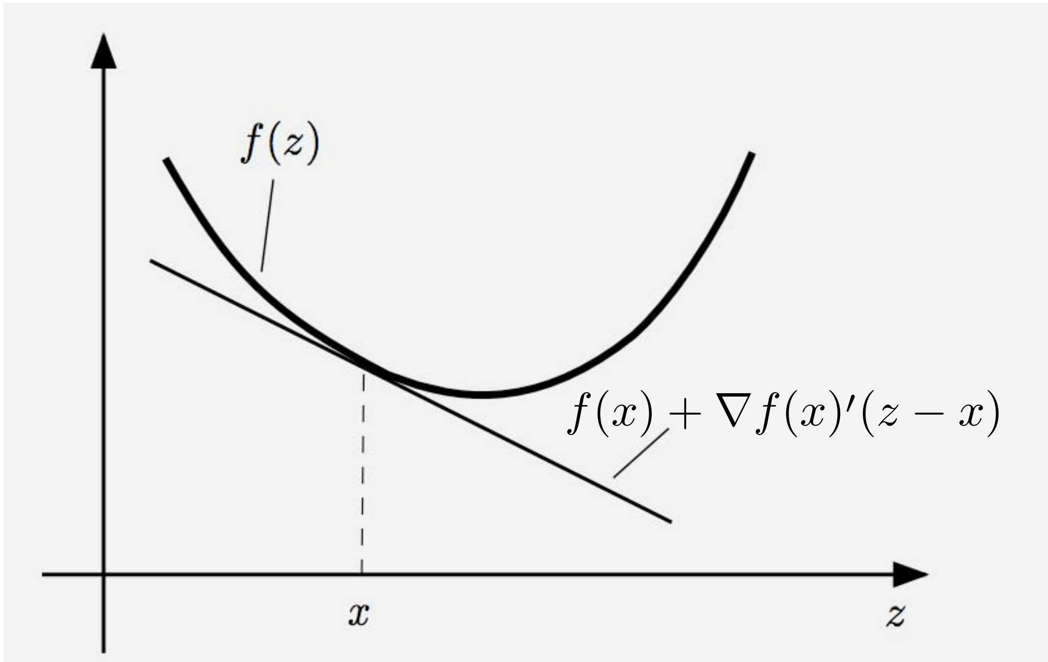
LECTURE 3

LECTURE OUTLINE

- Differentiable Convex Functions
- Convex and Affine Hulls
- Caratheodory's Theorem
- Relative Interior

DIFFERENTIABLE CONVEX FUNCTIONS



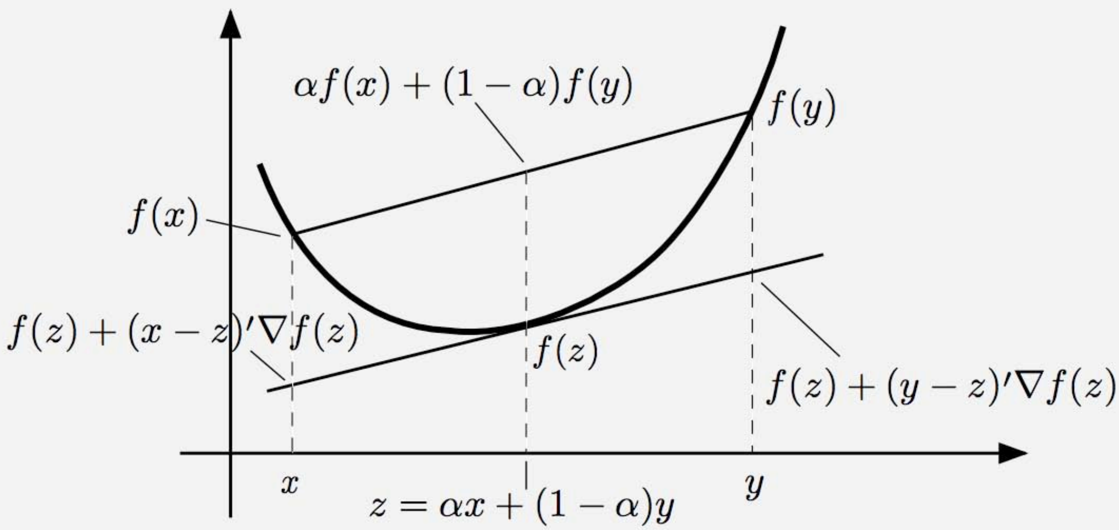
- Let $C \subset \Re^n$ be a convex set and let $f : \Re^n \mapsto \Re$ be differentiable over $\Re^n$.

