6.045: Automata, Computability, and Complexity
Or, Great Ideas in Theoretical Computer Science
Spring, 2010

Class 8

Nancy Lynch

Today

- More undecidable problems:
  - About Turing machines: Emptiness, etc.
  - About other things: Post Correspondence Problem.
- Topics:
  - Undecidable problems about Turing machines.
  - The Post Correspondence Problem: Definition
  - Computation histories
  - First proof attempt
  - Second attempt: Undecidability of modified PCP (MPCP)
  - Finish undecidability of PCP
- Reading: Sipser Sections 4.2, 5.1.

Undecidable Problems about Turing Machines

Undecidable Problems about Turing Machines

- We already showed that Acc_{TM} and $Halt_{TM}$ are not Turing-decidable (and their complements are not even Turing-recognizable).
- Now consider some other problems:
  - $Acc01_{TM}$ = { <M> | M is a TM that accepts the string 01 }
  - $Empty_{TM}$ = { <M> | M is a TM that accepts no strings}
  - Reg_{TM} = { <M> | M is a TM and L(M) is regular}
  - EQ_{TM}, equivalence for TMs, = { < M_1, M_2 > | M_1 and M_2 are TMs and $L(M_1) = L(M_2)$ }

$Acc01_{TM}$

- $Acc01_{TM}$ = { <M> | M accepts the string 01 }
- Theorem 1: $Acc01_{TM}$ is not Turing-decidable.
- This might seem surprising---it seems simpler than the general acceptance problem, since it involves just one particular string.
- Proof attempt:
  - Try a reduction---show if you could decide $Acc01_{TM}$ then you could decide general acceptance problem Acc_{TM}.
  - Let R be a TM that decides $Acc01_{TM}$.; design S to decide Acc_{TM}.
  - S: On input <M,w>:
    - Run R on <M>.
    - If R accepts… ??? Gives useful information only if w = 01.
    - Doesn’t work.

$Acc01_{TM}$

- Theorem 1: $Acc01_{TM}$ is not Turing-decidable.
- Proof attempt:
  - Let R be a TM that decides $Acc01_{TM}$.
  - S: On input <M,w>:
    - Run R on <M>.
    - If R accepts… ???
    - Doesn't work.
- How can we use information about what a machine does on 01 to help decide what a given machine M will do on an arbitrary w?
- Idea: Consider a different machine---modify M.

$Acc01_{TM}$

- Theorem 1: $Acc01_{TM}$ is not Turing-decidable.
- Proof:
  – Let R be a TM that decides $Acc01_{TM}$.; design S to decide Acc_{TM}.
  – S: On input <M,w>:
    - Instead of running M on w, S constructs a new machine $M'_{M,w}$ that depends on M and w.
    - $M'_{M,w}$: On any input x, ignores x and runs M on w.
    - Thus, the new machine is the same as M, but hard-wires in the given input w.
  – More precisely:

$Acc01_{TM}$

- Theorem 1: $Acc01_{TM}$ is not Turing-decidable.
- Proof:
  - R decides $Acc01_{TM}$.; design S to decide Acc_{TM}.
  - S: On input <M,w>:
    - Step 1: Construct a new machine <$M'_{M,w}$>, where
      - $M'_{M,w}$: On input x:
        - Run M on w and accept/reject if M does.
    - Step 2: Run R on <$M'_{M,w}$>, and accept/reject if R does.
  - Note that S can construct <$M'_{M,w}$> algorithmically, from inputs M and w.

$Acc01_{TM}$

- Theorem 1: $Acc01_{TM}$ is not Turing-decidable.
- Proof:
  - R decides $Acc01_{TM}$.; design S to decide Acc_{TM}.
  - S: On input <M,w>:
    - Step 1: Construct a new machine <$M'_{M,w}$>, where
      - $M'_{M,w}$: On input x:
        - Run M on w and accept/reject if M does.
    - Step 2: Run R on <$M'_{M,w}$>, and accept/reject if R does.
  - Running R on <$M'_{M,w}$> tells us whether or not $M'_{M,w}$ accepts 01.
  - Claim: $M'_{M,w}$ accepts 01 if and only if M accepts w.
    - $M'_{M,w}$ always behaves the same, ignoring its own input and simulating M on w.
    - If $M'_{M,w}$ accepts 01 (or anything else), then M accepts w.
    - If M accepts w, then $M'_{M,w}$ accepts 01 (and everything else).
  - So S gives the right answer for whether M accepts w.

