Lecture 20

Lecturer: Pablo A. Parrilo *Scribe: ???*

In this lecture we introduce Schmüdgen's theorem about the K-moment problem (or equivalently, on the representation of positive polynomials) and describe the basic elements in his proof. This approach combines both algebraic tools (using the Positivstellensatz to prove the boundedness of certain operators) and functional analysis (spectral measures of commuting families of operators and the Hahn-Banach theorem). We will also describe some alternative versions due to Putinar, as well as a related purely functional-analytic result due to Megretski.

For a comprehensive treatment and additional references, we mention [BCR98, Mar00, PD01] among others.

1 Positive polynomials

As we have seen, the Positivstellensatz allows us to obtain certificates of the emptiness of a basic semialgebraic set, explicitly given by polynomials. When looking for bounded degree certificates, this provides a natural hierarchy of SDP-based conditions [Par00, Par03].

What if we want to apply this for the particular case of optimization? As we have seen, it is relatively straightforward to convert a polynomial optimization problem to a one-parameter family of feasibility problems, by considering the sublevel sets, i.e., the sets $\{x \in \mathbb{R}^n \mid f(x) \leq \gamma\}$.

In the general case of constrained problems, however, using the full power of the Psatz will yield conditions that are not linear in the unknown parameter γ (because we need products between the constraints and objective function), and in principle, this presents a difficulty to the direct use of SDP. Notice nevertheless, that the problem is certainly an SDP for any fixed value of γ, and is thus quasiconvex (which is almost as good, except for the fact that we cannot use "standard" SDP solvers to solve it directly, but rather rely on methods such as bisection).

Of course, we can always produce specific families of certificates that are linear in γ, and use them for optimization (e.g., like we did in the copositivity case). However, in general it is unclear whether the desired family is "complete," in the sense that we will be able to prove arbitrarily good bounds on the optimal value as the degree of the polynomials grows to infinity.

2 Schmüdgen's theorem

In 1991, Schmüdgen presented a characterization of the moment sequences of measures supported on a compact semialgebraic K (the K-moment problem). As in the one-dimensional case we studied earlier the question is, given an (infinite) sequence of moments, decide whether it actually corresponds to a nonnegative measure with support on a given set K.

His solution combined both real algebraic methods (the Psatz), with some functional analytic tools (reproducing kernel Hilbert spaces, bounded operators, and the spectral theorem).

This characterization of moment sequences can be used, in turn, to produce an explicit description of the set of strictly positive polynomials on a compact semialgebraic set:

Theorem 1 ([Sch91])**.** *If $p(x)$ is strictly positive on $K = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0\}$, and K is compact, then $p(x) \in \mathbf{cone}\{f_1, \dots, f_m\}$.*

`expand` ToDo

There are several interesting ideas in the proof; a coarse description follows. The first step is to use the Positivstellensatz to produce an algebraic certificate of the compactness of the set K. Then the given moment sequence (which is a positive definite function on the semigroup of monomials) is used to construct a particular pre-Hilbert space and its completion (namely, the associated reproducing kernel Hilbert space). In this Hilbert space, we consider linear operators T_{x_i} given by multiplication by the coordinate variables, and use the algebraic certificate of compactness to prove that these are bounded. Now, the T_{x_i} are a finite collection of pairwise commuting, bounded, self-adjoint operators, and thus there exists a spectral measure for the family, from which a measure, only supported in K, can be extracted. Finally, a Hahn-Banach (separating hyperplane) argument is used to prove the final result.

2.1 Putinar's approach

The theorem in the previous section requires (in principle) all $2^m - 1$ squarefree products of constraints[1]. Putinar [Put93] presented a modified formulation (under stronger assumptions) for which the representation is *linear* in the constraints. We introduce the following concept:

Definition 2. *Let* $\{f_1, \ldots, f_m\} \subset \mathbb{R}[x]$*. The* preprime *generated by the* f_i*, and denoted by* $\mathbf{preprime}\{f_1, \ldots, f_m\}$ *is the set of all polynomials of the form* $s_0 + s_1 f_1 + \cdots + s_m f_m$*, where all the* s_i *are sums of squares.*

Notice that $\mathbf{preprime}\{f_i\} \subset \mathbf{cone}\{f_i\}$, and that every element in the preprime takes only nonnegative values on $\{x \in \mathbb{R}^n, f_i(x) \geq 0\}$.

