Engineering Risk Benefit Analysis

1.155, 2.943, 3.577, 6.938, 10.816, 13.621, 16.862, 22.82, ESD.72, ESD.721

RPRA 3. Probability Distributions in RPRA

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Overview

We need models for:

- **The probability that a component will start (fail) on demand.**
- **The probability that a component will run for a period of time given a successful start.**
- **The impact of repair on these probabilities.**
- **The frequency of initiating events.**

Failure to start

P[failure to start on demand] ≡ q ≡ unavailability

P[successful start on demand] ≡ p ≡ availability

Requirement: $p + q = 1$

The Binomial Distribution (1)

- Start with an "experiment" that can have only two outcomes: "success" and "failure" or {0, 1} with probabilities p and q, respectively.
- Consider N "trials," i.e., repetitions of this experiment with constant q. These are called *Bernoulli trials*.
- Define a new DRV: X = number of 1's in N trials
- Sample space of X: {0,1,2,...,N}
- What is the probability that there will be k 1's (failures) in N trials?

The Binomial Distribution (2)

<u>For the coin</u>: Assume 3 trials. We are interested in 1 failure.

•There are 3 such sequences: fss, sfs, ssf (mutually exclusive).

•The probability of each is qp^2.

•If order is unimportant, the probability of 1 failure in 3 trials is $3qp^2$.

$$\Pr[X = k] = \binom{N}{k} q^k (1 - q)^{N-k}$$

- This is the <u>probability mass function</u> of the Binomial Distribution.
- It is the probability of *exactly* k failures in N demands.
- The *binomial coefficient* is:

$$\binom{N}{k} \equiv \frac{N!}{k!(N-k)!}$$

The Binomial Distribution (3)

- **Mean number of failures: qN**
- **Variance: $q(1-q)N$**

Normalization:

$$\sum_{k=0}^{N} \binom{N}{k} q^{k} (1-q)^{N-k} = 1$$

$$\text{P[at most m failures]} = \sum_{k=0}^{m} \binom{N}{k} q^{k} (1-q)^{N-k} \equiv F(m) \qquad \text{CDF}$$

Example: 2-out-of-3 system

- **We found in slide 16 of RPRA 1 that the structure function is (using min cut sets):**
  $X_T = (X_A X_B + X_B X_C + X_C X_A) - 2 X_A X_B X_C$

- **The failure probability is P(failure) = $P(X_T = 1) = 3q^2 - 2q^3$**

- **Using the binomial distribution:**

- **Pr(system failure) = P[2 fail] + P[3 fail] = $3q^2(1-q) + q^3$**
  $= 3q^2 - 2q^3$

Notes: **1. We have assumed nominally identical and independent components in both cases.**

2. If the components are not nominally identical or independent, the structure function approach still works, but the binomial distribution is not applicable. Why?

The Poisson Distribution

- Used typically to model the occurrence of initiating events.
- DRV: number of events in (0, t)
- Rate λ is constant; the events are independent.
- The probability of exactly k events in (0, t) is (pmf):

$$\Pr[k] = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$k! \equiv 1*2*\ldots*(k-1)*k \qquad 0! = 1 \qquad m = \lambda t \qquad \sigma^2 = \lambda t$

Example of the Poisson Distribution

- *A component fails due to "shocks" that occur, on the average, once every 100 hours. What is the probability of exactly one replacement in 100 hours? Of no replacement?*

- **$\lambda t = 10^{-2} * 100 = 1$**
- **Pr[1 repl.] = $e^{-\lambda t} = e^{-1} = 0.37$ = Pr[no replacement]**
- **Expected number of replacements: 1**

$$\mathbf{Pr[2repl] = e^{-1}\frac{1^2}{2!} = \frac{e^{-1}}{2} = 0.185}$$

$$\mathbf{Pr[k \leq 2] = 0.37 + 0.37 + 0.185 = 0.925}$$

Failure while running

- T: the time to failure of a component.
- $F(t) = P[T < t]$: failure distribution (unreliability)
- $R(t) \equiv 1-F(t) = P[t < T]$: reliability
- m: mean time to failure (MTTF)
- f(t): failure density, $f(t)dt = P\{\text{failure occurs between } t \text{ and } t+dt\} = P\ [t < T < t+dt]$

The Hazard Function or Failure Rate

$$h(t) \equiv \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)} \qquad F(t) = 1 - \exp\left(-\int_0^t h(s)ds \right).$$

The distinction between h(t) and f(t) :

