6.852: Distributed Algorithms
Fall, 2009

Class 7

Today's plan

- Asynchronous systems
- Formal model
  - I/O automata
  - Executions and traces
  - Operations: composition, hiding
  - Properties and proof methods:
    - Invariants
    - Simulation relations
- Reading: Chapter 8
- Next:
  - Asynchronous network algorithms: Leader election, breadth-first search, shortest paths, spanning trees.
  - Reading: Chapters 14 and 15

Last time

- Finished synchronous network algorithms:
  - Lower bounds on number of rounds
  - k-agreement
- Commit:
  - 2-phase commit:
    - Weak termination only.
  - 3-phase commit:
    - Strong termination.
    - But depends strongly on synchrony:
      - Coordinator deduces that all processes are ready or failed, just by waiting sufficiently long so it knows that its messages have arrived.

Practical issues for 3-phase commit

- Depends on strong assumptions, which may be hard to guarantee in practice:
  - Synchronous model:
    - Could emulate with approximately-synchronized clocks, timeouts.
  - Reliable message delivery:
    - Could emulate with acks and retransmissions.
    - But if retransmissions add too much delay, then we can't emulate the synchronous model accurately.
    - Leads to unbounded delays, asynchronous model.
  - Accurate diagnosis of process failures:
    - Get this "for free" in the synchronous model.
    - E.g., 3-phase commit algorithm lets process that doesn't hear from another process i at a round conclude that i must have failed.
    - Very hard to guarantee in practice: In Internet, or even a LAN, how to reliably distinguish failure of a process from lost communication?
- Other consensus algorithms can be used for commit, including some that don't depend on such strong timing and reliability assumptions.

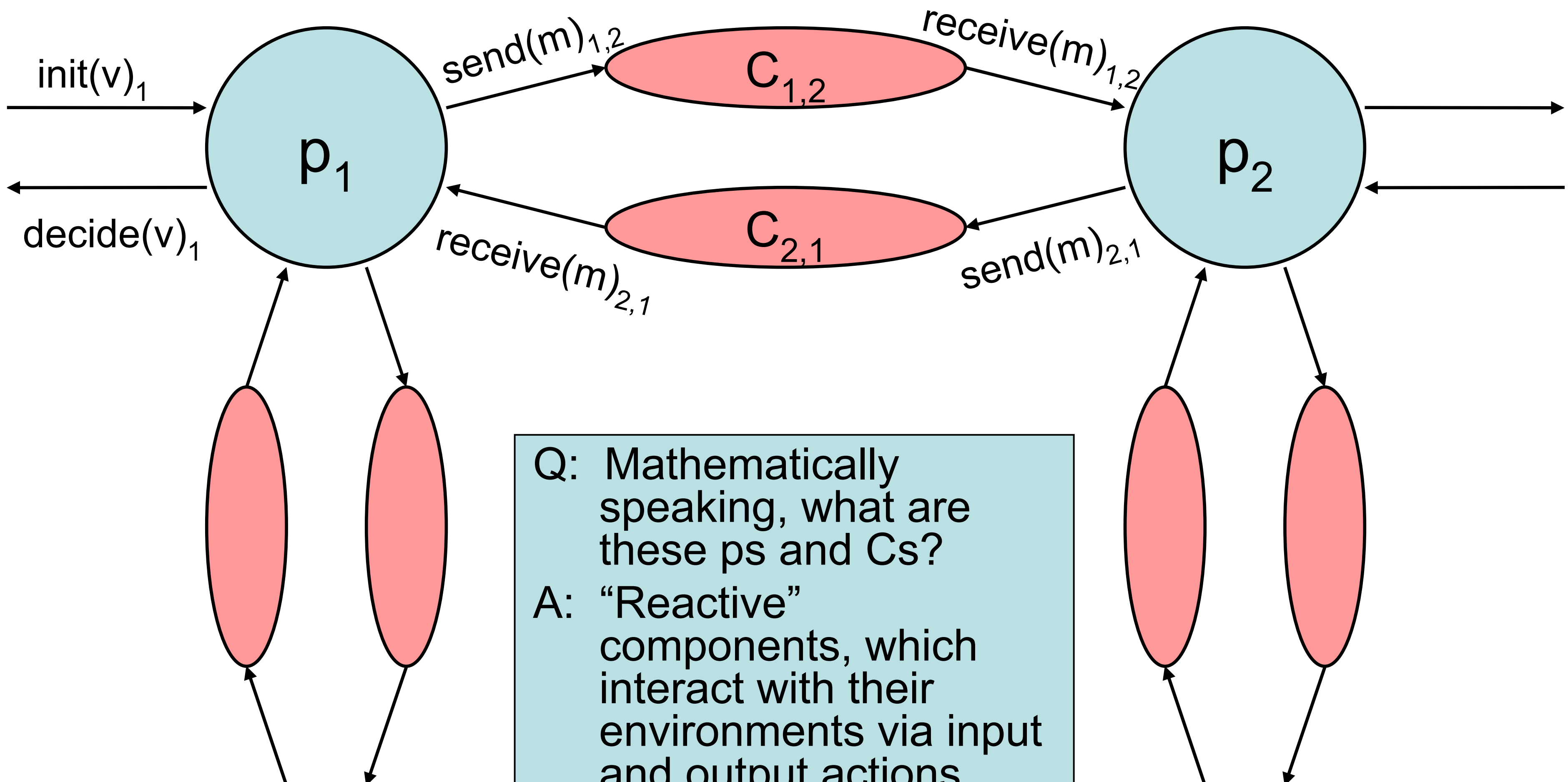
Paxos consensus algorithm

- A more robust consensus algorithm, could be used for commit.
- Tolerates process stopping and recovery, message losses and delays,…
- Runs in partially synchronous model.
- Based on earlier algorithm [Dwork, Lynch, Stockmeyer].
- Algorithm idea:
  - Processes use unreliable leader election subalgorithm to choose coordinator, who tries to achieve consensus.
  - Coordinator decides based on active support from majority of processes.
  - Does not assume anything based on not receiving a message.
  - Difficulties arise when multiple coordinators are active---must ensure consistency.
- Practical difficulties with fault-tolerance in the synchronous model motivate studying the asynchronous model.