$Acc01_{TM}$

- Theorem 1: $Acc01_{TM}$ is not Turing-decidable.
- Theorem: $Acc01_{TM}$ is Turing-recognizable.
- Corollary: $(Acc01_{TM})^c$ is not Turing-recognizable.

Empty_{TM}

- $\text{Empty}_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$
- Theorem 2: Empty_{TM} is not Turing-decidable.
- Proof:
  - Reduce Acc_{TM} to Empty_{TM}.
  - Modify the given machine M: Given <M,w>, construct a new machine M′ so that asking whether L(M′) = ∅ gives the right answer to whether M accepts w:
  - Specifically, M accepts w if and only if L(M′) ≠ ∅.
  - Use the same machine M′ as for Acc01_{TM}.
  - S: On input <M,w>:
    - Step 1: Construct < $M'_{M,w}$ > as before, which acts on every input just like M on w.
    - Step 2: Ask whether $L(M'_{M,w}) = \emptyset$ and output the opposite answer.

Empty_{TM}

- Theorem 2: Empty_{TM} is not Turing-decidable.
- Proof:
  - Reduce Acc_{TM} to Empty_{TM}.
  - S: On input <M,w>:
    - Step 1: Construct < $M'_{M,w}$ > as before, which acts on every input just like M on w.
    - Step 2: Ask whether $L(M'_{M,w}) = \emptyset$ and output the opposite answer.
  - Now M accepts w

    if and only if $M'_{M,w}$ accepts everything

    if and only if $M'_{M,w}$ accepts something

    if and only if $L(M'_{M,w}) \neq \emptyset$.
  - So S decides Acc_{TM}, contradiction.
  - So Empty_{TM} is not Turing-decidable.

$Empty_{TM}$

- Theorem 2: $Empty_{TM}$ is not Turing-decidable.
- Theorem: $(Empty_{TM})^c$ is Turing-recognizable.
- Proof: On input <M>, run M on all inputs, dovetailed, accept if any accept.
- Corollary: $Empty_{TM}$ is not Turing-recognizable

Reg_{TM}

- Reg_{TM} = { <M> | M is a TM and L(M) is regular}
- That is, given a TM, we want to know whether its language is also recognized by some DFA.
- For some, the answer is yes: TM that recognizes 0*1*
- For some, no: TM that recognizes $\{0^n1^n \mid n \geq 0\}$
- We can prove that there is no algorithm to decide whether the answer is yes or no.
- Theorem 3: Reg_{TM} is not Turing-decidable.
- Proof:
  - Reduce Acc_{TM} to Reg_{TM}.
  - Assume TM R that decides Reg_{TM}, design S to decide Acc_{TM}.
  - S: On input <M,w>:
    - Step 1: Construct a new machine < $M'_{M,w}$ > that accepts a regular language if and only if M accepts w.
    - Tricky…

Reg_{TM}

- Reg_{TM} = { <M> | L(M) is regular }
- Theorem 3: Reg_{TM} is not Turing-decidable.
- Proof:
  - Assume R decides Reg_{TM}, design S to decide Acc_{TM}.
  - S: On input <M,w>:
    - Step 1: Construct a new machine < $M'_{M,w}$ > that accepts a regular language if and only if M accepts w.
      - $M'_{M,w}$: On input x:
        - If x is of the form 0^n1^n, then accept.
        - If x is not of this form, then run M on w and accept if M accepts.
    - Step 2: Run R on input < $M'_{M,w}$ >, and accept/reject if R does.