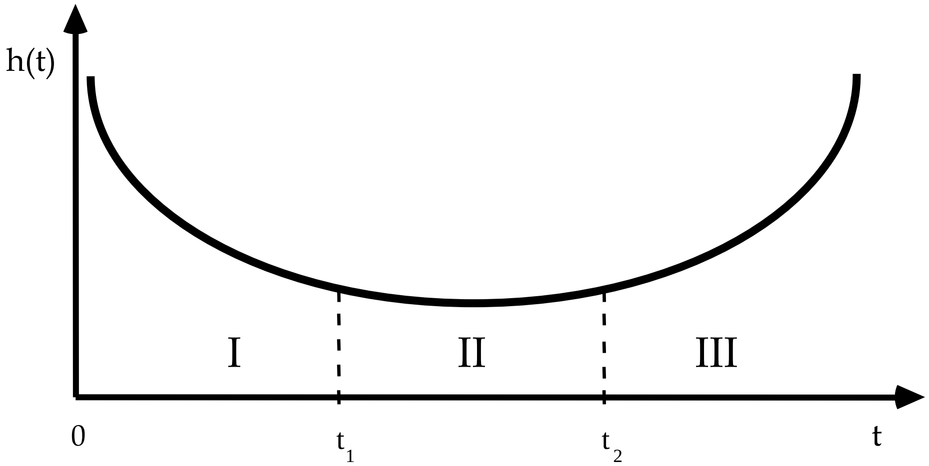
$f(t)dt$: <u>unconditional</u> probability of failure in $(t, t + dt)$,

$$f(t)dt = P\,[t < T < t + dt]$$

$h(t)dt$: <u>conditional</u> probability of failure in $(t, t + dt)$ <u>given</u> that the component has survived up to t.

$$h(t)dt = P\,[t < T < t + dt / \{ t < T \}]$$

The "Bathtub" Curve



I	**Infant Mortality**
II	**Useful Life**
III	**Aging (Wear-out)**

The Exponential Distribution

$$f(t) = \lambda e^{-\lambda t} \qquad \lambda > 0 \quad t > 0$$ (failure density)

- $F(t) = 1 - e^{-\lambda t}$ $\qquad$ $R(t) = e^{-\lambda t}$

- $h(t) = \lambda$ constant (no memory; the *only* pdf with this property) $\Rightarrow$ useful life on bathtub curve

$F(t) \cong \lambda t \quad$ when $\quad \lambda t < 0.1$ $\quad$ (*another* rare-event approximation)

Mean Time Between Failures: $\quad m = \dfrac{1}{\lambda} = \sigma$

Example: 2-out-of-3 system

Each sensor has a MTTF equal to 2,000 hours. What is the unreliability of the system for a period of 720 hours?

- ***Step 1: System Logic.***

$$X_T = (X_A X_B + X_B X_C + X_C X_A) - 2X_A X_B X_C$$

Example: 2-out-of-3 system (2)

Step 2: Probabilistic Analysis.

For nominally identical components:

$P(X_T) = 3q^2 - 2q^3$ **(slide 7 of this lecture)**

But $q(t) = 1 - e^{-\lambda t} \equiv F(t)$ **with** $\lambda = 5x10^{-4}$ hr^{-1}

System Unreliability: $F_T(t) = 3(1 - e^{-\lambda t})^2 - 2(1 - e^{-\lambda t})^3$

Example: 2-out-of-3 system (3)

For a failure rate of $5\text{x}10^{-4}$ hr^{-1} and for t = 720 hrs $\Rightarrow$ $\lambda t = 0.36$

$$\Rightarrow \quad q(t) = 1 - e^{-0.36} = 0.30$$

$F_T(720) = 3 \text{ x } 0.30^2 - 2 \text{ x } 0.30^3 = 0.216$

Since = 0.36 > 0.1 $\Rightarrow$ the rare-event approximation does not apply.