Asynchronous systems

- No timing assumptions
  - No rounds
- Two kinds of asynchronous models:
  - Asynchronous networks
    - Processes communicating via channels
  - Asynchronous shared-memory systems
    - Processes communicating via shared objects

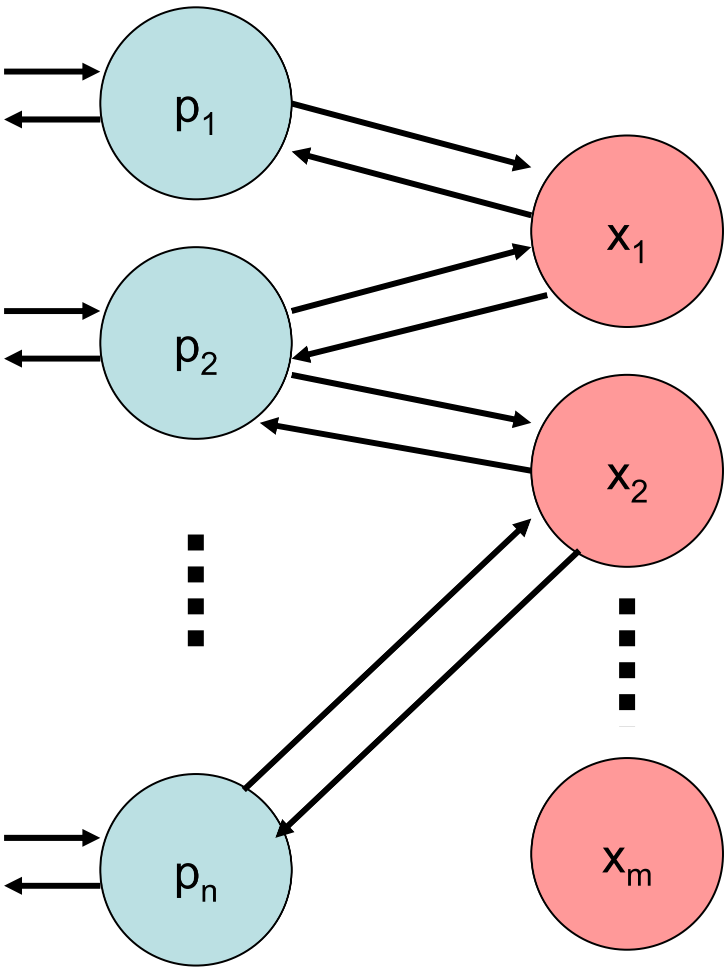
Asynchronous network: Processes and channels



Q: Mathematically speaking, what are these ps and Cs?

A: “Reactive” components, which interact with their environments via input and output actions.

Asynchronous shared-memory system: Processes and objects



These processes and objects are also "reactive" components.

In both cases, reactive components.

So, we give a general model for reactive components.

Specifying problems and systems

- Processes, channels, and objects are automata
  - Take actions while changing state.
  - Reactive
    - Interact with environment via input and output actions.
    - Not just functions from input values to output values, but more flexible interactions.
- Execution:
  - Sequence of actions
  - Interleaving semantics
- External behavior (trace):
  - We observe external actions.
  - State and internal actions are hidden.
  - Problems specify allowable traces.

I/O Automata

Input/Output Automata

- General mathematical modeling framework for reactive components.
  - Little structure---must add structure to specialize it for networks, shared-memory systems,…
- Designed for describing systems in a modular way:
  - Supports description of individual system components, and how they compose to yield a larger system.
  - Supports description of systems at different levels of abstraction, e.g.:
    - Detailed implementation vs. more abstract algorithm description.
    - Optimized algorithm vs. simpler, unoptimized version.
- Supports standard proof techniques:
  - Invariants
  - Simulation relations (like running 2 algorithms side-by-side and relating their behavior step-by-step).
  - Compositional reasoning (prove properties of individual components; use to infer properties for overall system).

Input/output automaton

- State transition system
  - Transitions labeled by actions
- Actions classified as input, output, internal
  - Input, output are external.
  - Output, internal are locally controlled.

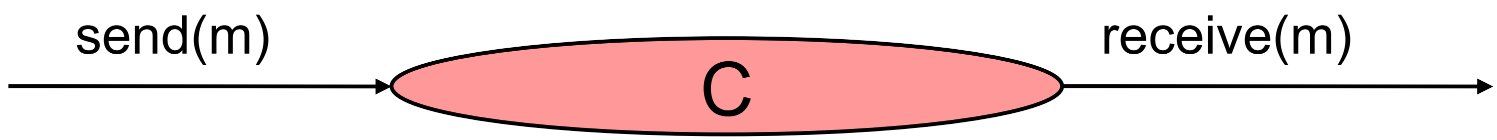
Input/output automaton

- sig = (in, out, int)
  - input, output, internal actions (disjoint)
  - acts = in $\cup$ out $\cup$ int
  - ext = in $\cup$ out
  - local = out $\cup$ int
- states: Not necessarily finite
- start $\subseteq$ states
- trans $\subseteq$ states $\times$ acts $\times$ states
  - Input-enabled: Any input “enabled” in any state.
- tasks, partition of locally controlled actions
  - Used for liveness.

Remarks

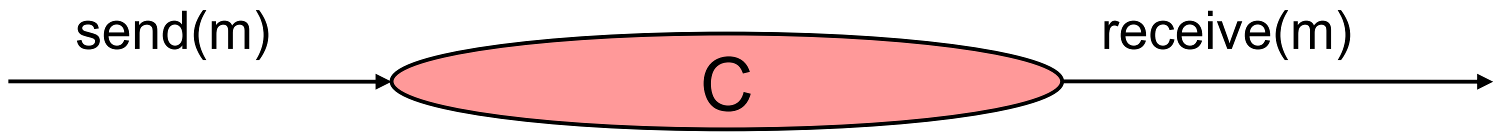
- A step of an automaton is an element of trans.
- Action π is enabled in a state s if there is a step (s, π, s') for some s'.
- I/O automata must be input-enabled.
  - Every input action is enabled in every state.
  - Captures idea that an automaton cannot control inputs.
    - If we want restrictions, model the environment as another automaton and express restrictions in terms of the environment.
    - Could allow a component to detect bad inputs and halt, or exhibit unconstrained behavior for bad inputs.
- Tasks correspond to “threads of control”.
  - Used to define fairness (give turns to all tasks).
  - Needed to guarantee liveness properties (e.g., the system keeps making progress, or eventually terminates).

