (fixed cost) + 53 (transport cost)= 57
    - Assume all customers served from warehouse 0
  - Lower bound= 4 (fixed cost) + 21 (transport cost) = 25
    - Assume customers served from built and unknown warehouses
- **No further pruning is possible at this node**
- **Arbitrarily branch on warehouse 1. Set x_1= 1**

E node bounds



Pruning rules at E-node

- **Minimum savings at all warehouses:**

Warehouse	Fixed cost	Cost to ship to customer j					Minimum	Pruning
k	**f[k]**	**0**	**1**	**2**	**3**	**4**	**savings**	**decision**
0	*4*	*3*	*10*	*8*	*18*	*14*	*NA*	*NA*
1	*6*	*9*	*4*	*6*	*5*	*5*	*NA*	*NA*
2	6	12	6	10	4	8	1	None
3	8	8	6	5	12	9	1	None

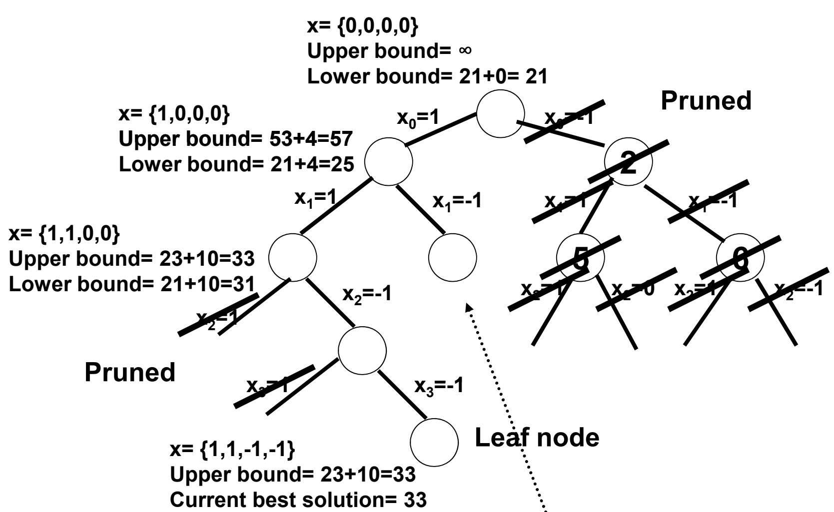
- **Maximum savings at all warehouses:**

Warehouse	Fixed cost	Cost to ship to customer j					Maximum	Pruning
k	**f[k]**	**0**	**1**	**2**	**3**	**4**	**savings**	**decision**
0	*4*	*3*	*10*	*8*	*18*	*14*	*NA*	*NA*
1	*6*	*9*	*4*	*6*	*5*	*5*	*NA*	*NA*
2	6	12	6	10	4	8	1	x2=-1
3	8	8	6	5	12	9	1	x3=-1

- **Thus we are able to prune the x_2 and x_3 branches of the tree**

Yellow is cheapest warehouse to serve customer; gray is next cheapest

E node bounds



We now have just one E-node left to explore

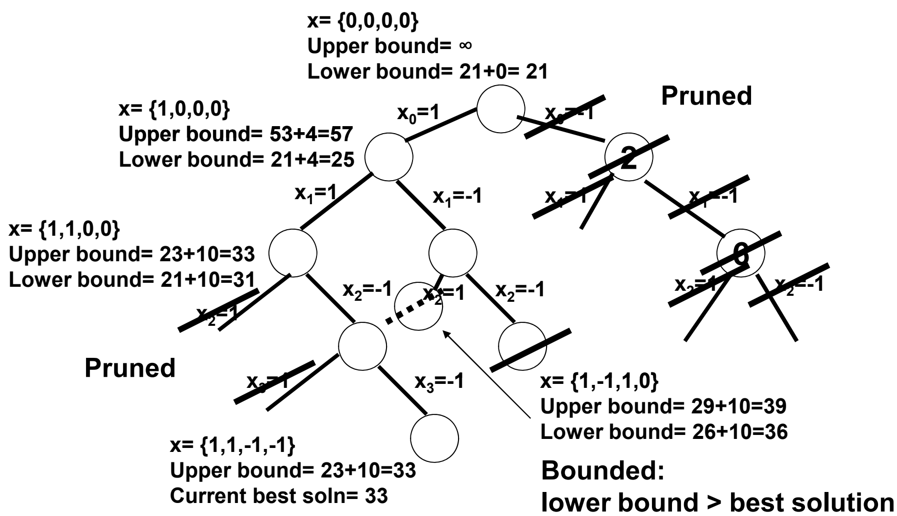
Pruning rules at E-node

- **Minimum savings at all warehouses:**

Warehouse	Fixed cost	Cost to ship to customer j					Minimum	Pruning
k	f[k]	0	1	2	3	4	savings	decision
0	*4*	*3*	*10*	*8*	*18*	*14*	*NA*	*NA*
1	*6*	*9*	*4*	*6*	*5*	*5*	*NA*	*NA*
2	6	12	6	10	4	8	8	x2=1
3	8	8	6	5	12	9	3	None

