Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

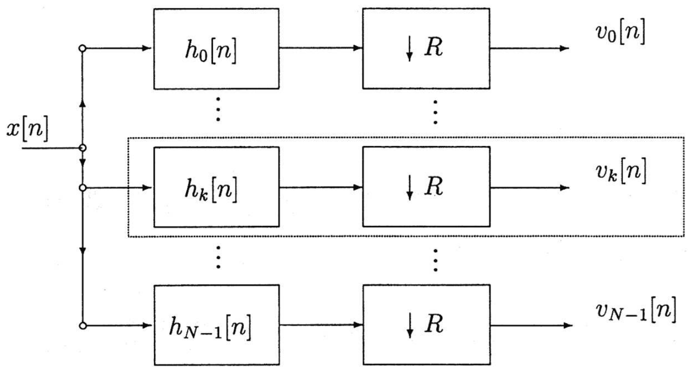
OpenCourseWare 2006

Lecture 22
Modulated Filter Bank

Reading: Review Section 4.7 of OSB; also carefully study the figures in this handout.

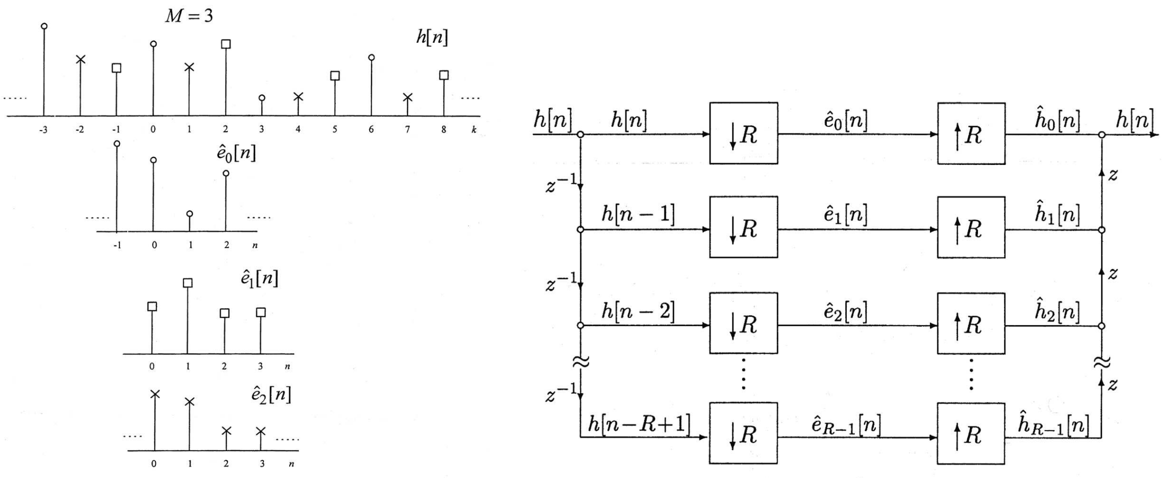
In this lecture, we will tie together some relationships between modulated filter banks (MFB) for time-dependent Fourier transform (TDFT), polyphase structure, and performing the discrete Fourier transform(DFT) through the fast Fourier transform(FFT).

Similarity between the general form of a branch of the MFB and the decimation filter suggests that a polyphase implementation of the MFB is possible. As a first step, consider a single filter-downsampler pair in the parallel structure:



A typical modulated filter bank.

Recall that a possible polyphase decomposition of an impulse response $h[n]$ is to downsample successively advanced (rather than delayed) versions of it:

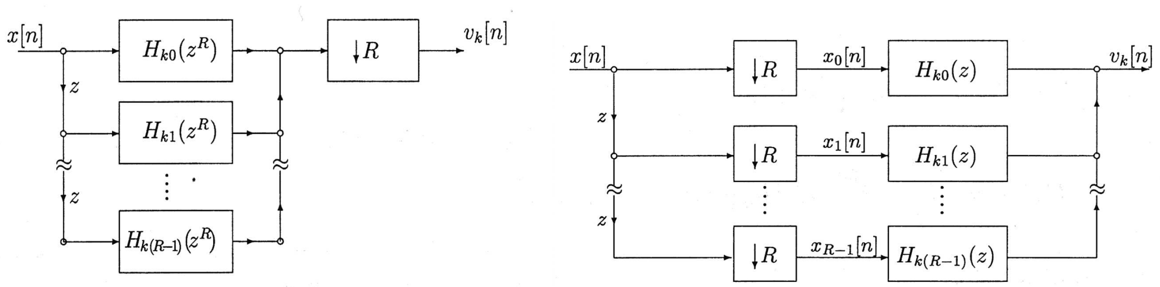


Polyphase decomposition of $h[n]$

Written analytically, such a decomposition applied to the prototype filter $H_0(z)$ and the modulated filters $H_k(z)$ is:

$$
\begin{aligned}
h_0[n] \Leftrightarrow H_0(z) &= H_{00}(z^R) + zH_{01}(z^R) + \ldots + z^{(R-1)}H_{0(R-1)}(z^R) \\
&= \sum_{p=0}^{R-1} z^p H_{0p}(z^R) \\
h_k[n] = e^{j\omega_k n} h_0[n] \Leftrightarrow H_k(z) = H_0(e^{j\frac{2\pi k}{N}} z) &= \sum_{p=0}^{R-1} e^{-j\frac{2\pi kp}{N}} z^p H_{0p}(e^{-j\frac{2\pi kR}{N}} z^R) \\
&= \sum_{p=0}^{R-1} z^p H_{kp}(z^R)\ ,
\end{aligned}
$$

where $H_{0p}(z)$ is the p-th polyphase component of $H_0(z)$ and $H_{kp}(z^R) = e^{-j\frac{2\pi kp}{N}} H_{0p}(e^{-j\frac{2\pi kR}{N}} z^R)$ is the p-th polyphase component of $H_k(z)$. This decomposition suggests implementing the k-th filter of the MFB with the following polyphase structure:

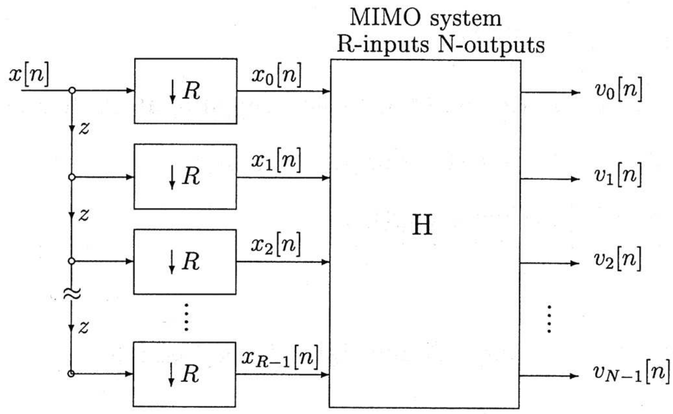


Polyphase implementation of the k-th branch of the MFB

The output $V_k(z)$ of the k-th branch is a linear combination of intermediate signals $x_k[n] \Leftrightarrow X_k(z)$,$0 \le k \le R-1$:

$$V_k(z) = \sum_{p=0}^{R-1} X_p(z) H_{kp}(z) \ , \qquad H_{kp}(z) = e^{-j\frac{2\pi kp}{N}} H_{0p}(e^{-j\frac{2\pi k}{N}} z) \ .$$

Observe that $x_0[n], x_1[n], \ldots, x_{R-1}[n]$ are the polyphase components of the input sequence $x[n]$. Because they are common to each parallel branch of the MFB, we can share them at the input to obtain the following amalgamated structure:



Polyphase implementation of the MFB

In this system:

$$\text{No. of filter bank channels } = N, \qquad \frac{\text{Filter bank output rate}}{\text{Filter bank input rate}} = \frac{N}{R} \ .$$