Channel automaton



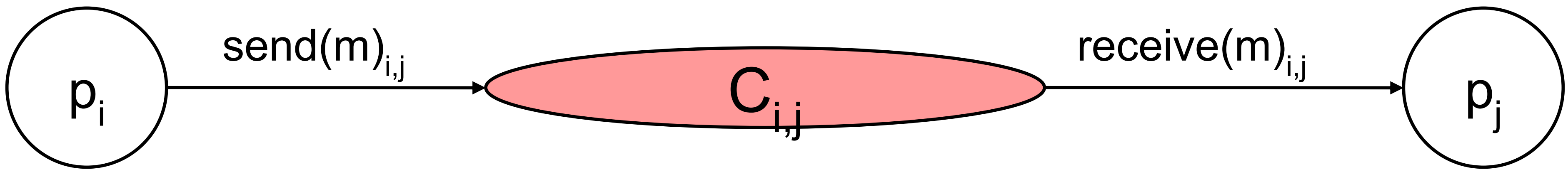
- Reliable unidirectional FIFO channel between two processes.
  - Fix message alphabet M.
- signature
  - input actions: send(m), $m \in M$
  - output actions: receive(m), $m \in M$
  - no internal actions
- states
  - queue: FIFO queue of M, initially empty

Channel automaton



- trans
  - send(m)
    - effect: add m to (end of) queue
  - receive(m)
    - precondition: m is at head of queue
    - effect: remove head of queue
- tasks
  - All receive actions in one task.

Channel automaton



- trans
  - $send(m)_{i,j}$
    - effect: add m to (end of) queue
  - $receive(m)_{i,j}$
    - precondition: m is at head of queue
    - effect: remove head of queue
- tasks
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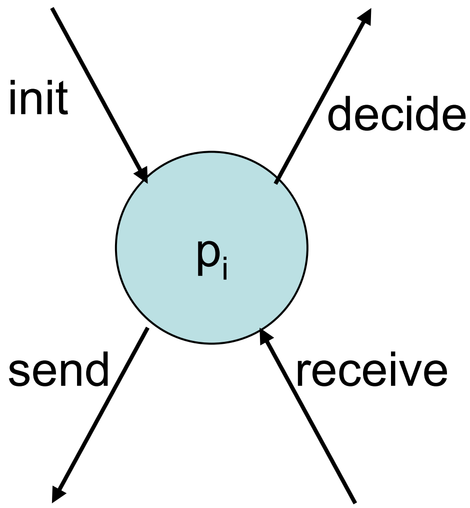
A process



- E.g., in a consensus protocol.
- See book, p. 205, for code details.
- Inputs arrive from the outside.
- Process sends/receives values, collects vector of values for all processes.
- When vector is filled, outputs a decision obtained as a function of the vector.
- Can get new inputs, change values, send and output repeatedly.
- Tasks for:
  - Sending to each individual neighbor.
  - Outputting decisions.

Executions

- An I/O automaton executes as follows:
  - Start at some start state.
  - Repeatedly take step from current state to new state.
- Formally, an execution is a finite or infinite sequence:
  - $s_0\ \pi_1\ s_1\ \pi_2\ s_2\ \pi_3\ s_3\ \pi_4\ s_4\ \pi_5\ s_5$ ... (if finite, ends in state)
  - s_0 is a start state
  - $(s_i, \pi_{i+1}, s_{i+1})$ is a step (i.e., in trans)

λ, send(a), a, send(b), ab, receive(a), b, receive(b), λ

Execution fragments

- An I/O automaton executes as follows:
  - Start at some start state.
  - Repeatedly take step from current state to new state.
- Formally, an ~~execution~~ execution fragment is a sequence:
  - $s_0\ \pi_1\ s_1\ \pi_2\ s_2\ \pi_3\ s_3\ \pi_4\ s_4\ \pi_5\ s_5\ \ldots$
  - ~~s_0 is a start state~~
  - $(s_i, \pi_{i+1}, s_{i+1})$ is a step.

Invariants and reachable states

- A state is reachable if it appears in some execution.
  - Equivalently, at the end of some finite execution
- An invariant is a predicate that is true for every reachable state.
  - Most important tool for proving properties of concurrent/distributed algorithms.
  - Typically proved by induction on length of execution.

Traces

- Allow us to focus on components' external behavior.
- Useful for defining correctness.
- A trace of an execution is the subsequence of external actions in the execution.
  - No states, no internal actions.
  - Denoted trace(α), where α is an execution.
  - Models "observable behavior".

λ, send(a), a, send(b), ab, receive(a), b, receive(b), λ

send(a), send(b), receive(a), receive(b)

Operations on I/O Automata

Operations on I/O automata

- To describe how systems are built out of components, the model has operations for composition, hiding, renaming.
- Composition:
  - "Put multiple automata together."
  - Output actions of one may be input actions of others.
  - All components having an action perform steps involving that action at the same time ("synchronize on actions").
- Composing finitely many or countably infinitely many automata A_i, $i \in I$:
- Need compatibility conditions:
  - Internal actions aren't shared:
    - $int(A_i) \cap acts(A_j) = \emptyset$
  - Only one automaton controls each output:
    - $out(A_i) \cap out(A_j) = \emptyset$
  - But output of one automaton can be an input of one or more others.
  - No action is shared by infinitely many A_is.