- **This locks in warehouse 2**
  - **We don't do the maximum savings calculation**
  - **We next compute the bounds at the new node (x_2= 1)**

E node bounds



Termination

- We are done:
  - There are no more live nodes
  - All have either been pruned
    - Maximum savings < fixed cost or
    - Minimum savings > fixed cost
  - Or bounded
    - Lower bound > best solution so far
- Optimal solution is the best solution found:
  - {1, 1, -1, -1}
  - Cost= 33
- We examined 7 nodes in tree out of 31
  - In larger problems, we can only examine a small fraction of total nodes, since there are 2^n nodes for n 0-1 variables

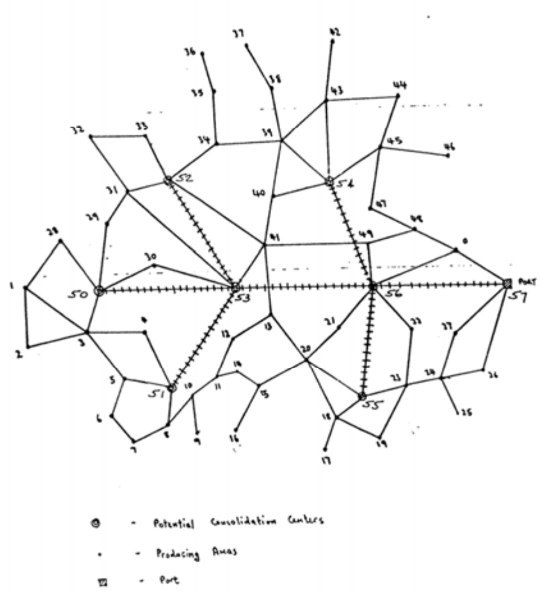
Algorithm pseudocode

```
public boolean branchAndBound() {
  upperBound= infinity;
  eNode= root;
  initialize queue empty; // Holds children of eNode
  bound(root);
  if (root is leaf) { upperBound= cost(root); answer= {root};};
  do {
    generate left and right child at first xi=0 of eNode;
    bound(left child);        // May also generate/prune nodes
    bound(right child);       // May also generate/prune nodes
    for each child w of eNode {
      if (lowerBound(w) < upperBound) {
        if (w is a leaf) { upperBound= lowerBound(w); answer= {w} };
        else {
          add w to queue;
          if (upperBound(w) + TOLERANCE < upperBound)
            upperBound= upperBound(w) + TOLERANCE;
      }
    }
    do {
      if (queue empty) return;
      delete eNode from front of queue;
    } while (lowerBound(eNode) >= upperBound);
  } while (number of children < maximum number of children);
```

Algorithm operation

- We can change algorithm from classical breadth first search (BFS) to D-search or LC-search by substituting a stack or heap for the queue
  - LC search can use lower bound, upper bound or other criteria as priority to explore child
- We don’t store parent of node
  - We store answer array, from which we generate children
  - BFS, D-search and LC-search never backtrack, so parent is not needed
  - If you want to backtrack, then store parent
  - Our bound() methods can generate several children during their calculations
    - This makes it more convenient for us to store the answer array
- One special case is not handled in our code:
  - If all costs from warehouses to customers are equal, maximum savings from any warehouse will be zero and all warehouses will be closed at root node
    - If this occurs, we know only one warehouse needs to be open
    - Pick the cheapest.

Branch and bound example



Branch and bound example

- Shipping sugar harvest from Brazil for export
- Warehouse customers: nodes 0 through 49
  - Ship product to warehouse
  - Each customer has a quantity produced and shipped
  - Each arc in highway network has a cost
- Warehouses: nodes 50 through 57
  - Warehouses are on rail lines, ship to port by rail
  - Each warehouse has fixed cost, if built
  - No capacity constraint
- Which warehouses do we build to minimize cost?
  - What customers ship to each warehouse?
  - What are flows, costs for each customer and warehouse?