Reg_{TM}

- Theorem 3: Reg_{TM} is not Turing-decidable.
- Proof:
  - S: On input <M,w>:
    - Step 1: Construct a new machine < $M'_{M,w}$ > that accepts a regular language if and only if M accepts w.
      - $M'_{M,w}$: On input x:
        - If x is of the form 0^n1^n, then accept.
        - If x is not of this form, then run M on w and accept if M accepts.
    - Step 2: Run R on input < $M'_{M,w}$ >, and accept/reject if R does.
  - If M accepts w, then $M'_{M,w}$ accepts everything, hence recognizes the regular language $\{0,1\}^*$.
  - If M does not accept w, then $M'_{M,w}$ accepts exactly the strings of the form 0^n1^n, which constitute a non-regular language.
  - Thus, M accepts w iff $M'_{M,w}$ recognizes a regular language.

And more questions

- Many more questions about what TMs compute can be proved undecidable using the same method.
- One more example: $EQ_{TM} = \{<M_1, M_2> \mid M_1$ and M_2 are basic TMs that recognize the same language $\}$
- Theorem 4: EQ_{TM} is not Turing-decidable.
- Proof:
  - Reduce $Empty_{TM}$ to EQ_{TM}.
  - Assume R is a TM that decides EQ_{TM}; design S to decide $Empty_{TM}$.
  - Define any particular TM $M_\varnothing$ with $L(M) = \varnothing$ (M accepts nothing).
  - S: On input <M>:
    - Run R on input $<M, M_\varnothing>$; accept/reject if R does.
  - R tells whether $<M, M_\varnothing> \in EQ_{TM,}$ that is, whether $L(M) = L(M_\varnothing) = \varnothing$.

An Undecidable Problem not involving Turing Machines

Post Correspondence Problem

- A simple string-matching problem.
- Given a finite set of “tile types”, e.g.:

$$\left\{ \begin{pmatrix} a \\ a\,b \end{pmatrix} \quad \begin{pmatrix} c\,a \\ a\,b \end{pmatrix} \quad \begin{pmatrix} b \\ c \end{pmatrix} \quad \begin{pmatrix} b\,d \\ d \end{pmatrix} \right\}$$

- Is there a nonempty finite sequence of tiles (allowing repetitions, and not necessarily using all the tile types) for which the concatenation of top strings = concatenation of bottom strings?
- Example: $\begin{pmatrix} a \\ a\,b \end{pmatrix} \begin{pmatrix} b\,d \\ d \end{pmatrix}$ or $\begin{pmatrix} a \\ a\,b \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} \begin{pmatrix} c\,a \\ a\,b \end{pmatrix} \begin{pmatrix} b\,d \\ d \end{pmatrix}$

- No limit on length, but must be finite.
- Call such a sequence a match, or correspondence.
- Post Correspondence Problem (PCP) =
{ < T > | T is a finite set of tile types that has a match }

Post Correspondence Problem

- Given a finite set of tile types, is there a nonempty finite sequence of tiles for which the concatenation of top strings = concatenation of bottom strings?
- Call sequence a match, or correspondence.
- Post Correspondence Problem (PCP) = { < T > | T is a finite set of tile types that has a match }.
- Theorem: PCP is undecidable.
- Proof:
  - Reduce Acc_{TM} to PCP.
  - Previous reductions involved reducing one question about TMs (usually Acc_{TM}) to another question about TMs.
  - Now we reduce TM acceptance to a question about matching strings.
  - Do this by encoding TM computations using strings…