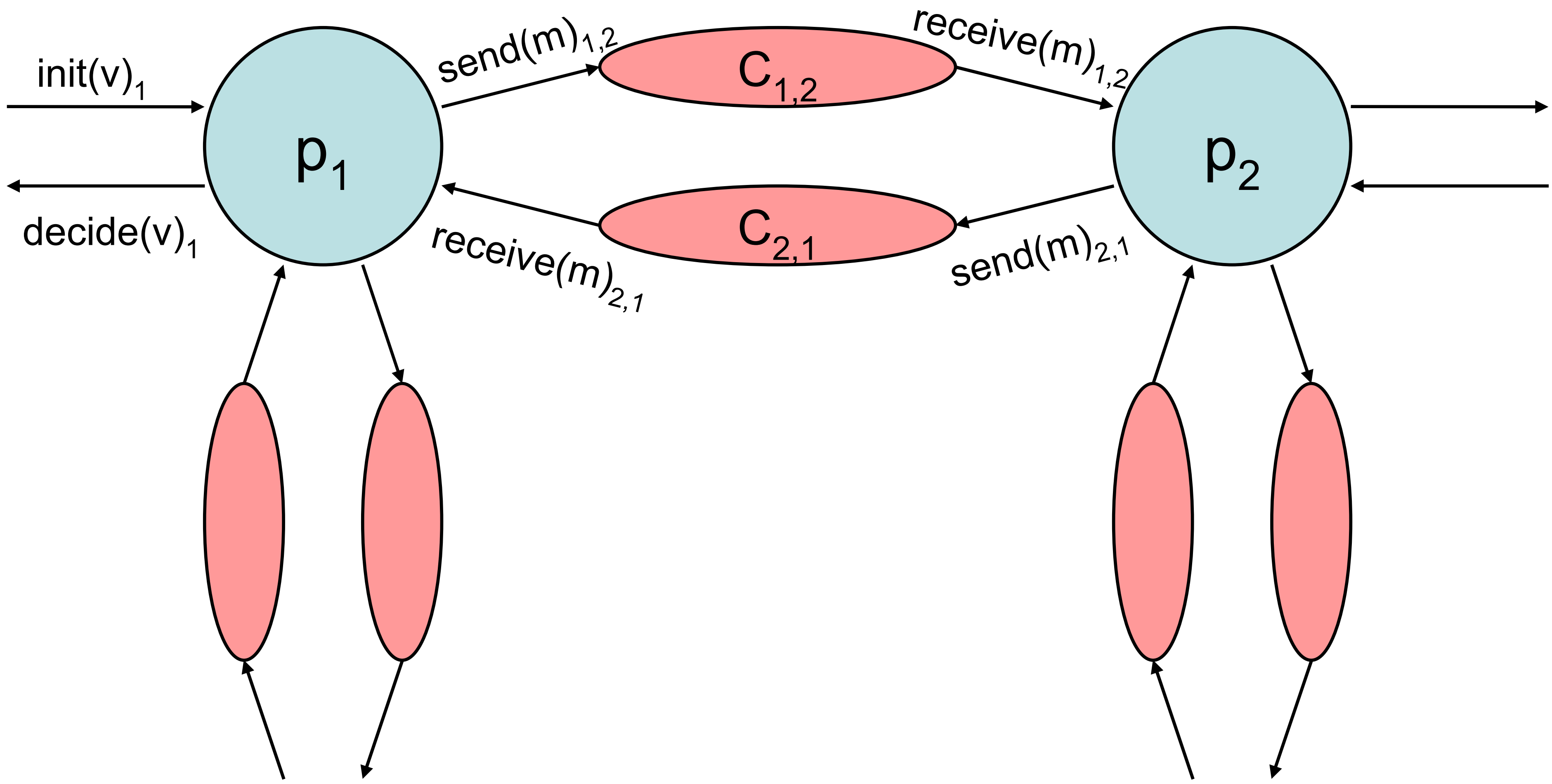
Operations on I/O automata

Composition of compatible automata

- Compose two automata A and B (see book for general case).
- $out(A \times B) = out(A) \cup out(B)$
- $int(A \times B) = int(A) \cup int(B)$
- $in(A \times B) = in(A) \cup in(B) - (out(A) \cup out(B))$
- $states(A \times B) = states(A) \times states(B)$
- $start(A \times B) = start(A) \times start(B)$
- $trans(A \times B)$: includes (s, π, s') iff
  - $(s_A, \pi, s'_A) \in trans(A)$ if $\pi \in acts(A)$; $s_A = s'_A$ otherwise.
  - $(s_B, \pi, s'_B) \in trans(B)$ if $\pi \in acts(B)$; $s_B = s'_B$ otherwise.
- $tasks(A \times B) = tasks(A) \cup tasks(B)$

- Notation: $\Pi_{i \in I} A_i$, for composition of $A_i : i \in I$ (I countable)

Composition of channels and consensus processes



Composition: Basic results

- Projection
  - Execution of composition “looks good” to each component.
- Pasting
  - If execution “looks good” to each component, it is good overall.
- Substitutivity
  - Can replace a component with one that implements it.

Composition: Basic results

Theorem 1: Projection

- If $\alpha \in execs(\Pi A_i)$ then $\alpha|A_i \in execs(A_i)$ for every i.
- If $\beta \in traces(\Pi A_i)$ then $\beta|A_i \in traces(A_i)$ for every i.

Composition: Basic results

Theorem 2: Pasting

Suppose β is a sequence of external actions of Π A_i.

- If $\alpha_i \in$ execs(A_i) and $\beta|A_i$ = trace(α_i) for every i, then there is an execution α of Π A_i such that β = trace(α) and $\alpha_i = \alpha|A_i$ for every i.
- If β $|A_i \in$ traces(A_i) for every i then $\beta \in$ traces(Π A_i).

Composition: Basic results

Theorem 3: Substitutivity

- Suppose A_i and A'_i have the same external signature, and $\text{traces}(A_i) \subseteq \text{traces}(A'_i)$ for every i.
  - A kind of "implementation" relationship.
- Then $\text{traces}(\Pi\ A_i) \subseteq \text{traces}(\Pi\ A'_i)$ (assuming compatibility).

Proof:

- Follows from trace pasting and projection, Theorems 1 and 2.

Other operations on I/O automata

- Hiding
  - Make some output actions internal.
  - Hides internal communication among components of a system.
- Renaming
  - Change names of some actions.
  - Action names are important for specifying component interactions.
  - E.g., define a "generic" automaton, then rename actions to define many instances to use in a system.
    - As we did with channel automata.

Fairness

Fairness

- Task T (set of actions) corresponds to a “thread of control”.
- Used to define “fair” executions: a task that is continuously enabled gets to take a step.
- Needed to prove liveness properties, e.g., that something eventually happens, like an algorithm terminating.

- Formally, execution (or fragment) α of A is fair to task T if one of the following holds:
  - α is finite and T is not enabled in the final state of α.
  - α is infinite and contains infinitely many events in T.
  - α is infinite and contains infinitely many states in which T is not enabled.
- Execution of A is fair if it is fair to all tasks of A.
- Trace of A is fair if it is the trace of a fair execution of A.

Example

- Channel
  - Only one task (all receive actions).
  - A finite execution of Channel is fair iff queue is empty at the end.
  - Q: Is every infinite execution of Channel fair?
- Consensus process
  - Separate tasks for sending to each other process, and for output.
  - Means it “keeps trying” to do these forever.

