1.00 Lecture 23

Systems of Linear Equations

Reading for next time: Numerical Recipes, pp. 129-139

Linear Systems

$$a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + \ldots + a_{0,n-1}x_{n-1} = b_0$$

$$a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,n-1}x_{n-1} = b_1$$

...

$$a_{m-1,0}x_0 + a_{m-1,1}x_1 + a_{m-1,2}x_2 + \ldots + a_{m-1,n-1}x_{n-1} = b_{m-1}$$

- **If n=m, we try to solve for a unique set of x. Obstacles:**
  - **If any row (equation) or column (variables) is linear combination of others, matrix is degenerate or not of full rank. No solution. Your underlying model is probably wrong; you'll need to fix it.**
  - **If rows or columns are nearly linear combinations, roundoff errors can make them linearly dependent during computations. We'll fail to find a solution, even though one may exist.**
  - **Roundoff errors can accumulate rapidly. While you may get a solution, when you substitute it into your equation system, you'll find it's not a solution. (Right sides don't quite equal left sides.)**
- **Large linear systems tend to be close to singular (degenerate). Beware!**

Systems of Linear Equations

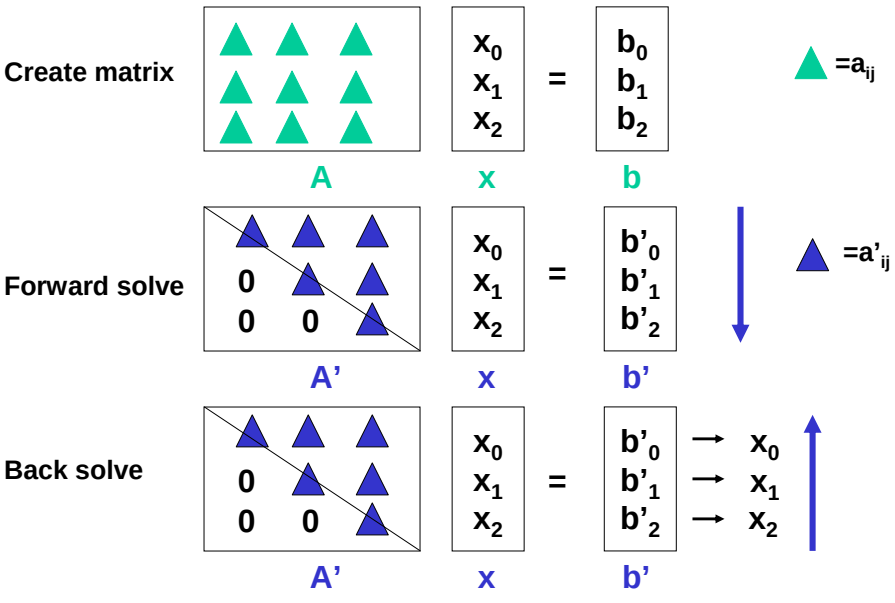
$$3x_0 + x_1 - 2x_2 = 5$$
$$2x_0 + 4x_1 + 3x_2 = 35$$
$$x_0 - 3x_1 = -5$$

$$\begin{vmatrix} 3 & 1 & -2 \\ 2 & 4 & 3 \\ 1 & -3 & 0 \end{vmatrix} \begin{vmatrix} x_0 \\ x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 5 \\ 35 \\ -5 \end{vmatrix}$$

A x = b

3 x 3 3 x 1 3 x 1

Algorithm to Solve Linear System



Gaussian Elimination: Forward Solve

$$Q = \left[\begin{array}{ccc|c} 3 & 1 & -2 & 5 \\ 2 & 4 & 3 & 35 \\ 1 & -3 & 0 & -5 \end{array}\right] \qquad \begin{matrix} A & & & b \end{matrix}$$

Form Q for convenience
Do elementary row ops:
Multiply rows
Add/subtract rows

Make column 0 have zeros below diagonal

Pivot= 2/3
Pivot= 1/3

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 5 \\ 0 & 10/3 & 13/3 & 95/3 \\ 0 & -10/3 & 2/3 & -20/3 \end{array}\right]$$

Row 1'= row 1 - (2/3) row 0
Row 2'= row 2 - (1/3) row 0

Make column 1 have zeros below diagonal

Pivot= 1

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 5 \\ 0 & 10/3 & 13/3 & 95/3 \\ 0 & 0 & 15/3 & 75/3 \end{array}\right]$$

Row 2''= row 2' + 1 * row 1

Gaussian Elimination: Back Solve

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 5 \\ 0 & 10/3 & 13/3 & 95/3 \\ 0 & 0 & 15/3 & 75/3 \end{array}\right]$$

$(15/3)x_2=(75/3)$ $\quad x_2 = 5$

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 5 \\ 0 & 10/3 & 13/3 & 95/3 \\ 0 & 0 & 15/3 & 75/3 \end{array}\right]$$

$(10/3)x_1 + (13/3)*5= (95/3)$ $\quad x_1 = 3$

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 5 \\ 0 & 10/3 & 13/3 & 95/3 \\ 0 & 0 & 15/3 & 75/3 \end{array}\right]$$

$3x_0 + 1*3 - 2*5 = 5$ $\quad x_0 = 4$

A Complication

0	1	-2	5
2	4	3	35
1	-3	0	-5

Row 1'= row 1 - (2/0) row 0

Exchange rows: put largest pivot element in row:

2	4	3	35
0	1	-2	5
1	-3	0	-5

Do this as we process each column.