Indeed,

$F_T(720) \cong 3\text{x}0.36^2 - 2\text{x}0.36^3 = 0.295 > 0.216$

A note on the calculation of the MTTF

$$\mathrm{MTTF} = \int_0^\infty R(t)dt$$

Proof

$$\mathrm{MTTF} = \int_0^\infty t f(t)dt = \int_0^\infty t\left(-\frac{dR}{dt}\right)dt = -\int_0^\infty t\,dR =$$

$$= -tR\Big|_0^\infty + \int_0^\infty R(t)dt = \int_0^\infty R(t)dt$$

A note on the calculation of the MTTF (cont.)

since

$$f(t) = \frac{dF}{dt} = \frac{d(1-R)}{dt} = -\frac{dR}{dt}$$

and

$$R(t \to \infty) \to 0 \quad \text{faster} \quad \text{than} \quad t \to \infty$$

MTTF Examples: Single Exponential Component

$$R(t) = \exp(-\lambda t)$$

$$MTTF = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}$$

MTTF Examples: The Series System

Step 1: System Logic

(minimal path sets)

$$Y_T = \prod_{1}^{M} Y_j$$

Step 2: Probabilistic Analysis

$P(Y_T = 1) = p^M$ **but** $p(t) = e^{-\lambda t} \equiv P(T > t)$

$$P(Y_T = 1) = R_S(t) = e^{-(M\lambda)t}$$

$\Rightarrow$ **The system is exponential**

$$MTBF = \frac{1}{M\lambda} \equiv \frac{1}{\lambda_{system}}$$

MTTF Examples: 1-out-of-2 System

Step 1: System Logic $X_T = X_1 X_2$ (slide 9 of RPRA 1)

Step 2: Probabilistic Analysis

$$F_S(t) = (1 - e^{-\lambda t})^2$$

$$R_S(t) = 1 - F_S(t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

$$\text{MTTF} = \frac{2}{\lambda} - \frac{1}{2\lambda} = \frac{3}{2\lambda}$$

(compare with the MTTF for a single component, $\frac{1}{\lambda}$)

MTTF Examples: 2-out-of-3 System

Using the result for $F_T(t)$ on slide 15, we get

$$\mathrm{MTTF} = \int_0^\infty R_T(t)dt = \int_0^\infty [1 - 3(1 - e^{-\lambda t})^2 + 2(1 - e^{-\lambda t})^3]dt$$

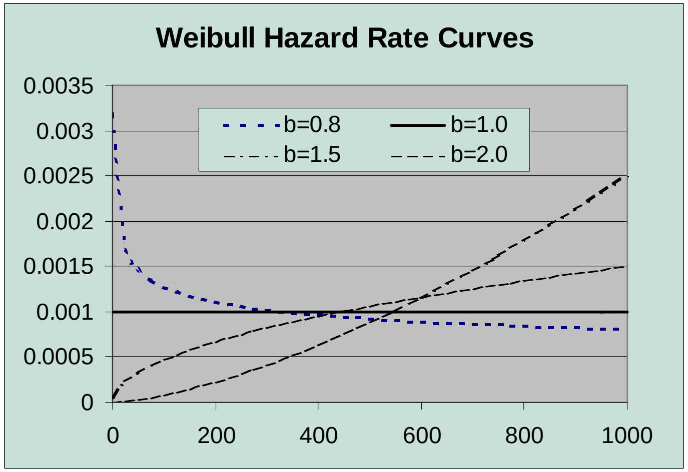
$$\mathrm{MTTF} = \frac{1}{2\lambda} + \frac{1}{3\lambda} = \frac{5}{6\lambda}$$

The MTTF for a single exponential component is: $\dfrac{1}{\lambda}$

$\Rightarrow$ The 2-out-of-3 system is slightly worse.

The Weibull failure model

Adjusting the value of b, we can model any part of the bathtub curve.