LC-search branch and bound

- LCBB.java (least-cost search) imports

```
import src.dataStructures.Heap;  // LC-search
import src.greedy.Graph;         // Used by all BB codes
```

  - Use nested class BBNode to allow LC-search to use lower bound, upper bound or answer array to select next node to search
- DBB.java (D-search) and BFSBB (breadth-first search) import

```
import src.dataStructures.Stack; // For D-search only
import src.dataStructures.Queue; // For BFS search only
import src.greedy.Graph;         // Used by all BB codes
```

  - Stack and queue implementations use BBNode as well, to demonstrate interchangeability of code
- BBArray.java uses BFS and ‘raw’ arrays rather than a BBNode (branch and bound) nested class like the first three
  - There are extensive comments in the BBArray.java file
  - All classes use java.io.* and java.util.*

Code outline

```
Graph class: constructor, shortHK()
LCBB class:
  LCBB data members: input, calculation, output
  BBNode inner class: data members, constructor, compareTo()
  LCBB() constructor
    bbNetwork(): read warehouse.txt input data
  branchAndBound():
    setC(): call g.shortHK() on all warehouses, create costs
    initializeBB(): cost initialization, create root of BB tree
    bb(): branch-and-bound algorithm
      bound(): compute min, max savings, lower/upper bounds
        warehouseBound(): compute min, max savings at 1 warehouse
    bbAssign(): postprocess output, assign customer to warehouse
    bbOutput(): prints out solution, costs, flows
  main():
    create Graph object g
    create LCBB object w
      call w.branchAndBound()
```

LCBB data members

```
public class LCBB {
   // Input data
   private int nw;     // Number of potential warehouses
   private int nc;     // Number of customers
   private int[] f;    // Fixed cost of each potential warehouse
   private int[][] c;  // Cost from customer to warehouse
   private int[] railcost;    // Cost by rail, warehouse to port
   private int[] prod; // Production volume from each customer
   private final static int EPS= 1;   // Epsilon, tolerance
   private final static int MAXBBNODES= 10000;

   // Data used by branch and bound calculations
   private int[] savMax;      // Calculated by warehouseBound()
   private int[] savMin;      // Calculated by warehouseBound()
   private BBNode[] nodes;    // Branch and bound nodes
   private Heap h;            // Keeps nodes to be visited still
                              // Stack or Queue in other versions
   // Solution
   private int[] ans;       // Solution: 1 if in,-1 if not,0 unknown
   private int upperBound;    // Global upper bound
   private boolean optimumFound;
   int[] whAssign;            // Warehouse assgd to customer
   int[] flow;                // Flow through each warehouse
```

BBNode nested class

```
private class BBNode implements Comparable {
     private int[] x;       // Solution
     private int upBound;   // Upper bound (cost) estimate
     private int lowBound;  // Lower bound (cost) estimate

     public BBNode() {
            x= new int[nw];
     }
     // Place node with lowest lower bound at top of heap
     public int compareTo(Object other) {
            BBNode o= (BBNode) other;
            if (lowBound < o.lowBound)
                   return 1;
            else if (lowBound > o.lowBound)
                   return -1;
            else
                   return 0;
     }
}    // Can create general rule for which node is at top
```

LCBB constructor

```
public LCBB(String filename) {
     // Input data
     bbNetwork(filename);
     c= new int[nw+1][nc];  // Cost matrix, cust-whse
                            // Last row holds max cost
     // Data used by branch and bound calculations
     savMax= new int[nw];
     savMin= new int[nw];
     nodes= new BBNode[MAXBBNODES];
     // Allocate all BBNode memory first
     for (int i= 0; i < MAXBBNODES; i++)
            nodes[i]= new BBNode();
     h= new Heap();         // Or Stack or Queue

     // Solution
     ans= new int[nw];
     upperBound= Integer.MAX_VALUE;
     whAssign= new int[nc];
     flow= new int[nw];
}
```

bbNetwork: read warehouse input file

```
public void bbNetwork(String filename) {
  try {
    FileReader fin= new FileReader(filename);
    BufferedReader in= new BufferedReader(fin);
    nc= Integer.parseInt(in.readLine());
    nw= Integer.parseInt(in.readLine());
    f= new int[nw];
    railcost= new int[nw];
    prod= new int[nc+nw];
    for (int i=0; i < nw; i++) {
      String str = in.readLine();
      StringTokenizer t = new StringTokenizer(str, ",");
      int wNumber= (Integer.parseInt(t.nextToken()));
      railcost[i]= (Integer.parseInt(t.nextToken()));
      f[i]= (Integer.parseInt(t.nextToken()));
    }
    for (int i= 0; i < nc; i++) {
      String str = in.readLine();
      StringTokenizer t = new StringTokenizer(str, ",");
      int cNumber= (Integer.parseInt(t.nextToken()));
      prod[i]= (Integer.parseInt(t.nextToken()));
    }
    in.close(); …     // Catch exception, and end method
```