(a) The function f is convex over C iff

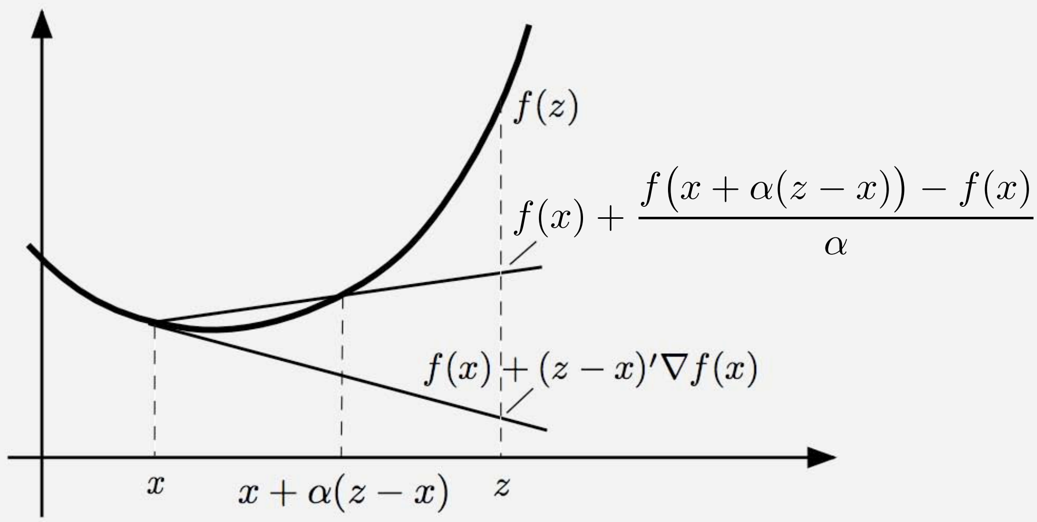
$$f(z) \geq f(x) + (z - x)'\nabla f(x), \qquad \forall\, x, z \in C$$

(b) If the inequality is strict whenever $x \neq z$, then f is strictly convex over C.

PROOF IDEAS



(a)



(b)

OPTIMALITY CONDITION

- Let C be a nonempty convex subset of $\Re^n$ and let $f : \Re^n \mapsto \Re$ be convex and differentiable over an open set that contains C. Then a vector $x^* \in C$ minimizes f over C if and only if

$$\nabla f(x^*)'(z - x^*) \geq 0, \qquad \forall\, z \in C.$$

Proof: If the condition holds, then

$$f(z) \geq f(x^*) + (z - x^*)'\nabla f(x^*) \geq f(x^*), \quad \forall\, z \in C,$$

so x^* minimizes f over C.

Converse: Assume the contrary, i.e., x^* minimizes f over C and $\nabla f(x^*)'(z - x^*) < 0$ for some $z \in C$. By differentiation, we have

$$\lim_{\alpha \downarrow 0} \frac{f\big(x^* + \alpha(z - x^*)\big) - f(x^*)}{\alpha} = \nabla f(x^*)'(z - x^*) < 0$$

so $f\big(x^* + \alpha(z - x^*)\big)$ decreases strictly for sufficiently small $\alpha > 0$, contradicting the optimality of x^*. **Q.E.D.**

TWICE DIFFERENTIABLE CONVEX FNS

- Let C be a convex subset of $\Re^n$ and let $f : \Re^n \mapsto \Re$ be twice continuously differentiable over $\Re^n$.

  (a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then f is convex over C.

  (b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex over C.

  (c) If C is open and f is convex over C, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.

Proof: (a) By mean value theorem, for $x, y \in C$

$$f(y) = f(x)+(y-x)'\nabla f(x)+\tfrac{1}{2}(y-x)'\nabla^2 f\big(x+\alpha(y-x)\big)(y-x)$$

for some $\alpha \in [0, 1]$. Using the positive semidefiniteness of $\nabla^2 f$, we obtain

$$f(y) \geq f(x) + (y - x)'\nabla f(x), \qquad \forall\, x, y \in C$$

From the preceding result, f is convex.

(b) Similar to (a), we have $f(y) > f(x) + (y - x)'\nabla f(x)$ for all $x, y \in C$ with $x \neq y$, and we use the preceding result.

(c) By contradiction ... similar.

CONVEX AND AFFINE HULLS

- Given a set $X \subset \Re^n$:

- A *convex combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where $x_i \in X$, $\alpha_i \geq 0$, and $\sum_{i=1}^m \alpha_i = 1$.