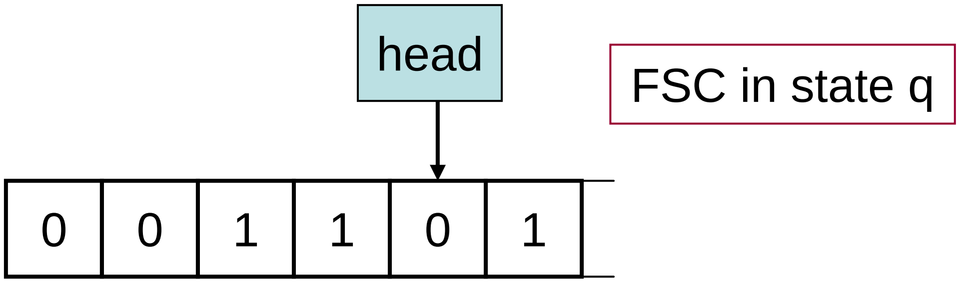
Computation Histories

Computation Histories

- Computation History (CH): A formal, stylized way of representing the computation of a TM on a particular input.
- Configuration:
  - Instantaneous snapshot of the TM's computation.
  - Includes current state, current tape contents, current head position.
  - Write in standard form: $w_1\ q\ w_2$, where w_1 and w_2 are strings of tape symbols and q is a state.
  - Meaning:
    - $w_1\ w_2$ is the string on the non-blank portion of the tape, perhaps part of the blank portion (rest assumed blank).
    - w_1 is the portion of the string strictly to the left of the head.
    - w_2 is the portion directly under the head and to the right.
    - q is the current state.

Configurations

- Configuration:
  - w_1 q w_2, where w_1 and w_2 are strings of tape symbols and q is a state.
  - Meaning:
    - w_1 w_2 is the string on the non-blank portion of the tape, perhaps part of the blank portion (rest assumed blank).
    - w_1 is the portion of the string strictly to the left of the head.
    - w_2 is the portion directly under the head and to the right.
    - q is the current state.
- Example: 0011q01 represents TM configuration:



Computation Histories

- TM begins in a starting configuration, of the form q_0 w, where w is the input string, and moves through a series of configurations, following the transition function.
- Computation History of TM M on input w:
  - A (finite or infinite) sequence of configs C_1, C_2, C_3, …, C_k,…, where
    - C_1, C_2, … are configurations of M.
    - C_1 is the starting configuration with input w.
    - Each C_{i+1} follows from C_i using M's transition function.
- Accepting CH: Finite CH ending in accepting configuration.
- Rejecting CH: Finite CH ending in rejecting configuration.
- Represent CH as a string # C_1 # C_2 # … # C_k #, where # is a special separator symbol.
- Claim: M accepts w iff there is an accepting CH of M on w.

Undecidability of PCP: First Attempt

First attempt

- Theorem: PCP is undecidable.
- Proof attempt:
  - Reduce Acc_{TM} to PCP, that is, show that, if we can decide PCP, then we can decide Acc_{TM}.
  - Given <M,w>, construct a finite set $T_{M,w}$ of tile types such that $T_{M,w}$ has a match iff M accepts w.
  - That is, $T_{M,w}$ has a match iff there is an accepting CH of M on w.
  - Write the accepting CH twice:

$$\# \; C_1 \; \# \; C_2 \; \# \; C_3 \; \# \dots \# \; C_k \; \#$$
$$\# \; C_1 \; \# \; C_2 \; \# \; C_3 \; \# \dots \# \; C_k \; \#$$

  - Split along boundaries of successive configurations:

$$\# \; C_1 \; \# \; C_2 \; \# \; C_3 \; \# \dots \# \; C_k \; \#$$
$$\# \; C_1 \; \# \; C_2 \; \# \; C_3 \; \# \dots \# \; C_k \; \#$$

First attempt

- Given <M,w>, construct a finite set $T_{M,w}$ of tile types s.t. $T_{M,w}$ has a match iff there is an accepting CH of M on w.
- Write the accepting CH twice, and split along boundaries of successive configurations:

$$\left|\begin{array}{c} \# \; C_1 \; \# \; C_2 \; \# \; C_3 \; \# \; \ldots \; \# \; C_k \; \# \\ \# \; C_1 \; \# \; C_2 \; \# \; C_3 \; \# \; \ldots \; \# \; C_k \; \# \end{array}\right|$$

  – What tiles do we need?

  – Try $T_{M,w} = \left\{ \begin{bmatrix} \# \\ \# \, C_1 \end{bmatrix}, \begin{bmatrix} C_k \, \# \\ \# \end{bmatrix}, \begin{bmatrix} C_i \;\; \# \\ \# \;\; C_j \end{bmatrix} \right\}$ where

  - C_1 = starting configuration for M on w,
  - C_k = accepting configuration (can assume unique, because we can assume accepting machine cleans up its tape).
  - C_j follows from C_i by rules of M (one step).