The transfer function matrix H satisfies

$$N \left\{ \begin{bmatrix} \\ & H_{kp}(z) & \\ \\ \end{bmatrix} \right. \begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{R-1}(z) \end{bmatrix} = \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{N-1}(z) \end{bmatrix}$$

$$H_{kp}(z) = e^{-j\frac{2\pi kp}{N}} H_{0p}(e^{-j\frac{2\pi k}{N}} z) \ .$$

Such a polyphase implementation is preferred over the direct implementation of the MFB because it is more efficient in terms of computational complexity. To see why this is true, assume our system operates with complex arithmetic. In the original system:

$$\left.\begin{array}{l}\text{Input } x[n] \text{ clocked at 1 sample/unit time} \\ \text{Each modulated filter } h_k[n] \text{ is of length } L \\ N \text{ modulated filters } h_k[n] \text{ in the MFB}\end{array}\right\} \Rightarrow \begin{array}{l}\text{total of } LN \\ \text{complex multiplies/input sample}\end{array}$$

In the polyphase implementation:

$$\left.\begin{array}{l}\text{Input subsequence } x_k[n] \text{ clocked at } \frac{1}{R} \text{ sample/unit time} \\ \text{Each polyphase component filter } h_{kp}[n] \text{ is of length } \frac{L}{R} \\ R \text{ polyphase components for each modulated filter } h_k[n] \\ N \text{ modulated filters } h_k[n] \text{ in the MFB}\end{array}\right\} \Rightarrow \begin{array}{l}(\frac{1}{R})(\frac{L}{R})(R)(N) = \frac{LN}{R} \\ \text{multiplies/input sample}\end{array}$$

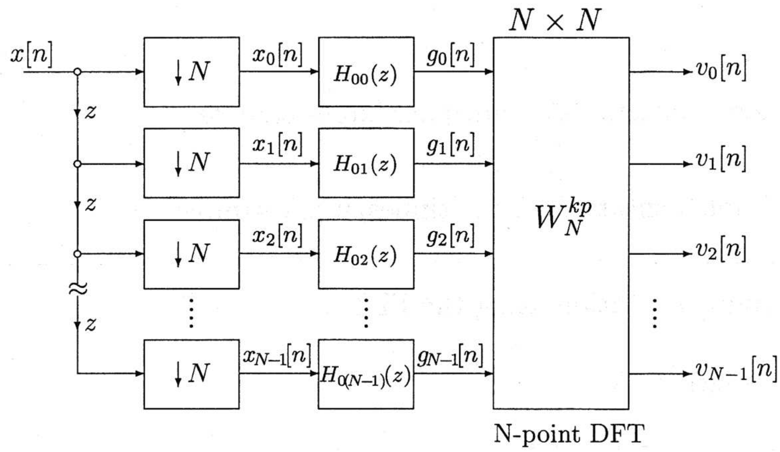
A polyphase structure hence gives $\frac{1}{R}$ reduction in the required number of multiplications. To take one step further, let's consider in more detail the R-input, N-output MIMO structure on the previous page. If we let $R = N$, each polyphase component filter simplifies to

$$H_{kp}(z) = e^{-j\frac{2\pi kp}{N}} H_{0p}(z) , \qquad 0 \le k \le N-1 , \quad 0 \le p \le R-1 = N-1 .$$

Such a system is said to be critically sampled. Its outputs are

$$V_k(z) = \sum_{p=0}^{N-1} X_p(z) H_{0p}(z) e^{-j\frac{2\pi kp}{N}} = \sum_{p=0}^{N-1} \underbrace{X_p(z) H_{0p}(z)}_{G_p(z) \Leftrightarrow g_p[n]} W_N^{kp} , \qquad 0 \le k \le N-1 .$$

Looks familiar? This is the N-point DFT of $g_p[n]$. It gives rise to the following structure:



Polyphase implementation of the MFB using DFT/FFT

$$v_k[n] = \sum_{p=0}^{N-1} g_p[n] W_N^{kp} = N\text{-point DFT of } g_p[n] \text{ at each } n$$

The N-point DFT in this system can be efficiently computed using the FFT. With both the polyphase structure and the FFT in place:

$$\left.\begin{array}{r} \text{Input } x_k[n] \text{ clocked at } \frac{1}{N} \text{ sample/unit time} \\ \text{Each polyphase component filter } h_{kp}[n] \text{ is of length } \frac{L}{N} \\ N \text{ channels in the MFB} \end{array}\right\} \Rightarrow \begin{array}{l} \frac{L}{N} \text{ mul/input sample} \\ \text{to compute } g_p[n], \\ 0 \leq p \leq N-1 \end{array}$$

$$\left.\begin{array}{r} g_p[n], 0 \leq p \leq N-1, \text{ clocked at } \frac{1}{N} \text{ sample/unit time} \\ N \log_2 N \text{ multiplies for the FFT} \end{array}\right\} \Rightarrow \begin{array}{l} \log_2 N \text{ mul/input sample} \\ \text{for the FFT} \end{array}$$