Fairness and composition

- Fairness "behaves nicely" with respect to composition---results analogous to non-fair results:

Theorem 4: Projection

- If $\alpha \in$ fairexecs($\Pi\ A_i$) then $\alpha|A_i \in$ fairexecs(A_i) for every i.
- If $\beta \in$ fairtraces($\Pi\ A_i$) then $\beta|A_i \in$ fairtraces(A_i) for every i.

Theorem 5: Pasting

Suppose β is a sequence of external actions of $\Pi\ A_i$.

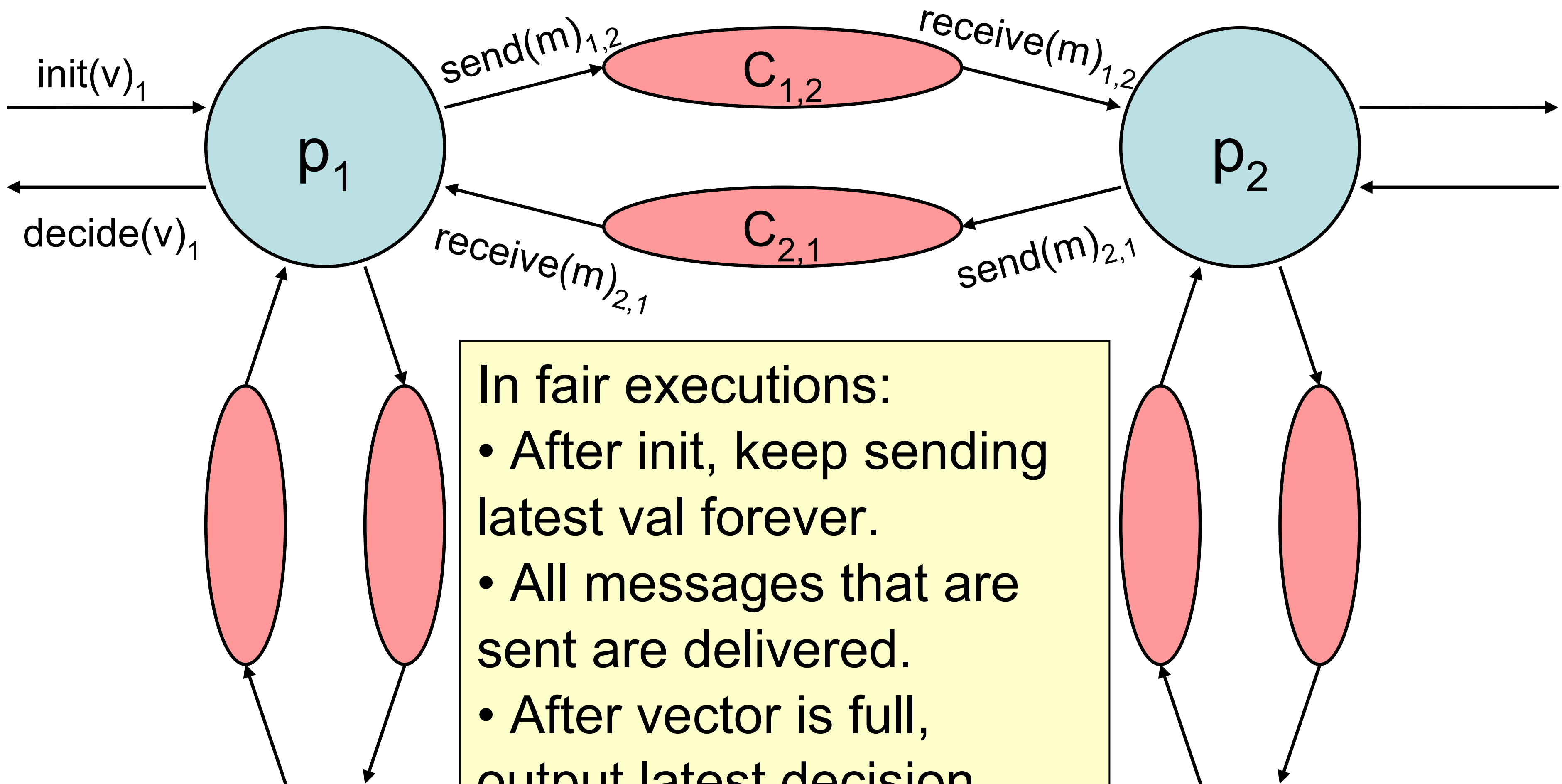
- If $\alpha_i \in$ fairexecs(A_i) and $\beta|A_i$ = trace(α_i) for every i, then there is a fair execution α of $\Pi\ A_i$ such that β = trace(α) and $\alpha_i = \alpha|A_i$ for every i.
- If $\beta\ |A_i \in$ fairtraces(A_i) for every i then $\beta \in$ fairtraces($\Pi\ A_i$).

Fairness and composition

Theorem 6: Substitutivity

- Suppose A_i and A'_i have the same external signature, and fairtraces(A_i) $\subseteq$ fairtraces(A'_i) for every i.
  - Another kind of “implementation” relationship.
- Then fairtraces($\Pi\ A_i$) $\subseteq$ fairtraces($\Pi\ A'_i$).

Composition of channels and consensus processes



In fair executions:
- After init, keep sending latest val forever.
- All messages that are sent are delivered.
- After vector is full, output latest decision forever.

Properties and Proof Methods

- Compositional reasoning
- Invariants
- Trace properties
- Simulation relations

Compositional reasoning

- Use Theorems 1-6 to infer properties of a system from properties of its components.
- And vice versa.

Invariants

- A state is reachable if it appears in some execution (or, at the end of some finite execution).
- An invariant is a predicate that is true for every reachable state.
- Most important tool for proving properties of concurrent and distributed algorithms.
- Proving invariants:
  - Typically, by induction on length of execution.
  - Often prove batches of inter-dependent invariants together.
  - Step granularity is finer than round granularity, so proofs are harder and more detailed than those for synchronous algorithms.