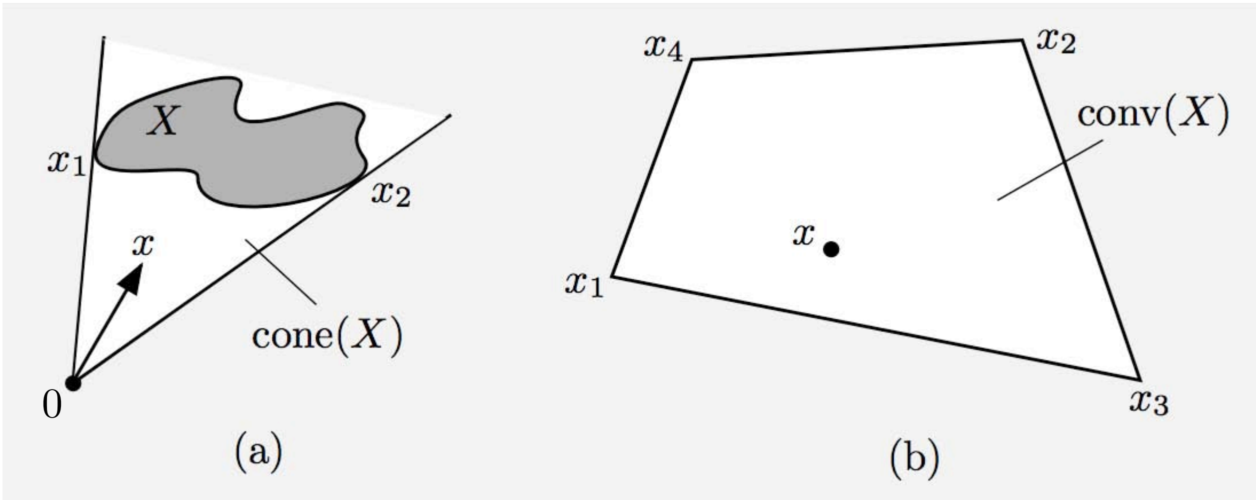
- The *convex hull* of X, denoted $\text{conv}(X)$, is the intersection of all convex sets containing X. (Can be shown to be equal to the set of all convex combinations from X).

- The *affine hull* of X, denoted $\text{aff}(X)$, is the intersection of all affine sets containing X (an affine set is a set of the form $x + S$, where S is a subspace).

- A *nonnegative combination* of elements of X is a vector of the form $\sum_{i=1}^m \alpha_i x_i$, where $x_i \in X$ and $\alpha_i \geq 0$ for all i.

- The *cone generated by* X, denoted $\text{cone}(X)$, is the set of all nonnegative combinations from X:
  - It is a convex cone containing the origin.
  - It need not be closed!
  - If X is a finite set, $\text{cone}(X)$ is closed (nontrivial to show!)

CARATHEODORY'S THEOREM



- Let X be a nonempty subset of $\Re^n$.
  - (a) Every $x \neq 0$ in cone(X) can be represented as a positive combination of vectors $x_1, \ldots, x_m$ from X that are linearly independent (so $m \leq n$).
  - (b) Every $x \notin X$ that belongs to conv(X) can be represented as a convex combination of vectors $x_1, \ldots, x_m$ from X with $m \leq n + 1$.

PROOF OF CARATHEODORY'S THEOREM

(a) Let x be a nonzero vector in $\text{cone}(X)$, and let m be the smallest integer such that x has the form $\sum_{i=1}^{m} \alpha_i x_i$, where $\alpha_i > 0$ and $x_i \in X$ for all $i = 1, \ldots, m$. If the vectors x_i were linearly dependent, there would exist $\lambda_1, \ldots, \lambda_m$, with