First attempt

– $T_{M,w} = \left\{ \begin{bmatrix} \# \\ \# \; C_1 \end{bmatrix}, \begin{bmatrix} C_k \; \# \\ \# \end{bmatrix}, \begin{bmatrix} C_i \;\; \# \\ \# \;\; C_j \end{bmatrix} \right\}$

  - C_1 = starting configuration for M on w,
  - C_k = accepting configuration.
  - C_j follows from C_i by rules of M (one step).

- M accepts w iff $T_{M,w}$ has a match.
- But there is a problem:
  – $T_{M,w}$ has infinitely many tile types $T_{M,w}$, because M has infinitely many configurations.
  – Configuration has tape contents, state, head position---infinitely many possibilities.
  – Of course, in any particular accepting computation, only finitely many configurations appear.
  – But we don't know what these are ahead of time.
  – So we can't pick a single finite set of tiles.

First attempt

- M accepts w iff $T_{M,w}$ has a match.
- But:
  - $T_{M,w}$ has infinitely many tile types $T_{M,w}$, because M has infinitely many configurations.
  - In any particular accepting computation, only finitely many configurations appear.
  - But we can't pick a single finite set of tiles for all computations.
- New insight:
  - Represent infinitely many configurations with finitely many tiles.
  - Going from one configuration to the next involves changing only a few "local" things:
    - State
    - Contents of one tape cell
    - Position of head, by at most 1
  - So let tiles represent small pieces of configs, not entire configs.

Undecidability of Modified PCP

Undecidability of Modified PCP

- Modified PCP (MPCP): Like PCP, but we're given not just a finite set of tiles, but also a designated tile that must start the match.
- MPCP = { <T, t > | T is a finite set of tiles, t is a tile in T, and there is a match for T starting with t }.
- Theorem: MPCP is undecidable.
- Later, we remove the requirement to start with t:
- Theorem: PCP is undecidable.
- Proof:
  - By reducing MPCP to PCP.
  - If PCP were decidable, MPCP would be also, contradiction.

MPCP is undecidable

- Reduce Acc_{TM} to MPCP.
- Given <M,w>, construct $(T_{M,w}, t_{M,w})$, an instance of MPCP.
- 7 kinds of tiles:
- Type 1 tile: $\left(\dfrac{\#}{\#\ q_0\ w_1\ w_2\ \ldots\ w_n\ \#} \right)$
  - $w = w_1\ w_2\ \ldots\ w_n$
  - $q_0\ w_1\ w_2\ \ldots\ w_n$ is the starting configuration for input w.
  - Bottom string is long, but there's only one tile like this.
  - Tile depends on w, which is OK.
  - Make this the initial tile $t_{M,w}$.

MPCP is undecidable

- Now consider how M goes from one configuration to the next.
- E.g., by moving right: $\delta(q,a) = (q',b,R)$.
- Config changes using this transition look like (e.g.):
  - $w_1\ w_2\ q\ a\ w_3 \rightarrow w_1\ w_2\ b\ q'\ w_3$.
  - Only change is to replace q a by b q′.
- Type 2 tiles:
  - For each transition of the form $\delta(q,a) = (q',b,R)$:

$$\left(\begin{array}{c} q\ a \\ b\ q' \end{array}\right)$$

MPCP is undecidable

- E.g., moving left: $\delta(q,a) = (q',b,L)$.
- Type 3 tile:
  - For each transition of the form $\delta(q,a) = (q',b,L)$, and every symbol c in the tape alphabet Γ:

$$\left[\begin{array}{c} c\ q\ a \\ q'\ c\ b \end{array}\right]$$

  - Include arbitrary c because it could be anything.

- Notice, only finitely many tiles (so far).

MPCP is undecidable

- Now, to match unchanged portions of 2 consecutive configurations:
- Type 4 tile:
  - For every symbol a in the tape alphabet Γ:

$$\begin{pmatrix} a \\ a \end{pmatrix}$$

- Still only finitely many tiles.