setC()

```
public void setC(Graph g) {
     int[][] DW= new int[nw+1][nc+nw];
     int[][] PW= new int[nw+1][nc+nw];
     int nodes= g.getNodes();

     for (int root= nc; root < (nc + nw); root++) {
       g.shortHK(root);
       int[] D= g.getD();
       int[] P= g.getP();
       for (int i= 0; i < nodes; i++) {
         DW[root-nc][i]= D[i];
         PW[root-nc][i]= P[i];
       }
     }
     for (int k= 0; k < nw; k++)
       for (int j= 0; j < nc; j++)
         c[k][j]= (DW[k][j] + railcost[k])* prod[j];
}
```

initializeBB()

```
public void initializeBB() {
     for (int m= 0; m < nc; m++) { // Write highest cost
       int temp= 0;
       for (int j= 0; j < nw; j++)
         if (c[j][m] > temp)
             temp= c[j][m];
         c[nw][m]= temp;
     }
                           // bound returns true if leaf
     if (bound(0)) {       // Find upper, lower bounds
       upperBound= nodes[0].lowBound;
       for (int k= 0; k < nw ; k++)
         ans[k]= nodes[0].x[k];
     }
     // If all warehouses closed at root, select cheapest
     // one. This special case not handled.
}
```

bb()

```
public boolean bb() {
  BBNode eNode= nodes[0];      // Root, node 0, is the first e-node
  int i= 0;                    // Root is 0th node in tree
  int inOut= 1;                // Toggles between -1 and +1
  do {                         // Infinite loop until queue empty
    int w= -1;
    do { w++;         // Find first warehouse with unknown status
    } while (!(eNode.x[w] == 0 || w >= nw));
    if (w < nw) {     // If unknown warehouse found, gen children
      for (int z=0; z <=1; z++) {
        i++;          // Generate child
        for (int j= 0; j < nw; j++)
          nodes[i].x[j]= eNode.x[j]; // Copy parent's solution
        nodes[i].x[w]= -inOut;       // Set unknown whse state
        boolean leaf= bound(i);      // Bound this child (t if leaf)
        if (nodes[i].lowBound < upperBound) { // If worth going
          if (leaf) {     // If child is leaf, we have new optimum
            upperBound= nodes[i].lowBound;   // Update upper bound
            for (int k= 0; k < nw; k++)
              ans[k]= nodes[i].x[k];         // Update solution
          } else {                           // Child is not leaf
            h.insert(nodes[i]);              // Add to heap
            if (nodes[i].upBound + EPS < upperBound)
              upperBound= nodes[i].upBound + EPS;   // Update upper
    } } } }    // Continues on next slide
```

bb(), p.2

```
    do {                      // Find new e-node
      if (h.isEmpty())        // If heap empty, we're done
        return true;          // Found optimum
      eNode= (BBNode) h.delete(); // Get e-node from heap
    } while (eNode.lowBound >= upperBound);
  } while (i < MAXBBNODES-2);
  return false;  // Generated maximum nodes w/o finding optimum
}
```

bound()

```
private boolean bound(int i) {     // Returns true if leaf node
  boolean change;
  do {      // Lock in/out warehouses based on max/min savings
    change= false;
    for (int k= 0; k < nw; k++)
      if (nodes[i].x[k] == 0)
        warehouseBound(i, k); // Find min, max savings for k
    for (int k= 0; k < nw; k++) {
      if (nodes[i].x[k] == 0) {
        if (savMin[k] - f[k] >= 0) {
          change= true;
          nodes[i].x[k]= 1;   // Lock in warehouse
          for (int j= 0; j < nc; j++)
            if (c[k][j] < c[nw][j])
              c[nw][j] = c[k][j];
        }
        if (savMax[k] - f[k] <= 0) {
          change= true;
          nodes[i].x[k]= -1;     // Lock out warehouse
  } } } } while (change);
```

