MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Lecture 19,20 Real-Time Optimization, Dynamic Programming
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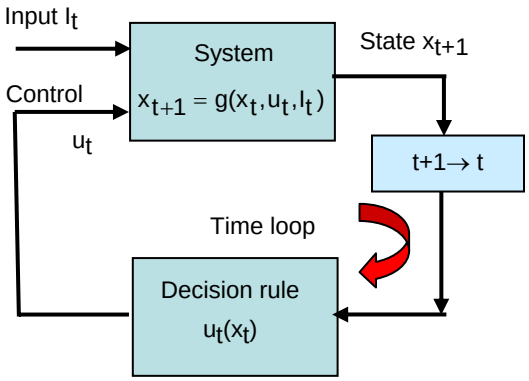
Real-time optimization

Real-time optimization problems rely on **decision rules** that specify how decisions should maximize future benefit, given the current **state** of a system. State dependence provides a convenient way to deal with **uncertainty**. Some examples:

- Reservoir releases – Decision rule specifies how current release should depend on current storage. Primary uncertainty is future reservoir inflow.
- Water treatment – Decision rule specifies how current operating conditions (e.g. temperature or chemical inputs) should depend on current concentration in treatment tank. Primary uncertainty is future influent concentration.
- Irrigation management - Decision rule specifies how current applied irrigation water should depend on current soil moisture and temperature. Primary uncertainties are future meteorological variables.

Real-time optimization can be viewed as a feedback control process:



At each time step:
- Observe state
- Derive control from decision rule
- Apply control.

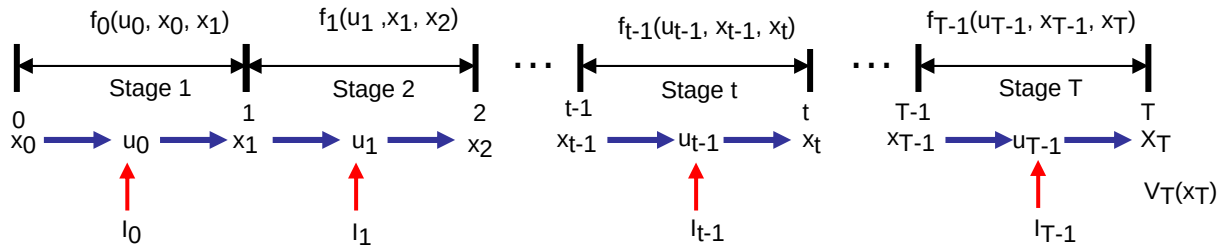
$t = 0,..., T-1$

State variables: x_t (decision variables, depend on controls and inputs)
Control variables: u_t (decision variables, selected to maximize benefit)
Input variables: I_t (inputs, assigned specified values)
Decision rule: $u_t(x_t)$ (function that gives u_t for any x_t)

State equation: $x_{t+1} = g(x_t, u_t, I_t)$ initial condition: x_0

Dynamic Programming

Dynamic programming provides a general framework for deriving decision rules. Most dynamic programming problems are divided into stages (e.g. time periods, spatial intervals, etc.):



Benefit accrued over Stage t+1 is $f_t(u_t, x_t, x_{t+1})$:

Optimization problem:

Select $u_t,...,u_{T-1}$ that **maximizes benefit-to-go** (benefit accrued from current time t through terminal time T) at each time t:

$$\underset{u_t,...u_{T-1}}{Max} \quad F_t(x_t,...,x_{T-1},u_t,...,u_{T-1}) \quad t = 0,\ldots,T\text{-}1$$

where benefit-to-go at t is terminal benefit (**salvage value**) $V_T(x_T)$ plus sum of benefits for stages t through T-1:

$$: \; F_t(x_t,...,x_{T-1},u_t,...,u_{T-1}) = \sum_{i=t}^{T-1} f_i(u_i,x_i,x_{i+1}) \quad + \quad V_T(x_T)$$

Benefit-to-go — **Benefit from remaining stages** — **Terminal benefit**

subject to:

$$x_{i+1} = g_t(x_i,u_i,I_i) \quad ; \quad i = t,...,T-1 \quad \text{(state equation)}$$

and other constraints on the decision variables:

$\{x_t,u_t\} \in \Gamma_t$, $\{x_T\} \in \Gamma_T$ (decision variables lie within some set Γ_t at t = 0,…,T).