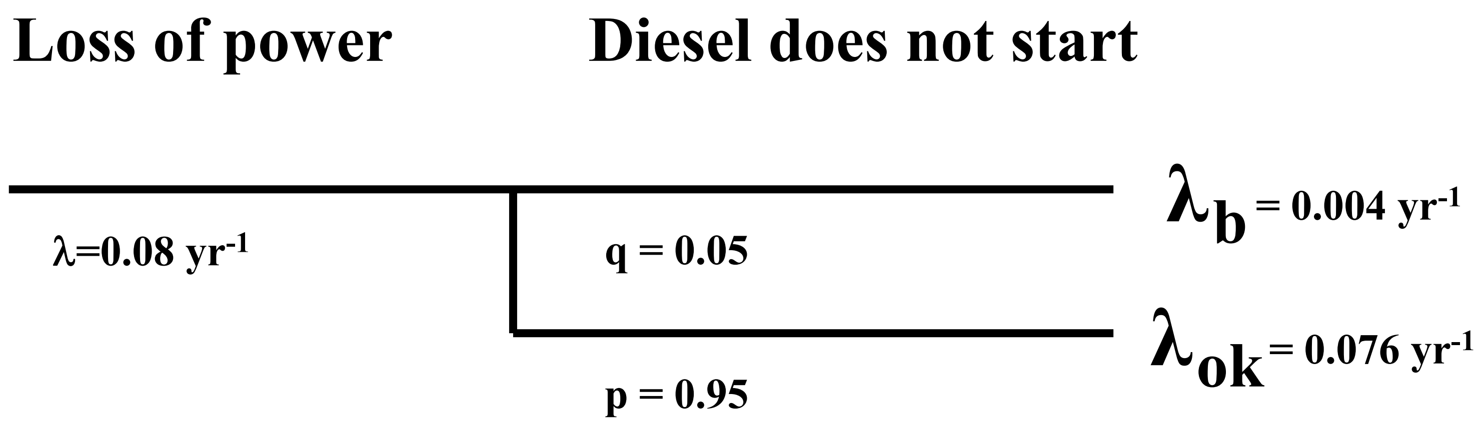
$$h(t) = b\lambda^b t^{b-1}$$

$$R(t) = e^{-(\lambda t)^b}$$

For b = 1 ⇒ the exponential distribution.

A Simple Calculation

- *The average rate of loss of electric power in a city is 0.08 per year. A hospital has an emergency diesel generator whose probability of starting successfully given loss of power is 0.95.*
- *i. What is the rate of occurrence of blackouts at this hospital?*



A Simple Calculation (cont.)

λ_b = 0.08 (power losses per year)x 0.05 (Diesel fails given a power loss) = 4×10^{-3} per year.

ii. ***What is the probability that the hospital will have no blackouts in a period of five years? Exactly one blackout? At least one blackout?***

$$\lambda_b t = 0.004 \times 5 = 0.02$$

Use the Poisson distribution:

P(no blackouts in 5 yrs) = exp(-0.02) = 0.9802

P(exactly one blackout in 5 yrs) = (0.02)x0.9802 = 0.0196

P(at least one blackout in 5 yrs) = P(1 or 2 or 3…) =

= 1 - P(no blackouts in 5 yrs) = 1 – 0.9802 = 0.0198

The Normal (Gaussian) distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right]$$

$$-\infty < x < \infty \qquad -\infty < \mu < \infty$$

$$0 < \sigma < \infty$$

Mean: μ **Standard Deviation: σ**

The Normal (Gaussian) distribution (2)

Standard Normal Variable:

$$Z = \frac{X - \mu}{\sigma}$$

Standard Normal Distribution:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right]$$

Area under the Standard Normal Curve (from 0 to a)

a	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0	0.004	0.008	0.012	0.016	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.091	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.148	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.17	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.195	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.219	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.258	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.291	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.334	0.3365	0.3389
1	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.377	0.379	0.381	0.383
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.398	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.437	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.475	0.4756	0.4761	0.4767
2	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.483	0.4834	0.4838	0.4842	0.4846	0.485	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.489
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.492	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.494	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.496	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.497	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.498	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.499	0.499

Example of the normal distribution

- μ = 10,000 hr (MTTF) σ = 1,000 hr
- Pr [X > 11,000 hr] = Pr [Z > 1] = 0.50 – 0.34= 0.16

$$Z = \frac{11{,}000 - 10{,}000}{1{,}000} = 1$$

An Example

A capacitor is placed across a power source. Assume that surge voltages occur on the line at a rate of one per month and they are normally distributed with a mean value of 100 volts and a standard deviation of 15 volts. The breakdown voltage of the capacitor is 130 volts.

An Example (2)

i. Find the mean time to failure (MTTF) for this capacitor.

λ_{sv} = 1 per month

$P_{d/sv}$ = conditional probability of damage given a surge voltage

= P (surge voltage>130 volts/surge voltage) =

$$= P\left(Z > \frac{130 - 100}{15}\right) = P(Z > 2) =$$

$$= 1 - P(Z < 2) = 1 - 0.9772 = 0.0228$$

An Example (3)

Therefore, the rate of damaging surge voltages is

$$\lambda_d = \lambda_{sv} x P_{d/sv} = 1x0.0228 = 2.28x10^{-2} \quad (\text{month})^{-1}$$

Equivalently, the capacitor's failure time follows an exponential distribution with the above rate.