Trace properties

- A trace property is essentially a set of allowable external behavior sequences.
- A trace property P is a pair of:
  - sig(P): External signature (no internal actions).
  - traces(P): Set of sequences of actions in sig(P).
- Automaton A satisfies trace property P if (two different notions):
  - extsig(A) = sig(P) and traces(A) $\subseteq$ traces(P)
  - extsig(A) = sig(P) and fairtraces(A) $\subseteq$ traces(P)

Safety and liveness

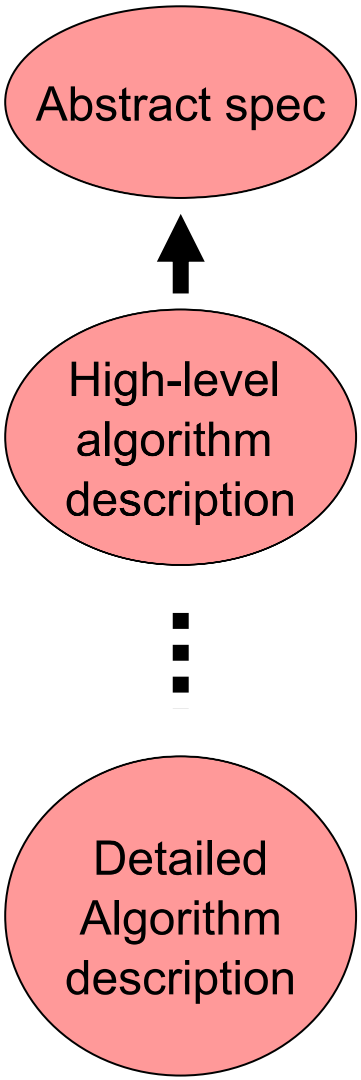
- Safety property: "Bad" thing doesn't happen:
  - Nonempty (null trace is always safe).
  - Prefix-closed: Every prefix of a safe trace is safe.
  - Limit-closed: Limit of sequence of safe traces is safe.
- Liveness property: "Good" thing happens eventually:
  - Every finite sequence over acts(P) can be extended to a sequence in traces(P).
  - "It's never too late."
- Can define safety/liveness for executions similarly.
- Fairness can be expressed as a liveness property for executions.

Automata as specifications

- Every I/O automaton specifies a trace property (extsig(A), traces(A)).
- So we can use an automaton as a problem specification.
- Automaton A "implements" automaton B if
  - extsig(A) = extsig(B)
  - traces(A) $\subseteq$ traces(B)

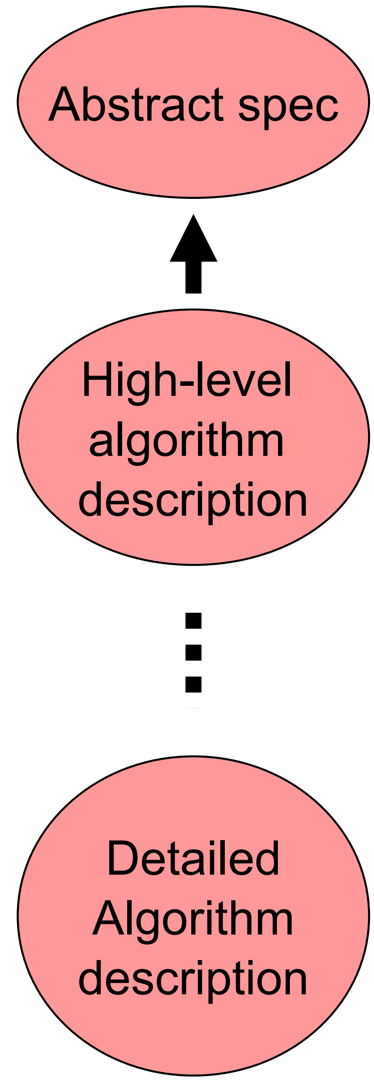
Hierarchical proofs

- Important strategy for proving correctness of complex asynchronous distributed algorithms.
- Define a series of automata, each implementing the previous one ("successive refinement").
- Highest-level automaton model captures the "real" problem specification.
- Next level is a high-level algorithm description.
- Successive levels represent more and more detailed versions of the algorithm.
- Lowest level is the full algorithm description.



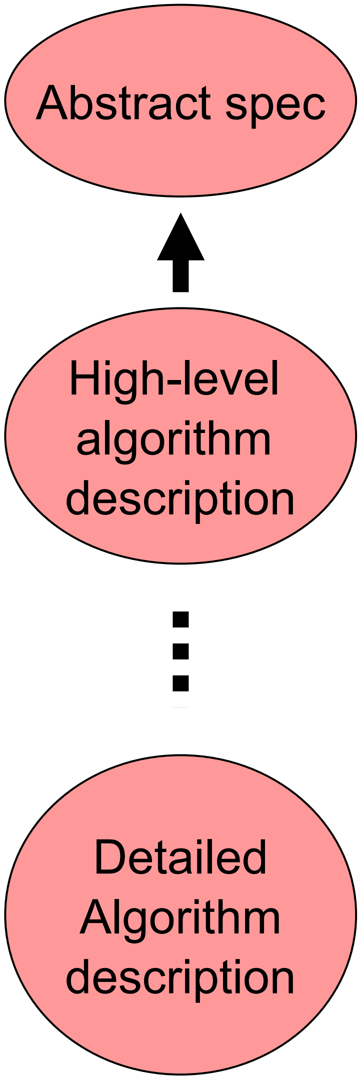
Hierarchical proofs

- For example:
  - High levels centralized, lower levels distributed.
  - High levels inefficient but simple, lower levels optimized and more complex.
  - High levels with large granularity steps, lower levels with finer granularity steps.
- In all these cases, lower levels are harder to understand and reason about.
- So instead of reasoning about them directly, relate them to higher-level descriptions.
- Method similar to what we saw for synchronous algorithms.



Hierarchical proofs

- Recall, for synchronous algorithms:
  - Optimized algorithm runs side-by-side with unoptimized version, and "invariant" proved to relate the states of the two algorithms.
  - Prove using induction.
- For asynchronous systems, things become harder:
  - Asynchronous model has more nondeterminism (in choice of new state, in order of steps).
  - So, harder to determine which execs to compare.
- One-way implementation relationship is enough:
  - For each execution of the lower-level algorithm, there is a corresponding execution of the higher-level algorithm.
  - "Everything the algorithm does is allowed by the spec."
  - Don't need the other direction: doesn't matter if the algorithm does everything that is allowed.