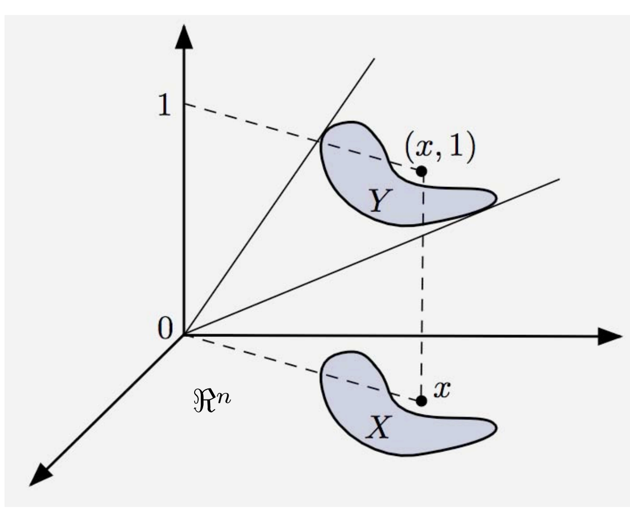
$$\sum_{i=1}^{m} \lambda_i x_i = 0$$

and at least one of the λ_i is positive. Consider

$$\sum_{i=1}^{m} (\alpha_i - \gamma \lambda_i) x_i,$$

where γ is the largest γ such that $\alpha_i - \gamma \lambda_i \geq 0$ for all i. This combination provides a representation of x as a positive combination of fewer than m vectors of X – a contradiction. Therefore, $x_1, \ldots, x_m$, are linearly independent.

(b) Use "lifting" argument: apply part (a) to $Y = \{(x, 1) \mid x \in X\}$.



AN APPLICATION OF CARATHEODORY

- The convex hull of a compact set is compact.

Proof: Let X be compact. We take a sequence in $\mathrm{conv}(X)$ and show that it has a convergent subsequence whose limit is in $\mathrm{conv}(X)$.

By Caratheodory, a sequence in $\mathrm{conv}(X)$ can be expressed as $\left\{\sum_{i=1}^{n+1} \alpha_i^k x_i^k\right\}$, where for all k and i, $\alpha_i^k \geq 0$, $x_i^k \in X$, and $\sum_{i=1}^{n+1} \alpha_i^k = 1$. Since the sequence

$$\left\{(\alpha_1^k, \ldots, \alpha_{n+1}^k, x_1^k, \ldots, x_{n+1}^k)\right\}$$

is bounded, it has a limit point

$$\left\{(\alpha_1, \ldots, \alpha_{n+1}, x_1, \ldots, x_{n+1})\right\},$$

which must satisfy $\sum_{i=1}^{n+1} \alpha_i = 1$, and $\alpha_i \geq 0$, $x_i \in X$ for all i.

The vector $\sum_{i=1}^{n+1} \alpha_i x_i$ belongs to $\mathrm{conv}(X)$ and is a limit point of $\left\{\sum_{i=1}^{n+1} \alpha_i^k x_i^k\right\}$, showing that $\mathrm{conv}(X)$ is compact. **Q.E.D.**

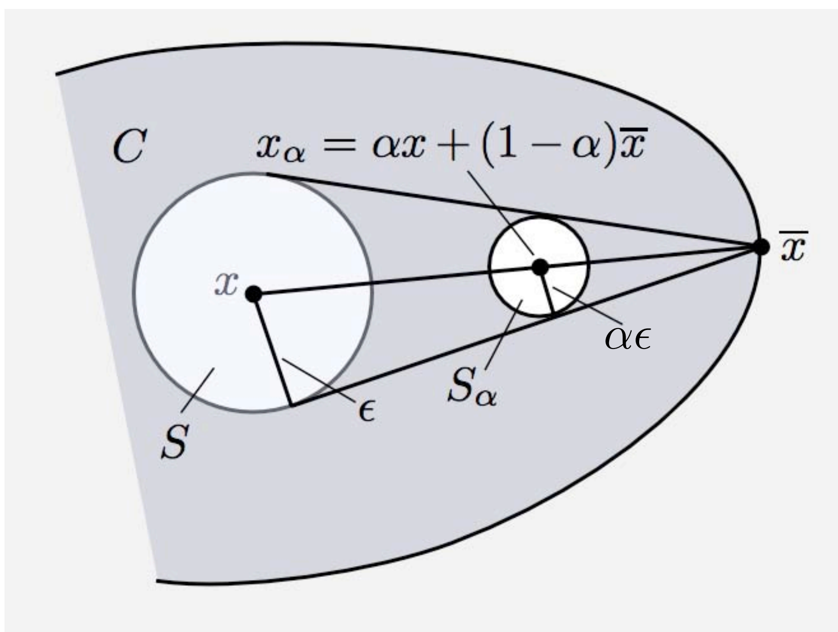
- Note that the convex hull of a closed set need not be closed!

RELATIVE INTERIOR

- x is a *relative interior point* of C, if x is an interior point of C relative to aff(C).

- ri(C) denotes the *relative interior of* C, i.e., the set of all relative interior points of C.

- **Line Segment Principle**: If C is a convex set, $x \in \text{ri}(C)$ and $x \in \text{cl}(C)$, then all points on the line segment connecting x and x, except possibly x, belong to ri(C).



- Proof of case where $x \in C$: See the figure.