MPCP is undecidable

- What can we do with the tiles we have so far?
- Example: Partial match
  - Suppose the starting configuration is q_0 1 1 0 and the first move is $(q_0, 1) \rightarrow (q_4, 0, R)$.
  - Then the next configuration is 0 q_4 1 0.
  - We can start the match with tile 1: $\left[\dfrac{\#}{\#\, q_0\, 110\, \#}\right]$
  - Continue with type 2 tile: $\left[\dfrac{q_0\ 1}{0\ \ q_4}\right]$
  - Use type 4 tiles for the 2 unchanged symbols: $\left[\dfrac{1}{1}\right]$ $\left[\dfrac{0}{0}\right]$
  - Yields: $\#\ |\ q_0\ 1\ |\ 1\ |\ 0\ |\ \#$

$$\#\ q_0\ 1\ 1\ 0\ \#\ |\ 0\ q_4\ |\ 1\ |\ 0\ |\ \#$$

MPCP is undecidable

- Now we put in the separators.
- Type 5 tiles:

$$\begin{pmatrix} \# \\ \# \end{pmatrix} \qquad \begin{pmatrix} \# \\ \text{--}\ \# \end{pmatrix}$$

Allows us to add extra spaces at right end as needed---lets the configuration size grow.

- Example: Extend previous match:

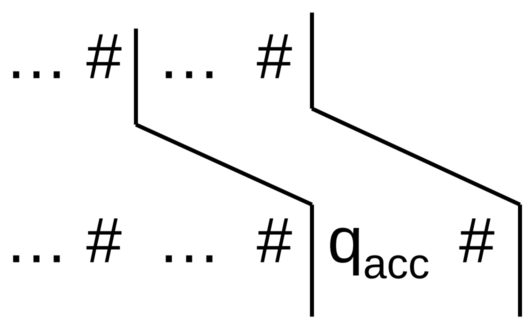
$\#\ q_0\ 1\ 1\ 0\ \#$

$\#\ q_0\ 1\ 1\ 0\ \#\ 0\ q_4\ 1\ 0\ \#$

MPCP is undecidable

- How does this end?
- Type 6 tiles:
  - For every a in Γ:
  - A trick…
  - Adds “pseudo-steps” to the end of the computation, where the state “eats” adjacent symbols in the top row.
  - Yields one symbol less in each successive bottom configuration.
  - Do this until the remaining bottom “configuration” is q_{acc} #:

$$\left[\begin{array}{c} a\ q_{acc} \\ q_{acc} \end{array}\right] \quad \left[\begin{array}{c} q_{acc}\ a \\ q_{acc} \end{array}\right]$$



MPCP is undecidable

- To finish off:
- Type 7 tile:

$$\left[\frac{q_{acc}\#\#}{\#}\right]$$

- That completes the definition of $T_{M,w}$ and $t_{M,w}$.
- Note that $T_{M,w}$, for a given M and w, is a finite set of tiles.