Objective may be rewritten if we repeatedly apply state equation to write all x_i ($i > t$) as functions of $x_t,u_t,...,u_{T-1},I_t,...,I_{T-1}$:

$$F_t(x_t,...,x_T,u_t,...,u_{T-1}) = F_t(x_t,u_t,...,u_{T-1},I_t,...,I_{T-1})$$

Decision rule $u_t(x_t)$ at each t is obtained by finding sequence of controls $u_t,...,u_{T-1}$ that maximizes $F_t(x_t,u_t,...,u_{T-1},I_t,...,I_{T-1})$ for a given state x_t and a given set of specified inputs $I_t,...,I_{T-1}$.

Backward Recursion for Benefit-to-go

Dynamic programming uses a **recursion** to construct decision rule $u_t(x_t)$:

Define **return function** $V_t(x_t)$ to be maximum benefit-to-go from *t* through *T*:

$$V_t(x_t) = \underset{u_t,\ldots,u_{T-1}}{Max} \left[F_t(x_t, u_t, \ldots, u_{T-1}, I_t, \ldots, I_{T-1})\right]$$

Separate benefit term for Stage t+1:

$$V_t(x_t) = \underset{u_t}{Max}\left[f_t(u_t, x_t, x_{t+1}) + \underset{u_{t+1},\ldots u_{T-1}}{Max} F_{t+1}(x_{t+1}, u_{t+1}, \ldots, u_{T-1}, I_{t+1}, \ldots, I_{T-1}) \right]$$

Replace second term in brackets with definition for $V_{t+1}(x_{t+1})$:

$$V_t(x_t) = \underset{u_t}{Max}\left[f_t(u_t, x_t, x_{t+1}) + V_{t+1}(x_{t+1}) \right]$$

Substitute state equation for x_{t+1}:

$$V_t(x_t) = \underset{u_t}{Max}\left\{ f_t[u_t, x_t, g_t(x_t, u_t, I_t)] + V_{t+1}[g_t(x_t, u_t, I_t)] \right\}$$

This equation is a **backward recursion** for $V_t(x_t)$, initialized with terminal benefit $V_{T+1}[g(x_t, u_t, I_t)]$. Expression in braces depends only on u_t [which is varied to find the maximum], x_t [the argument of $V_t(x_t)$], and I_t [the specified input].

At each stage find the u_t that maximizes $V_t(x_t)$ for all possible x_t.

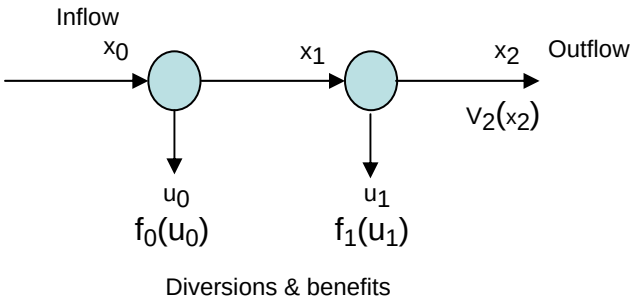
Store the results (e.g. as a function or table) to obtain the desired decision rule $u_t(x_t)$.

Example 1: Aqueduct Diversions

Maximize benefits from water diverted from 2 locations along an aqueduct.
Here the stages correspond to aqueduct sections rather than time (T = 2).

State equation:

$$x_{t+1} = x_t - u_t \qquad t = 0, 1, 2$$



Here stage numbers refer to **diversion index** rather than time.
Benefit at each stage depends only on control (diversion), not on state.
Terminal benefit depends on system outflow. No input included in state equation.