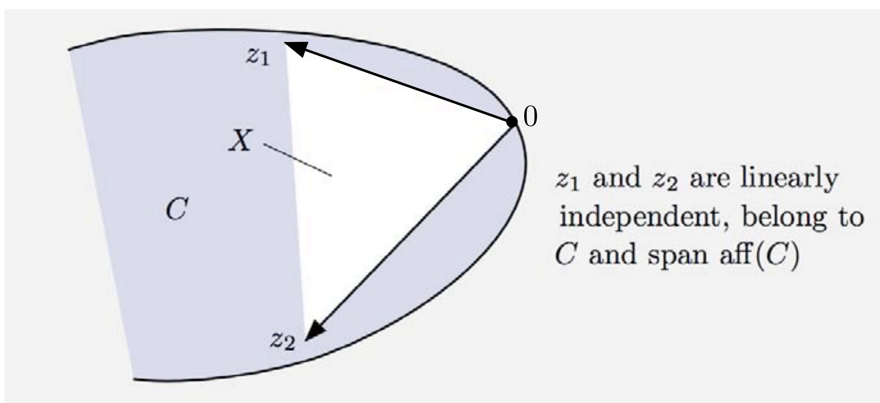
- Proof of case where $x \notin C$: Take sequence $\{x_k\} \subset C$ with $x_k \to x$. Argue as in the figure.

ADDITIONAL MAJOR RESULTS

- Let C be a nonempty convex set.

  (a) $\text{ri}(C)$ is a nonempty convex set, and has the same affine hull as C.

  (b) **Prolongation Lemma:** $x \in \text{ri}(C)$ if and only if every line segment in C having x as one endpoint can be prolonged beyond x without leaving C.



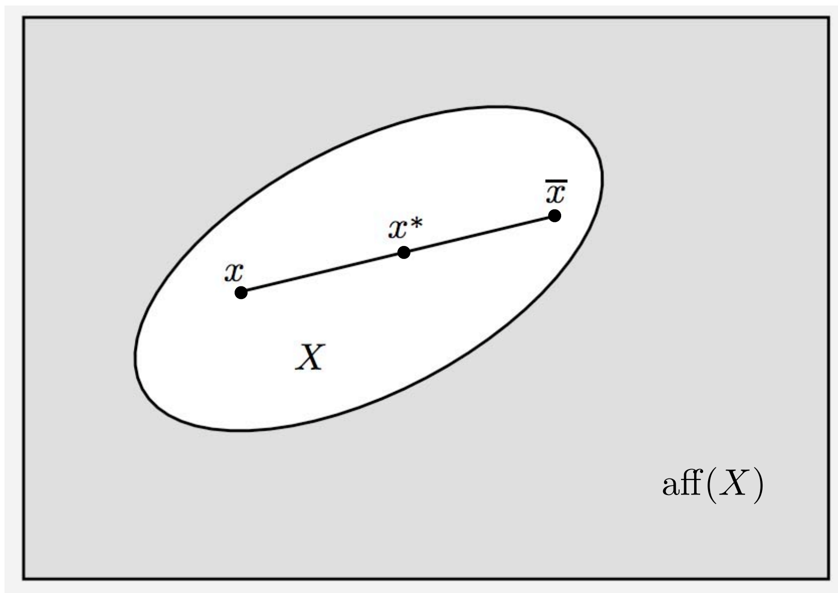
Proof: (a) Assume that $0 \in C$. We choose m linearly independent vectors $z_1, \ldots, z_m \in C$, where m is the dimension of $\text{aff}(C)$, and we let

$$X = \left\{ \sum_{i=1}^{m} \alpha_i z_i \;\middle|\; \sum_{i=1}^{m} \alpha_i < 1,\ \alpha_i > 0,\ i = 1, \ldots, m \right\}$$

(b) => is clear by the def. of rel. interior. Reverse: take any $x \in \text{ri}(C)$; use Line Segment Principle.

OPTIMIZATION APPLICATION

- A concave function $f : \Re^n \mapsto \Re$ that attains its minimum over a convex set X at an $x^* \in \text{ri}(X)$ must be constant over X.



Proof: (By contradiction) Let $x \in X$ be such that $f(x) > f(x^*)$. Prolong beyond x^* the line segment x-to-x^* to a point $x \in X$. By concavity of f, we have for some $\alpha \in (0, 1)$

$$f(x^*) \geq \alpha f(x) + (1 - \alpha) f(x),$$

and since $f(x) > f(x^*)$, we must have $f(x^*) > f(x)$ - a contradiction. **Q.E.D.**

- **Corollary:** A linear function can attain a minimum only at the boundary of a convex set.

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