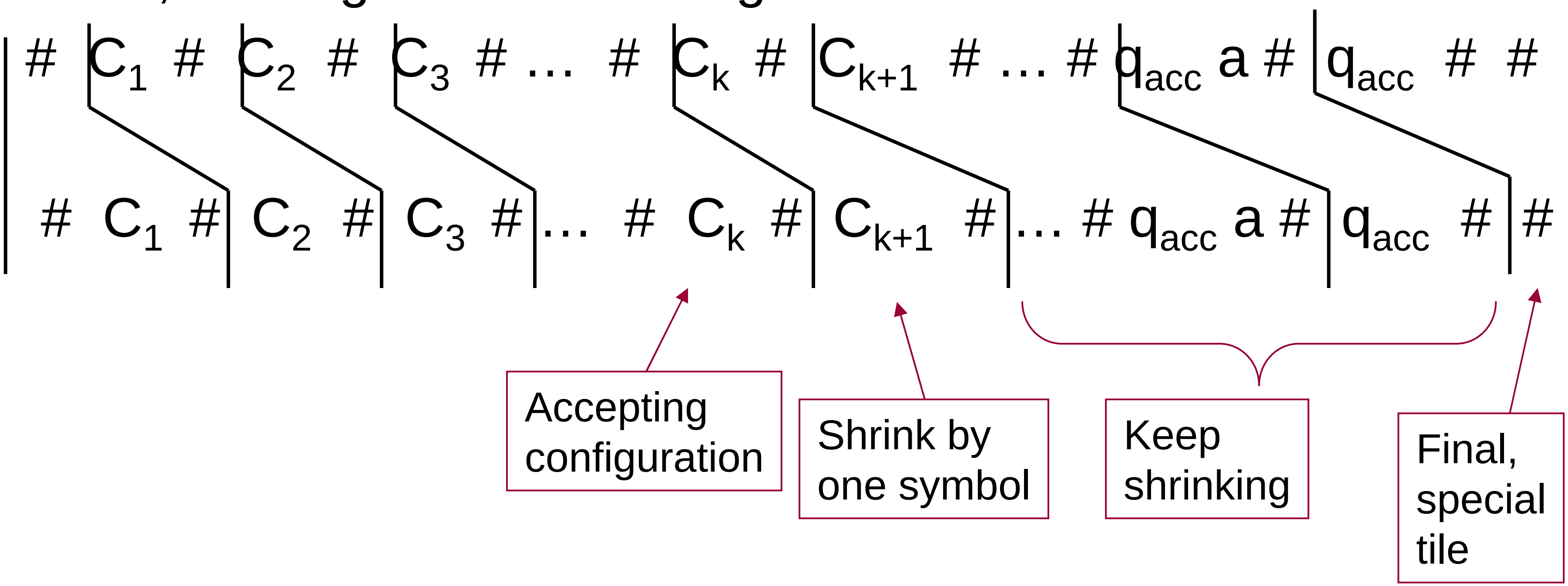
MPCP is undecidable

- That completes the definition of $T_{M,w}$ and $t_{M,w}$.
- Note that $T_{M,w}$, for a given M and w, is a finite set of tiles.
- Why does this work?
- Must show:
  - If M accepts w, then $T_{M,w}$ has a match beginning with $t_{M,w}$, that is, $<T_{M,w,}\ t_{M,w}> \in$ MPCP.
  - If $<T_{M,w,}\ t_{M,w}> \in$ MPCP, then M accepts w.

- If M accepts w, then there is an accepting computation history, which can be described by a match using the given tiles, starting from the distinguished initial tile:

MPCP is undecidable

- If M accepts w, then there is an accepting computation history, which can be described by a match using the given tiles, starting from the distinguished initial tile:



- So $T_{M,w}$ has a match beginning with $t_{M,w}$, that is, $<T_{M,w}, t_{M,w}> \in$ MPCP.

MPCP is undecidable

- If $<T_{M,w}, t_{M,w}> \in$ MPCP, that is, if $T_{M,w}$ has a match beginning with the designated tile $t_{M,w}$, then M accepts w.
- The rules are designed so the only way we can get a match beginning with the designated tile:

$$\left(\begin{array}{c} \# \\ \# \; q_0 \; w_1 \; w_2 \; \ldots \; w_n \; \# \end{array}\right)$$

is to have an actual accepting computation of M on w. Hand-wave, in the book, LTTR.

- Combining the two directions, we get:

M accepts w iff $<T_{M,w}, t_{M,w}> \in$ MPCP, that is,

$<M, w> \in Acc_{TM}$ iff $<T_{M,w}, t_{M,w}> \in$ MPCP.

MPCP is undecidable

- $<M, w> \in Acc_{TM}$ iff $<T_{M,w}, t_{M,w}> \in MPCP$.