This yields a total of $\frac{L}{N} + \log_2 N$ multiplications/input sample. Clearly the polyphase structure with FFT gives a large efficiency improvement in comparison to the direct implementation of MFB. Summarizing, the total number of multiplications required by each implementation is:

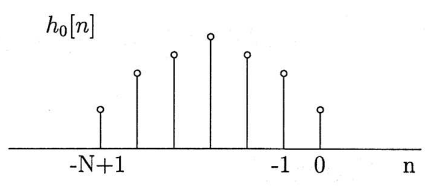
Direct implementation	(LN)	multiplies/input sample
Polyphase implementation	$(\frac{LN}{R})$	multiplies/input sample
Polyphase, $N = R$, FFT	$(\frac{L}{N} + \log_2 N)$	multiplies/input sample

As a conclusion to our discussion on the polyphase implementation of the MFB with FFT, here are two special cases of the system we have derived:

Special Case 1: $L = N = R$

- L length of the window, length of the prototype filter
- N desired number of modulated filters, sampling rate of the TDFT in the frequency domain
- R amount of shift by the window at each time step, down sampling rate of the TDFT in the time domain

Assume the prototype filter has the following impulse response:



Its polyphase decomposition is

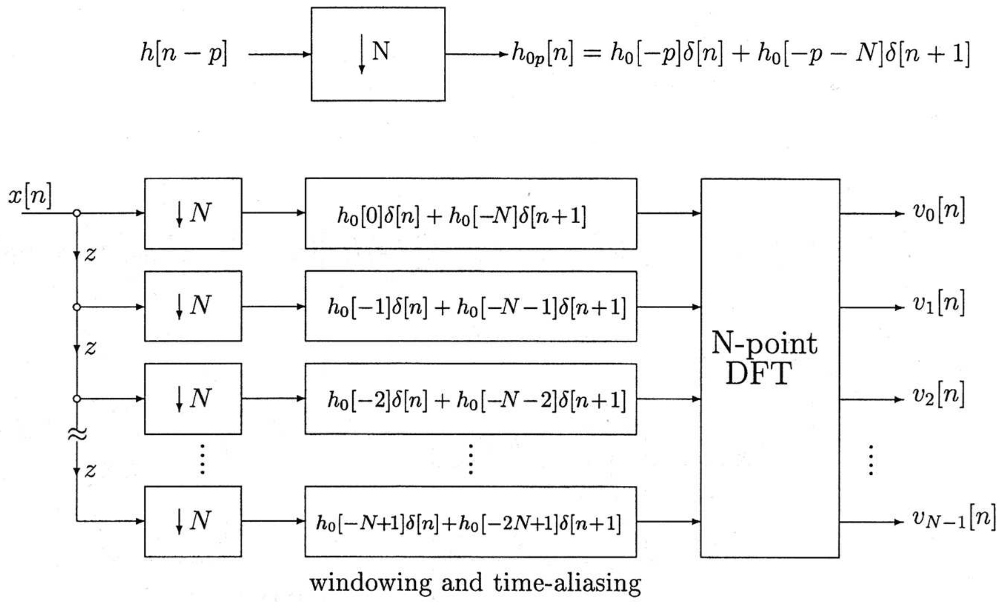
$$h_{00}[n] = h_0[0]\delta[n], \quad \ldots, \quad h_{0p}[n] = h_0[-p]\delta[n] .$$

The polyphase FFT implementation is then

Polyphase implementation of the MFB with FFT, $L = N = R$

Special Case 2: $L = 2N = 2R$

Here the polyphase components are 2-points long. Since $L = 2R$, time aliasing occurs when the N-point DFT is performed through FFT.



Polyphase implementation of the MFB with FFT, $L = 2N = 2R$