bound(), p.2

```
// Compute lower and upper bound. Start by adding up
// transportation costs over all customers to non-closed
// warehouses (lower bound) and to open warehouses (upper)
int lowc= 0, minc= 0, uppc= 0, maxc= 0;
  for (int j= 0; j < nc; j++) {
    minc= Integer.MAX_VALUE;
    maxc= c[nw][j];
    for (int k= 0; k < nw; k++)
      if ((nodes[i].x[k] != -1) && (c[k][j] < minc))
        minc= c[k][j];  // Find min transportation cost
    if (minc == Integer.MAX_VALUE)
      minc= 0;
    lowc += minc;
    uppc += maxc;
  }
```

bound(), p.3

```
  // Add fixed costs of open warehouses to lower and upper
  // bounds, fixed cost of unknown warehouses to upper bound
  boolean leaf= true;
  for (int k= 0; k < nw; k++) {
    if (nodes[i].x[k] == 1) {
      lowc += f[k];
      uppc += f[k];
    }
    if (nodes[i].x[k] == 0) {
      leaf= false;
      uppc += f[k];
    }
  }
  nodes[i].lowBound = lowc;
  nodes[i].upBound = uppc;
  return leaf;
}
```

warehouseBound()

```
private void warehouseBound(int i, int wh) {
  // Find minimum and maximum savings for a warehouse
  // i= current node, wh= warehouse being examined
  int minSav= 0;
  int maxSav= 0;
  for (int h= 0; h < nc; h++) {     // Loop thru each customer
    minSav= Integer.MAX_VALUE;
    if (c[wh][h] < c[nw][h])
      maxSav= c[nw][h] - c[wh][h];
    else
      maxSav= 0;
    for (int g= 0; g < nw ; g++)   // Loop thru each warehouse
      if ((g != wh) && (nodes[i].x[g] != -1) &&
         ((c[g][h] - c[wh][h]) < minSav))
        minSav= c[g][h] - c[wh][h];
    if (minSav == Integer.MAX_VALUE || minSav < 0 )
      minSav= 0;
    savMin[wh] += minSav;
    savMax[wh] += maxSav;
  }
}
```

bbAssign()

```
// Output method, after solution is computed
public void bbAssign() {
     for (int k= 0; k < nc; k++) {
             int temp= Integer.MAX_VALUE;
             for (int j= 0; j < nw; j++) {
                     if (c[j][k] < temp  && ans[j]== 1) {
                             temp= c[j][k];
                             whAssign[k]= j;
                     }
             }
     }
     for (int k=0; k < nc; k++)
             flow[whAssign[k]] += prod[k];
}
```

bbOutput()

```
public void bbOutput() {      // More output
  System.out.println("Optimum found? "+ optimumFound);
  if (!optimumFound) {        // This code only lightly tested
    System.out.println("Upper bound: " + upperBound);
    // Go through all E nodes in heap to find lowest lower bound
    int lowerBound= Integer.MAX_VALUE;
    while (!h.isEmpty()) {
      BBNode n= (BBNode) h.delete();
      if (n.lowBound < lowerBound)
        lowerBound= n.lowBound;
    }
    System.out.println("Lower bound: " + lowerBound);
  }

  // If no leaf node visited yet, answer array will be all zeros.
  // Can insert code here to set ans array= x array of node with
  // best lower bound. Not done.

  // Continues on next slide
```

bbOutput(), p.2

```
  int constr= 0;
  System.out.println("\nCenter \tConstruct? \tFixed Cost");
  for (int j= 0; j < nw; j++) {
    System.out.println(j+"\t\t"+ ans[j]+ "\t\t"+ f[j]);
    if (ans[j] == 1)
      constr += f[j];
  }
  int trans= upperBound - constr;
  System.out.println("\nTransport cost: "+ trans +
       " fixed cost: "+ constr);
  System.out.println("\nFlow through consolidation centers");
  System.out.println("Center\tTons");
  for (int j= 0; j < nw ; j++)
    System.out.println(j + "\t"+ flow[j]);
  for (int j= 0; j < nw; j++)
    if (ans[j] == 1) {
      System.out.println("\nAreas that ship to center "+ j);
      for (int k= 0; k < nc; k++)
        if (whAssign[k] == j)
          System.out.print(" " + k);
    }
}
```

branchAndBound(), main()

```
    public void branchAndBound(Graph g) {
         setC(g);
         initializeBB();
         optimumFound= bb();
         bbAssign();
         bbOutput();
    }

    public static void main(String[] args) {
         Graph g= new Graph("src/bb/graph.txt");
         LCBB w= new LCBB("src/bb/warehouse.txt");
         w.branchAndBound(g);
    }
}
```

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