- Theorem: MPCP is undecidable.
- Proof:
  - By contradiction.
  - Assume MPCP is decidable, and decide Acc_{TM}, using S:
  - S: On input $<M, w>$:
    - Step 1: Construct $<T_{M,w}, t_{M,w}>$, instance of MPCP, as described.
    - Step 2: Use MPCP to decide if $T_{M,w}$ has a match beginning with $t_{M,w}$. If so, accept; if not, reject.
  - Thus, if MPCP is decidable, then also Acc_{TM} is decidable, contradiction.

Undecidability of (Unmodified) PCP

Undecidability of PCP

- We showed that MPCP, in which the input is a set of tiles + designated input tile, is undecidable, by reducing Acc_{TM} to MPCP.
- Now we want:
- Theorem: PCP is undecidable.
- Why doesn't our construction reduce Acc_{TM} to PCP?
- $T_{M,v}$ has trivial matches, e.g., just

$$\begin{bmatrix} a \\ a \end{bmatrix}$$

- Proof of the theorem:
  - To show that PCP is undecidable, reduce MPCP to PCP, that is, show that if PCP is decidable, then so is MPCP.

Undecidability of PCP

- Theorem: PCP is undecidable.
- Proof:
  - Reduce MPCP to PCP.
  - To decide MPCP using PCP, suppose we are given:
    - T: $\left\{ \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \cdots\cdots \begin{bmatrix} u_k \\ v_k \end{bmatrix} \right\}$
    - t: $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$
  - We want to know if there is a match beginning with t.
  - Construct an instance T′ of ordinary PCP that has a match (starting with any tile) iff T has a match starting with t.

Undecidability of PCP

- Given T: $\left\{ \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \cdots\cdots \begin{pmatrix} u_k \\ v_k \end{pmatrix} \right\}$ t: $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$
- Construct an instance T′ of PCP that has a match iff T has a match starting with t.
- Construction (technical):
  - Add 2 new alphabet symbols, ♥ and ♦
  - If $u = u_1 u_2 \ldots u_n$ then define:
    - ♥ u = ♥ u_1 ♥ u_2 … ♥ u_n
    - u ♥ = u_1 ♥ u_2 … ♥ u_n ♥
    - ♥ u ♥ = ♥ u_1 ♥ u_2 … ♥ u_n ♥
  - Instance T′ of PCP:

$\left\{ \begin{pmatrix} ♥ u_1 \\ ♥ v_1 ♥ \end{pmatrix} \begin{pmatrix} ♥ u_1 \\ v_1 ♥ \end{pmatrix} \begin{pmatrix} ♥ u_2 \\ v_2 ♥ \end{pmatrix} \cdots\cdots \begin{pmatrix} ♥ u_k \\ v_k ♥ \end{pmatrix} \begin{pmatrix} ♥ ♦ \\ ♦ \end{pmatrix} \right\}$

Undecidability of PCP

- Claim: T has a match starting with t iff T′ has any match.

  $\Rightarrow$ Suppose T has a match starting with t: $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$

  Mimic this match with T′ tiles, starting with $\begin{pmatrix} \heartsuit\ u_1 \\ \heartsuit\ v_1\ \heartsuit \end{pmatrix}$

  and ending with $\begin{pmatrix} \heartsuit\ \diamondsuit \\ \diamondsuit \end{pmatrix}$

  Yields the same matching strings, with ♥s interspersed, and with ♦ at the end.

  $\Leftarrow$ If T′ has any match, it must begin with $\begin{pmatrix} \heartsuit\ u_1 \\ \heartsuit\ v_1\ \heartsuit \end{pmatrix}$ because that's the only tile in which top and bottom start with the same symbol.

  Other tiles are like T tiles but with extra ♥s.

  Stripping out ♥s yields match for T beginning with t.

Undecidability of PCP

- So, to decide MPCP using a decider for PCP:
- Given instance <T, t> for MPCP,
  - Step 1: Construct instance T′ for PCP, as above.
  - Step 2: Ask decider for PCP whether T′ has any match.
    - If so, answer yes for <T, t>.
    - If not, answer no.

- Since we already know MPCP is undecidable, so is PCP.

Next time…

- Mapping reducibility
- Rice's Theorem
- Reading:
  - Sipser Section 5.3, Problems 5.28-5.30.