If there is no nonzero element in a column, matrix is not full rank.

Gaussian Elimination

```
// In class Matrix, add:

public static Matrix gaussian(Matrix a, Matrix b) {
    int n = a.data.length;                    // Number of unknowns
    Matrix q = new Matrix(n, n + 1);

    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++)           // Form q matrix
            q.data[i][j]= a.data[i][j];
        q.data[i][n]= b.data[i][0];
    }

    forward_solve(q);                 // Do Gaussian elimination
    back_solve(q);                    // Perform back substitution

    Matrix x= new Matrix(n, 1);
    for (int i = 0; i < n; i++)
        x.data[i][0]= q.data[i][n];
    return x;
}
```

Forward Solve

```
private static void forward_solve(Matrix q) {
    int n = q.data.length;

    for (int i = 0; i < n; i++) { // Find row w/max element in this
        int maxr = i;             // column, at or below diagonal
        for (int k = i + 1; k < n; k++)
            if (Math.abs(q.data[k][i]) > Math.abs(q.data[maxr][i]))
                maxr = k;

        if (maxr != i)            // If row not current row, swap
            for (int j = i; j <= n; j++) {
                double t = q.data[i][j];
                q.data[i][j]= q.data[maxr][j];
                q.data[maxr][j]= t;
            }

        for (int j = i + 1; j < n; j++) {  // Calculate pivot ratio
            double pivot = q.data[j][i] / q.data[i][i];
            for (int k = i; k <= n; k++)  // Pivot operation itself
                q.data[j][k] -= q.data[i][k] * pivot;
        }
    }
}
```

Back Substitution

```
private static void back_solve(Matrix q) {
    int n = q.data.length;

    for (int j = n - 1; j >= 0; j--) {       // Start at last row
        double t = 0.0;                      // t- temporary
        for (int k = j + 1; k < n; k++)
            t += q.data[j][k] * q.data[k][n];
        q.data[j][n]= (q.data[j][n] - t) / q.data[j][j];
    }
}
```

Test Program

```
import javax.swing.*;

public class GaussTest {
    public static void main(String[] args) {
        int i, j;
        double term;
        String input = JOptionPane.showInputDialog(
             "Enter number of unknowns");
        int n = Integer.parseInt(input);

        Matrix a = new Matrix(n, n);
        Matrix b = new Matrix(n, 1);

        // Enter matrix a
        for (i = 0; i < a.getNumRows(); i++)
            for (j = 0; j < a.getNumCols(); j++) {
                input =JOptionPane.showInputDialog(
                     "Enter a[" + i + "][" + j + "]");
                term = Double.parseDouble(input);
                a.setElement(i, j, term);
            }       // Continued on next slide
```

Test Program, p.2

```
        // Enter vector b as 1-column matrix
        for (i = 0; i < b.getNumRows(); i++) {
              input = JOptionPane.showInputDialog(
                    "Enter b[" + i + "]");
              term = Double.parseDouble(input);
              b.setElement(i, 0, term);
        }
        Matrix x= Matrix.gaussian(a, b);       // Solve it!

        System.out.println("Matrix a:");
        a.print();
        System.out.println("Vector b:");
        b.print();
        System.out.println("Solution vector x:");
        x.print();
    }
}
```

Variations

Multiple right hand sides: augment Q, solve all eqns at once

$$\left|\begin{array}{ccc|ccc} 3 & 1 & -2 & 5 & 7 & 87 \\ 2 & 4 & 3 & 35 & 75 & -1 \\ 1 & -3 & 0 & -5 & 38 & 52 \end{array}\right|$$

Matrix inversion (rarely done in practice)

$$\underbrace{\left|\begin{array}{ccc|ccc} 3 & 1 & -2 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right|}_{Q} \longrightarrow \left|\begin{array}{ccc|ccc} \# & \# & \# & @ & @ & @ \\ 0 & \# & \# & @ & @ & @ \\ 0 & 0 & \# & @ & @ & @ \end{array}\right|$$