Simulation relations

- Most common method of proving that one automaton implements another.
- Assume A and B have the same extsig, and R is a relation from states(A) to states(B).
- Then R is a simulation relation from A to B provided:
  - $s_A \in start(A)$ implies there exists $s_B \in start(B)$ such that s_A R s_B.
  - If s_A, s_B are reachable states of A and B, s_A R s_B and (s_A, π, s'_A) is a step, then there is an execution fragment β starting with s_B and ending with s'_B such that s'_A R s'_B and trace(β) = trace(π).

Simulation relations

$$\begin{tikzcd}
s_B \arrow[rr, dotted, "\beta"] \arrow[d, dash, "R"] & & s'_B \arrow[d, dotted, dash, "R"] \\
s_A \arrow[rr, "\pi"] & & s'_A
\end{tikzcd}$$

- R is a simulation relation from A to B provided:
  - $s_A \in start(A)$ implies $\exists s_B \in start(B)$ such that $s_A \; R \; s_B$.
  - If s_A, s_B are reachable states of A and B, $s_A \; R \; s_B$ and (s_A, π, s'_A) is a step, then $\exists \beta$ starting with s_B and ending with s'_B such that $s'_A \; R \; s'_B$ and $trace(\beta) = trace(\pi)$.

Simulation relations

- Theorem: If there is a simulation relation from A to B then traces(A) $\subseteq$ traces(B).
- This means all traces of A, not just finite traces.
- Proof: Fix a trace of A, arising from a (possibly infinite) execution of A.
- Create a corresponding execution of B, using an iterative construction.

$$s_{0,A} \xrightarrow{\pi_1} s_{1,A} \xrightarrow{\pi_2} s_{2,A} \xrightarrow{\pi_3} s_{3,A} \xrightarrow{\pi_4} s_{4,A} \xrightarrow{\pi_5} s_{5,A}$$

Simulation relations

- Theorem: If there is a simulation relation from A to B then traces(A) $\subseteq$ traces(B).

$$\begin{tikzcd}
s_{0,B} \arrow[d, dotted, dash, "R"] \\
s_{0,A} \arrow[r, "\pi_1"] & s_{1,A} \arrow[r, "\pi_2"] & s_{2,A} \arrow[r, "\pi_3"] & s_{3,A} \arrow[r, "\pi_4"] & s_{4,A} \arrow[r, "\pi_5"] & s_{5,A}
\end{tikzcd}$$

Simulation relations

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\begin{tikzcd}
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Simulation relations

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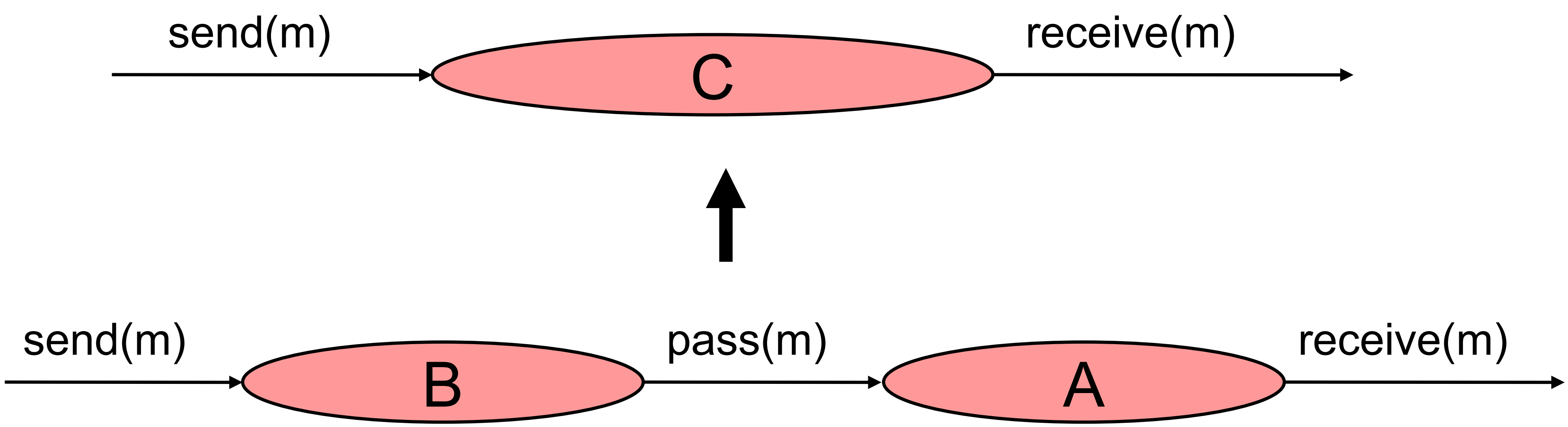
s_{0,A} \arrow[r, "\pi_1"] & s_{1,A} \arrow[r, "\pi_2"] & s_{2,A} \arrow[r, "\pi_3"] & s_{3,A} \arrow[r, "\pi_4"] & s_{4,A} \arrow[r, "\pi_5"] & s_{5,A}
\end{tikzcd}
$$

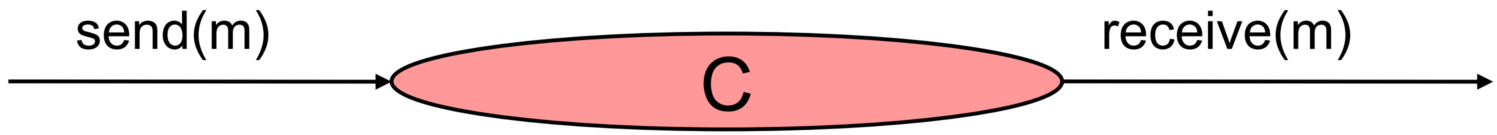
Example: Channels

- Show two channels implement one.



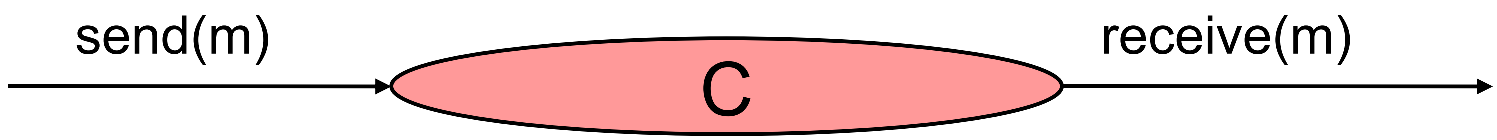
- Rename some actions.
- Claim that D = hide$_{\{pass(m)\}}$ A $\times$ B implements C, in the sense that traces(D) $\subseteq$ traces(C).