Theorem 3 ([Put93])**.** *Consider a set* $K = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0\}$*, such that there exists a* $q \in \mathbf{preprime}\{f_1, \ldots, f_m\}$ *and* $\{x \in \mathbb{R}^n, q(x) \geq 0\}$ *is compact (this implies that* K *is compact). Then,* $p(x) > 0$ *on* K *if and only if* $p(x) \in \mathbf{preprime}\{f_1, \ldots, f_m\}$*.*

Notice that here, the polynomial q serves as an algebraic certificate of the compactness of K, so in this case the Psatz is not needed.

Putinar's theorem was used by Lasserre to present a hierarchy of semidefinite relaxations for polynomial optimization, based on the dual moment interpretation [Las01].

2.2 Tradeoffs

In principle (and often, in practice) there is a tradeoff between how "expressive" our family of certificates is, the quality of the resulting bounds, and the complexity of finding proofs.

On one extreme, the most general method is the Psatz, as it encapsulates pretty much every possible "algebraic deduction," and will certainly provide the strongest bounds, since it includes the other techniques as special cases. For optimization, Schmüdgen's theorem provides the advantages of a linear representation, although (possibly) at the cost of having a large number of products between the constraints. Finally, the Putinar approach has a reduced number of constraints (and thus, SOS multipliers), although the obtained bounds can potentially be much weaker than the previous ones.

In the end, the decision concerning what approach to use should be dictated by the available computational resources, i.e., the size of the SDPs that we can solve in a reasonable time. It is not difficult to produce examples with significant gaps between the corresponding bounds; see for instance [Ste96] for a particularly simple example, that is trivial for the Psatz, but for which either the Schmüdgen or Putinar representations need large degree refutations.

`Add examples`	ToDo

[1] Recall that in practice, this may not be a issue at all, since the restriction on the degree of the certificates imposes a strict limit on how many products can be included.

2.3 Trigonometric case

Recently, Megretski [Meg03] analyzed the trigonometric case. We introduce the following notation: let $\mathbb{T}_n = \{z \in \mathbb{C}^n, |z_i| = 1\}$ be the n-dimensional torus, P_n is the set of multivariate Laurent polynomials, and $RP_n \subset P_n$ are the Laurent polynomials that are real-valued on $\mathbb{T}_n$.

Theorem 4 ([Meg03]). *Let $\{F, Q_1, \ldots, Q_m\} \subset RP_n$, such that $F(z) > 0$ for all $z \in \mathbb{T}_n$ satisfying $Q_1(z) = \ldots = Q_m(z) = 0$. Then there exist $V_1, \ldots, V_r \in P_n$, $H_1, \ldots, H_m \in RP_n$, such that*

$$F(z) = \sum_{i=1}^{r} |V_i(z)|^2 + \sum_{j=1}^{m} H_j(z)Q_i(z).$$

Notice that, by splitting into real and imaginary part, this corresponds to a special kind of (standard) polynomials, and a compact semialgebraic set (so in principle, any of the previous theorems would apply). Of course, the result exploits the complex structure for a more concise representation.

In particular, Megretski's proof is purely functional-analytic, the main tools being Bochner's theorem and Hahn-Banach. Bochner's theorem is an important result in harmonic analysis, that characterizes a positive definite function on an Abelian group in terms of the nonnegativity of its Fourier transform.

Notice that the theorem above deals only with the equality case (no inequalities), and the feasible set is compact (since so it $\mathbb{T}^n$). It essentially states that a positive polynomial is a sum of squares modulo the ideal generated by the Q_i. Recall we have proved similar results in the zero-dimensional case, and this theorem naturally generalizes these.

In simplified terms, one reason why trigonometric (or Laurent) polynomials are somewhat "easier" than the general case is because in this case there is a *group* structure, as opposed to the *semigroup* structure of regular monomials. For the group case, the corresponding theory is the classical harmonic analysis on abelian groups (e.g., [Rud90]); while for semigroups there is the newer, but well-developed characterizations of positive functions on (Abelian) semigroups; see for instance [BCR84].

We also mention that there are "purely algebraic" versions of these theorems, that do not use functional analytic ideas (e.g., [Mar00]). Roughly, the role played by the compactness of K in proving the boundedness of the operators T_{x_i} is replaced with a property called *Archimedeanity* of the corresponding preorder.

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