The mean time between failures of the capacitor is

$$MTBF = \frac{1}{2.28x10^{-2}} = 43.86 \quad \text{months}$$

An Example (4)

ii. Find the capacitor's reliability for a time period of three months.

$$R(3\ \text{mos}) = \exp(-\lambda_d \times 3) = \exp(-2.28 \times 10^{-2} x3) = 0.934$$

Observations

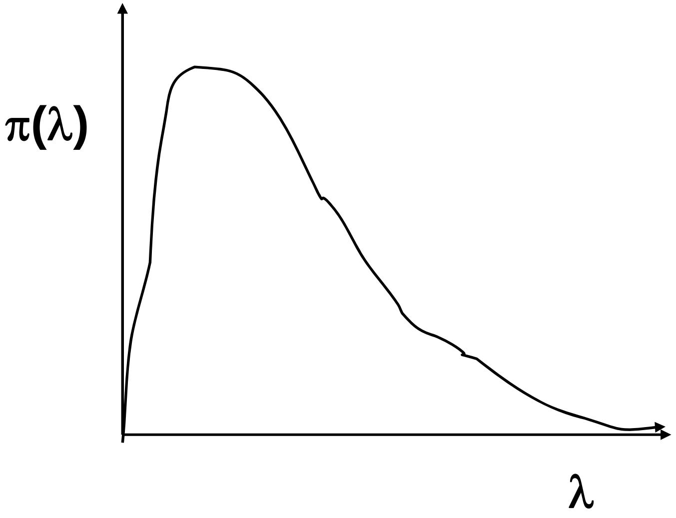
- Events ("shocks") occur in time according to the Poisson distribution [the losses of electric power on slide 24, the surge voltages on slide 30].
- There is a conditional probability that a given shock will be "lethal", i.e., will fail the component. This conditional probability was given on slide 24 as 0.05, while on slide 31 it was calculated from "reliability physics," i.e., from the normal distribution of the voltage (2.28×10^{-2}).
- We calculated the rate of lethal shocks as the product of the rate of shocks times the conditional probability of failure. The occurrence of lethal shocks is, then, modeled as a Poisson process with this rate.

The Poisson and Exponential Distributions

- Let the rate of lethal shocks be λ^*.
- P[no lethal shocks in (0, t)] = $e^{-\lambda^* t}$ (slide 8, k=0)
  The Poisson DRV is the number of lethal shocks.
- The component will not fail as long as no lethal shocks occur. So, we can also write
- P(failure occurs after t) = P[T > t] = $e^{-\lambda^* t}$ (slide 13)
  The exponential CRV is T, the failure time.

The Lognormal Distribution

$$\pi(\lambda) = \frac{1}{\sqrt{2\pi}\sigma\lambda} \exp\left[-\frac{(\ln\lambda - \mu)^2}{2\sigma^2} \right]$$



The Lognormal Distribution (2)

$$\text{mean}: m = \exp\left[\mu + \frac{\sigma^2}{2}\right]$$

$$\text{median}: \lambda_{50} = e^{\mu} \qquad \lambda_{95} = e^{\mu+1.645\sigma}$$

$$\lambda_{05} = e^{\mu-1.645\sigma}$$

Error Factor: $$EF = \frac{\lambda_{95}}{\lambda_{50}} = \frac{\lambda_{50}}{\lambda_{05}} = \sqrt{\frac{\lambda_{95}}{\lambda_{05}}}$$

Relationship with the Normal Distribution

If λ is a lognormal variable with parameters μ and σ, then:

$$Y \equiv \ln\lambda$$

is a normal variable with parameters

μ (mean) and

σ (standard deviation).

The 95th percentile

Since Y is a normal variable, its 95th percentile is

$$Y_{95} = \mu + 1.645\ \sigma$$

But, $Y \equiv \ln\lambda \quad \Rightarrow \ln\lambda_{95} = \mu + 1.645\,\sigma \quad \Rightarrow$

$$\lambda_{95} = e^{\mu+1.645\sigma}$$ **as in slide 37**

The Lognormal Distribution: An example

Suppose that μ = -6.91 and σ = 1.40

- **Median:** $\lambda_{50} = \exp(-6.91) \cong 10^{-3}$
- **Mean:** $m = \exp(\mu + \sigma^2/2) = 2.65 \times 10^{-3}$
- **95th percentile:** $\lambda_{95} = \exp(-6.91 + 1.645 \times 1.40) \cong 10^{-2}$
- **5th percentile:** $\lambda_{05} = \exp(-6.91 - 1.645 \times 1.40) \cong 10^{-4}$
- **Error Factor:** $EF = 10$