Recall: Channel automaton



- Reliable unidirectional FIFO channel.
- signature
  - Input actions: send(m), $m \in M$
  - output actions: receive(m), $m \in M$
  - no internal actions
- states
  - queue: FIFO queue of M, initially empty

Channel automaton



- trans
  - send(m)
    - effect: add m to queue
  - receive(m)
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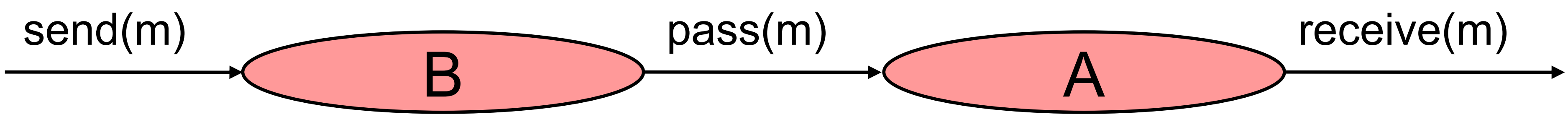
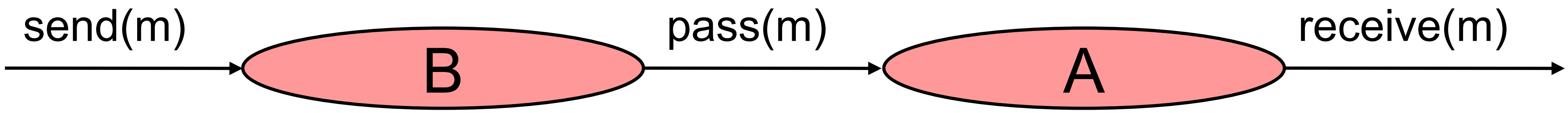
    - effect: remove head of queue
- tasks
  - All receive actions in one task

Composing two channel automata



- Output of B is input of A
  - Rename receive(m) of B and send(m) of A to pass(m).
- D = hide$_{\{ \text{pass}(m) \mid m \in M \}}$ A × B implements C
- Define simulation relation R:
  - For s ∈ states(D) and u ∈ states(C), s R u iff u.queue is the concatenation of s.A.queue and s.B.queue
- Proof that this is a simulation relation:
  - Start condition: All queues are empty, so start states correspond.
  - Step condition: Define "step correspondence":

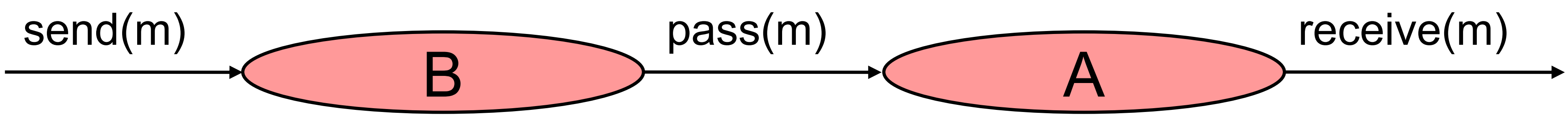
Composing two channel automata



s R u iff u.queue is concatenation of s.A.queue and s.B.queue

- Step correspondence:
  - For each step $(s, \pi, s') \in trans(D)$ and u such that s R u, define execution fragment β of C:
    - Starts with u, ends with u′ such that s′ R u′.
    - $trace(\beta) = trace(\pi)$
  - Here, actions in β happen to depend only on π, and uniquely determine post-state.
    - Same action if external, empty sequence if internal.

Composing two channel automata



s R u iff u.queue is concatenation of s.A.queue and s.B.queue

- Step correspondence:
  - π = send(m) in D corresponds to send(m) in C
  - π = receive(m) in D corresponds to receive(m) in C
  - π = pass(m) in D corresponds to λ in C
- Verify that this works:
  - Actions of C are enabled.
  - Final states related by relation R. case analysis.

– Routine case analysis:

Showing R is a simulation relation

s R u iff u.queue is concatenation of s.A.queue and s.B.queue

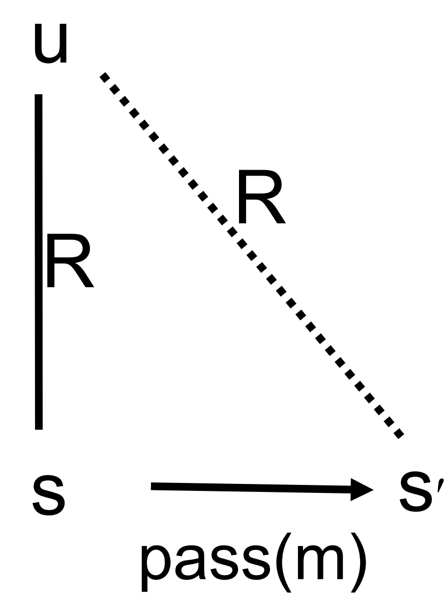
- Case: π = send(m)
  - No enabling issues (input).
  - Must check s′ R u′.
    - Since s R u, u.queue is the concatenation of s.A.queue and s.B.queue.
    - Adding the same m to the end of u.queue and s.B.queue maintains the correspondence.

- Case: π = receive(m)
  - Enabling: Check that receive(m), for the same m, is also enabled in u.
    - We know that m is first on s.A.queue.
    - Since s R u, m is first on u.queue.
    - So enabled in u.
  - s′ R u′: Since m removed from both s.A.queue and u.queue.

Showing R is a simulation relation

s R u iff u.queue is concatenation of s.A.queue and s.B.queue

- Case: π = pass(m)
  - No enabling issues (since no high-level steps are involved).
  - Must check s′ R u:
    - Since s R u, u.queue is the concatenation of s.A.queue and s.B.queue.
    - Concatenation is unchanged as a result of this step, so also u.queue is the concatenation of s′.A.queue and s′.B.queue.



Next lecture

- Basic asynchronous network algorithms:
  - Leader election
  - Breadth-first search
  - Shortest paths
  - Spanning trees.
- Reading:
  - Chapters 14 and 15

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6.852J / 18.437J Distributed Algorithms
Fall 2009

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