A I

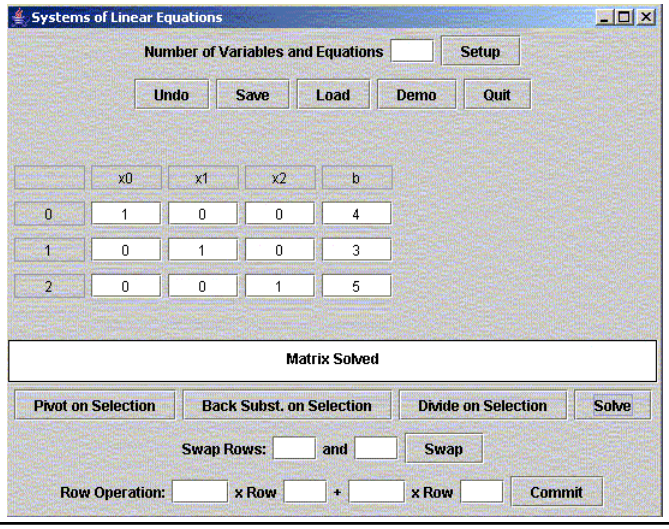
A^{-1}

$Ax=b$

$x= A^{-1} b$

Exercise

- **Download GElim and Matrix**
- **Compile and run GElim:**



Exercise

- Experiment with the following 3 systems:
  - Use pivot, back subst, divide on selection, etc. not solve
  - The 3x3 matrix example in the previous slides. Click on "Demo" to load it.

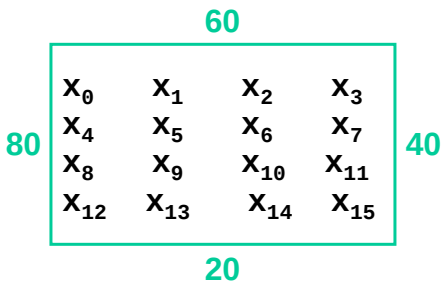
4	6	-3	10
2	5	9	12
8	8	-27	6

12	-13.5	3	0.5	2.75
8	-9	4	2.5	3.5
3	6	1.5	2	4.25
2	1.5	4	12	6

Using Linear Systems

- A common pattern in engineering, scientific and other analytical software:
  - Problem generator (model, assemble matrix)
    - Customized to specific application (e.g. heat transfer)
    - Use matrix multiplication, addition, etc.
  - Problem solution (system of simultaneous linear equations)
    - Usually "canned": either from library or written by you for a library
  - Output generator (present result in understandable format)
    - Customized to specific application (often with graphics, etc.)
- We did a pattern earlier: model-view-controller

Heat Transfer Example



4 by 4 grid of points on the plate produces 16 unknown temperatures x_0 through x_{15}

$$T = (T_{left} + T_{right} + T_{up} + T_{down})/4$$

Edge temperatures are known; interior temperatures are unknown
This produces a 16 by 16 matrix of linear equations

Heat Transfer Equations

- **Node 0:**
  $x_0 = (80 + x_1 + 60 + x_4)/4$ $\quad 4x_0 - x_1 - x_4 = 140$
- **Node 6:**
  $x_6 = (x_5 + x_7 + x_2 + x_{10})/4$ $\quad 4x_6 - x_5 - x_7 - x_2 - x_{10} = 0$
- **Interior node:**
  $x_i = (x_{i-1} + x_{i+1} + x_{i-n} + x_{i+n})/4$ $\quad 4x_i - x_{i-1} - x_{i+1} - x_{i-n} - x_{i+n} = 0$

Node	0	1	2	3	4	5	6	7
0	4	-1	0	0	-1	0	0	0
1	-1	4	-1	0	0	-1	0	0
2	0	-1	4	-1	0	0	-1	0
3	0	0	-1	4	0	0	0	-1
4	-1	0	0	0	4	-1	0	0
5	0	-1	0	0	-1	4	-1	0
6	0	0	-1	0	0	-1	4	-1
7	0	0	0	-1	0	0	-1	4

Ax= b

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
0	4	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	x0	140
1	-1	4	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0	x1	60
2	0	-1	4	-1	0	0	-1	0	0	0	0	0	0	0	0	0	x2	60
3	0	0	-1	4	0	0	0	-1	0	0	0	0	0	0	0	0	x3	100
4	-1	0	0	0	4	-1	0	0	-1	0	0	0	0	0	0	0	x4	80
5	0	-1	0	0	-1	4	-1	0	0	-1	0	0	0	0	0	0	x5	0
6	0	0	-1	0	0	-1	4	-1	0	0	-1	0	0	0	0	0	x6	0
7	0	0	0	-1	0	0	-1	4	0	0	0	-1	0	0	0	0	x7	40
8	0	0	0	0	-1	0	0	0	4	-1	0	0	-1	0	0	0	x8	80
9	0	0	0	0	0	-1	0	0	-1	4	-1	0	0	-1	0	0	x9	0
10	0	0	0	0	0	0	-1	0	0	-1	4	-1	0	0	-1	0	x10	0
11	0	0	0	0	0	0	0	-1	0	0	-1	4	0	0	0	-1	x11	40
12	0	0	0	0	0	0	0	0	-1	0	0	0	4	-1	0	0	x12	100
13	0	0	0	0	0	0	0	0	0	-1	0	0	-1	4	-1	0	x13	20
14	0	0	0	0	0	0	0	0	0	0	-1	0	0	-1	4	-1	x14	20
15	0	0	0	0	0	0	0	0	0	0	0	-1	0	0	-1	4	x15	60