Total benefit from diverted water and outflow x_3 is:

$$F(x_1, u_0, u_1) = f_0(u_0) + f_1(u_1) + V_2(x_2)$$

The stage benefit and terminal benefit are:

$$f_0(u_0) = \frac{1}{2}u_0 \qquad f_1(u_1) = u_1 \qquad V_2(x_2) = x_2(1 - x_2)$$

Additional constraints are:

$$u_0 \geq 0 \qquad u_1 \geq 0 \qquad 0 \leq x_2 \leq 1$$

Solve problem by proceeding backward, starting with last stage (t = 2):

Stage 2: Find $V_1(x_1)$ for specified x_1:

$$\begin{aligned} V_1(x_1) &= \underset{u_1}{Max}\left[f_1(u_1) + V_2(x_2)\right] \\ &= \underset{u_1}{Max}\left[f_1(u_1) + V_2(x_1 - u_1)\right] \\ &= \underset{u_1}{Max}\left[u_1 + (x_1 - u_1)(1 - x_1 + u_1)\right] \\ &= \underset{u_1}{Max}\left[-u_1^2 + 2x_1u_1 + x_1(1 - x_1)\right] \end{aligned}$$

Solution that satisfies constraints:

$$u_1 = x_1 \qquad V_1(x_1) = x_1$$

Stage 1: Maximize $V_0(x_0)$ for specified x_0:

$$\begin{aligned} V_0(x_0) &= \underset{u_0}{Max}\left[f_0(u_0) + V_1(x_1)\right] \\ &= \underset{u_0}{Max}\left[\frac{1}{2}u_0 + x_0 - u_0\right] \end{aligned}$$

Solution that satisfies constraints:

$$u_0 = 0 \qquad V_0(x_0) = x_0$$

Solution specifies that all water is withdrawn from second diversion point, where it is most valuable.

Discretization of Variables

In more complex problems the states, controls and inputs are frequently discretized into a finite number of compatible levels.

Simplest case:

$$x_t, u_t, I_t \quad \rightarrow \quad x_t^k, u_t^j, I_t^l \quad ; \quad j, k, l = 1, 2, \ldots, L \text{ where } L = \text{number of discrete levels.}$$

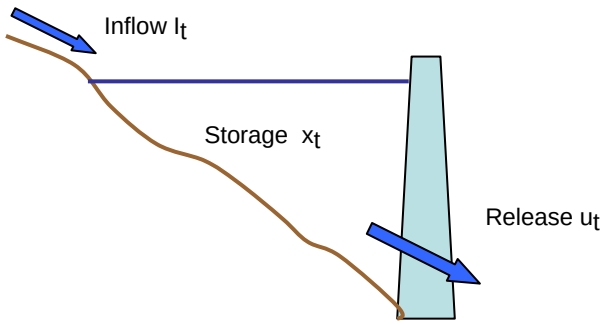
Then the optimization of problem at each stage reduces to an **exhaustive search** through all feasible levels of u_t^j to find the one that maximizes:

$$f_t[u_t^j, x_t^k, g_t(x_t^k, u_t^j, I_t^l)] + V_{t+1}[g_t(x_t^k, u_t^j, I_t^l)]$$

for each feasible x_t^k and a single specified input level I_t^l.

This is **discrete dynamic programming**.

Example 2: Reservoir Operations
Maximize benefits from water released from reservoir with variable inflows.
Stages correspond to 3 time periods (months, seasons, etc. $T = 3$).



State equation:

$$x_{t+1} = x_t - u_t + I_t \qquad t = 0, 1, 2$$

Total benefit from released water and final storage x_3:

$$F(x_0, u_0, ..., u_2) = f_0(u_0) + f_1(u_1) + f_2(u_2) + V_3(x_3)$$

Discretize all variables into compatible levels:

$u_t = \{0,1,2\}$ $x_t = \{0,1,2\}$ $I_t = \{0,1\}$ $t = 0, 1, 2$

Inflows:

I_0	I_1	I_2
1	0	1

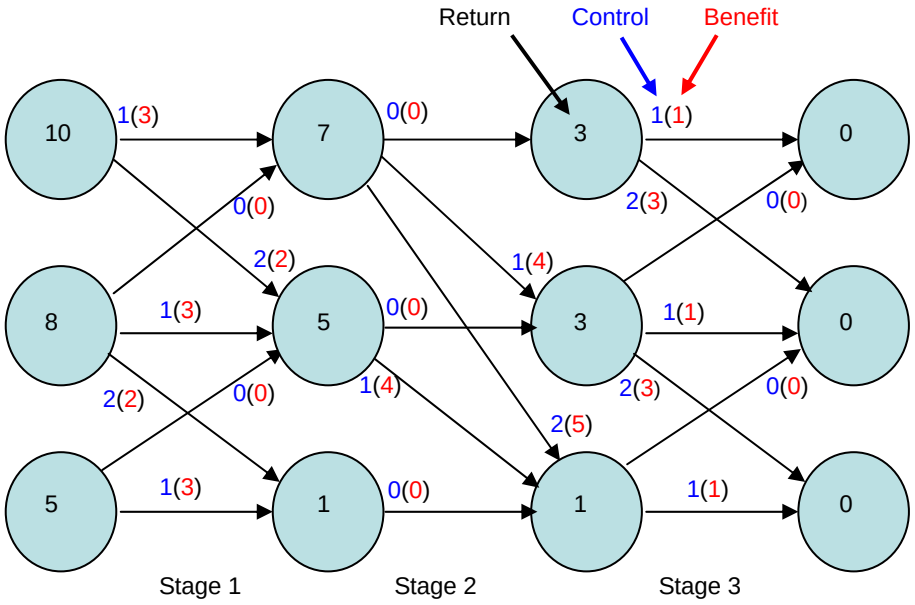
Terminal (outflow) benefits: $V_3(x_3) = 0$ for all x_3 values

Benefits for each release:

u_t	$f_0(u_0)$	$f_1(u_1)$	$f_2(u_2)$
0	0	0	0
1	3	4	1
2	2	5	3

Possible state transitions are derived from state equation, inputs, and permissible variable values:
Benefit is shown in parentheses after each feasible control value.

Show the possible transitions with a diagram where each feasible state level x_t^k is a circle and each feasible control level u_t^j is a line connecting circles:



Control, benefit, and return values

Solve series of 3 optimization problems defined by recursion equation for t = 2, 1, 0.
Start at last stage and move backward:

Stage 3: Find $V_2(x_2)$ for each level of x_2:

$$V_2(x_2) = \underset{u_2}{Max}\left[f_2(u_2) + V_3(x_3)\right] = \underset{u_2}{Max}\left[f_2(u_2) + V_3(x_2 - u_2 + I_2)\right]$$

Identify optimum $u_2(x_2)$ values for each possible x_2, $V_3(x_3)$ specified as an input:

X_2	$u_2(x_2)$	$f_2(u_2)$ +	$V_3(x_3)$		
0	0	0 +	0	= 0	
	1	1 +	0	= 1 = $V_2(x_2)$	← Optimum
1	0	0 +	0	= 0	
	1	1 +	0	= 1	
	2	3 +	0	= 3 = $V_2(x_2)$	← Optimum
2	1	1 +	0	= 1	
	2	3 +	0	= 3 = $V_2(x_2)$	← Optimum

Stage 2: Find $V_1(x_1)$ for each level of x_1:

$$V_1(x_1) = \underset{u_1}{Max}\left[f_1(u_1) + V_2(x_2)\right] = \underset{u_1}{Max}\left[f_1(u_1) + V_2(x_1 - u_1 + I_1)\right]$$

Identify optimum $u_1(x_1)$ value for each possible x_1, obtain $V_2(x_2)$ from Stage 2:

x_1	$u_1(x_1)$	$f_1(u_1)$ +	$V_2(x_2)$		
0	0	0 +	1	= 1 = $V_1(x_1)$	← Optimum
1	0	0 +	3	= 3	
	1	4 +	1	= 5 = $V_1(x_1)$	← Optimum
2	0	0 +	3	= 4	
	1	4 +	3	= 7 = $V_1(x_1)$	← Optimum
	2	5 +	1	= 6	

Stage 1: Find $V_0(x_0)$ for each level of x_0:

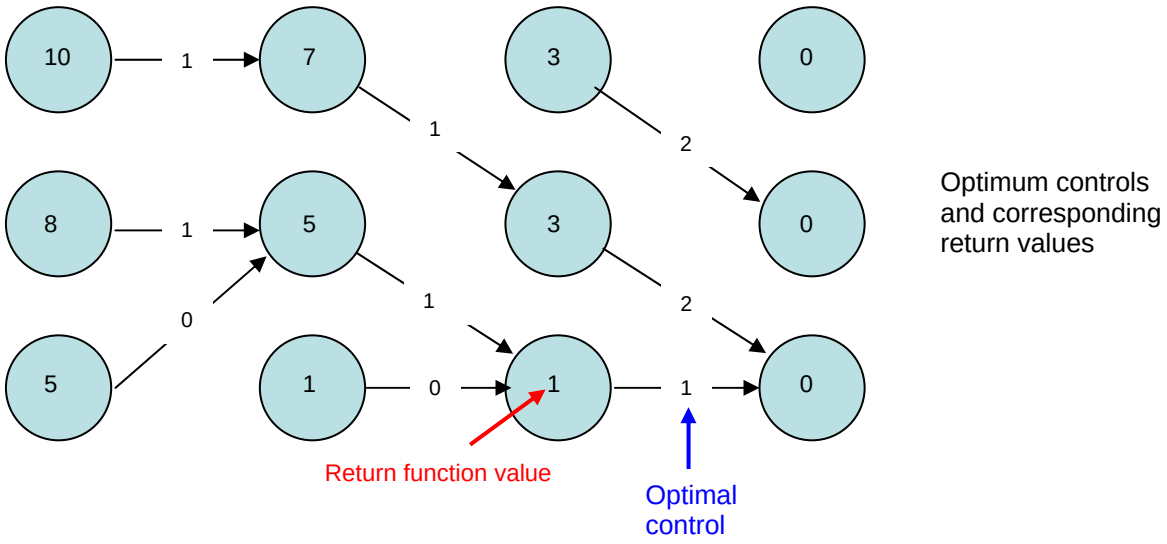
$$V_0(x_0) = \underset{u_0}{Max}\left[f_0(u_0) + V_1(x_1)\right] = \underset{u_0}{Max}\left[f_0(u_0) + V_1(x_0 - u_0 + I_0)\right]$$

Identify optimum $u_0(x_0)$ values for each x_0, obtain $V_1(x_1)$ from Stage 1:

x_1	$u_0(x_0)$	$f_0(u_0)$ +	$V_1(x_1)$		
0	0	0 +	5	= 5 = $V_0(x_0)$	← Optimum
	1	3 +	1	= 4	
1	0	0 +	7	= 7 = $V_0(x_0)$	← Optimum
	1	3 +	5	= 8	
	2	2 +	1	= 3	
2	1	3 +	7	= 10 = $V_0(x_0)$	← Optimum
	2	2 +	5	= 7	

The optimum $u_t(x_t)$ decision rules for t = 0, 1, 2 define a complete optimum decision strategy:

x_0	u_0	x_1	u_1	x_2	u_2
0	0	0	0	0	1
1	1	1	1	1	2
2	1	2	1	2	2



Note that there is a path leaving every state value. The optimum paths give a strategy for maximizing benefit-to-go from t onward, for any value of state x_t.

Optimal benefit for each possible initial storage is $V_0(x_0)$.

Computational Effort

The solution to the discretized optimization problem can be found by **exhaustive enumeration** (by comparing benefit-to-go for all possible $u_t(x_t)$ combinations).

Dynamic programming is much more efficient than enumeration since it divides the original T stage optimization problem into T smaller problems, one for each stage.

To compare computational effort of enumeration and dynamic programming assume:

- State dimension = M, Stages = T, Levels = L
- Equal number of levels for u_t and x_t at every stage
- All possible state transitions are permissible (i.e. L^2 transitions at each stage)

Then total number of V_t evaluations required is:

Exhaustive enumeration: $L^{M(T+1)}$

Dynamic Programming: TL^{2M}

For $M = 1$, $L = 10$, $T = 10$ the number of V_t evaluations required is:

Exhaustive enumeration: 10^{11}

Dynamic Programming: 10^3