Heat Transfer System

16 (rows) × 16 (columns)

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0,15} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1,15} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2,15} \\ & & \cdots & & \\ a_{24,0} & a_{24,1} & \cdots & & a_{24,15} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \cdots \\ x_{15} \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdots \\ b_{15} \end{bmatrix}$$

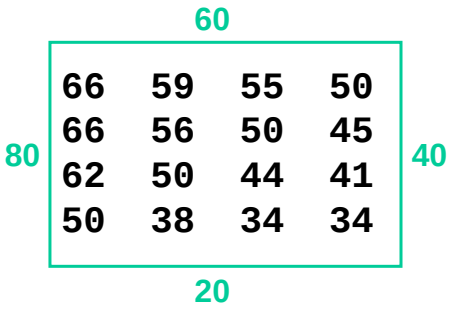
A x b

Contains 0, -1, 4 coefficients in (simple) pattern

Known temperatures (often 0 but use edge temperatures when close)

16 unknown interior temperatures

Heat Transfer Result



Heat Transfer Exercise, p.1

```
public class Heat {                    // Problem generator
  public static void main(String[] args) {
    double Te= 40.0, Tn=60.0, Tw=80.0, Ts=20.0; // Edge temps
    int col= 4;
    int row= 4;
    int n= col * row;
    Matrix a= new Matrix(n,n);
    for (int i=0; i < n; i++)
      for (int j=0; j < n; j++) {
        if (i==j)                      // Diagonal element (yellow)
          a.setElement(i, j, 4.0);
        else if (…)
         // Complete this code:
         //    Green elements (4, or ncols, away from diagonal)
         //    Blue elements (1 away from diagonal)
         //    Set blue and skip orange where we go to a point
         //    on the next row on the actual plate
    // Continued on next slide
```

Heat Transfer Exercise, p.2

```
   Matrix b= new Matrix(n, 1);          // Known temps
   for (int i=0; i < n; i++) {
     if (i < col)                       // Next to north edge
       b.setElement(i, 0, b.getElement(i,0)+Tn);
     if (…)
     // Complete this code for the other edges; no 'elses'!
     // Add edge temperature to b; you may add more than one
     // Look at the Ax=b example slide to find the pattern
   }
   Matrix x= Matrix.gaussian(a, b);           // Problem solution

   System.out.println("Temperature grid:");   // Output generator
   for (int i=0; i< row; i++) {
    for (int j=0; j < col; j++)
      System.out.print(Math.round(x.getElement((i*row+j),0)+" ");
    System.out.println();
   }
  }
}
```

Solution with 10 by 10 grid

60

80

69	65	62	60	59	58	57	56	54	50
72	67	63	60	58	56	55	53	50	46
74	68	63	60	57	55	52	50	47	44
74	68	63	59	55	52	50	48	45	43
73	67	61	57	53	50	48	45	44	42
72	65	59	54	50	47	45	43	42	41
70	62	55	50	46	43	41	40	40	40
67	57	50	45	41	39	37	37	37	38
62	50	43	38	35	33	32	32	33	35
50	38	33	30	28	27	26	26	28	31

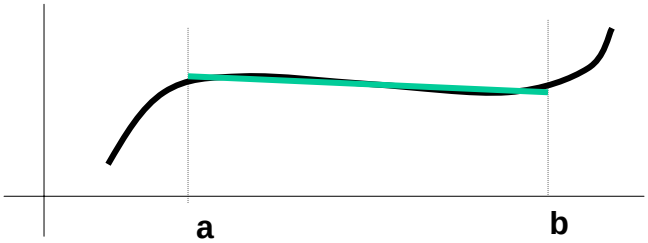
40

20

Same code as exercise; just change row= col= 10
You could create a grid of colors as Swing output
If you use 100 by 100 grid, you'll get very nice results

Other Applications

- **Solve systems with 1,000s or millions of linear equations or inequalities**
  - **Networks, mechanics, fluids, materials, economics**
  - **Often linearize systems in a defined range**



  - **Routines in this lecture are ok for a few hundred equations**
    - **They aren't very good at picking up collinear systems. Check first-see Numerical Recipes**
  - **Otherwise, see Numerical Recipes**