6.231 Dynamic Programming and Stochastic Control
Fall 2008

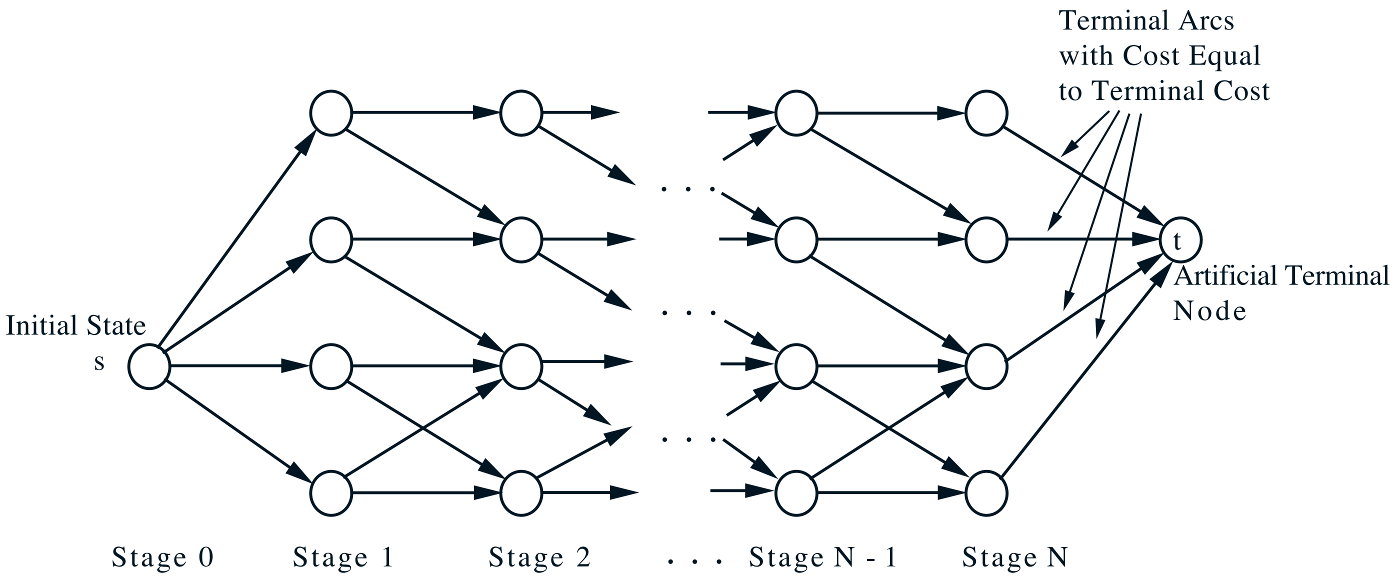
6.231 DYNAMIC PROGRAMMING

LECTURE 3

LECTURE OUTLINE

- Deterministic finite-state DP problems
- Backward shortest path algorithm
- Forward shortest path algorithm
- Shortest path examples
- Alternative shortest path algorithms

DETERMINISTIC FINITE-STATE PROBLEM



- States <==> Nodes
- Controls <==> Arcs
- Control sequences (open-loop) <==> paths from initial state to terminal states
- a_{ij}^k: Cost of transition from state $i \in S_k$ to state $j \in S_{k+1}$ at time k (view it as "length" of the arc)
- a_{it}^N: Terminal cost of state $i \in S_N$
- Cost of control sequence <==> Cost of the corresponding path (view it as "length" of the path)

BACKWARD AND FORWARD DP ALGORITHMS

- DP algorithm:

$$J_N(i) = a_{it}^N, \quad i \in S_N,$$

$$J_k(i) = \min_{j \in S_{k+1}} \left[a_{ij}^k + J_{k+1}(j) \right], \; i \in S_k, \; k = 0, \ldots, N-1$$

The optimal cost is $J_0(s)$ and is equal to the length of the shortest path from s to t

- Observation: An optimal path $s \rightarrow t$ is also an optimal path $t \rightarrow s$ in a "reverse" shortest path problem where the direction of each arc is reversed and its length is left unchanged

- Forward DP algorithm (= backward DP algorithm for the reverse problem):

$$\tilde{J}_N(j) = a_{sj}^0, \quad j \in S_1,$$

$$\tilde{J}_k(j) = \min_{i \in S_{N-k}} \left[a_{ij}^{N-k} + \tilde{J}_{k+1}(i) \right], \quad j \in S_{N-k+1}$$

The optimal cost is $\tilde{J}_0(t) = \min_{i \in S_N} \left[a_{it}^N + \tilde{J}_1(i) \right]$

- View $\tilde{J}_k(j)$ as *optimal cost-to-arrive* to state j from initial state s

A NOTE ON FORWARD DP ALGORITHMS

- There is no forward DP algorithm for stochastic problems
- Mathematically, for stochastic problems, we cannot restrict ourselves to open-loop sequences, so the shortest path viewpoint fails
- Conceptually, in the presence of uncertainty, the concept of "optimal-cost-to-arrive" at a state x_k does not make sense. The reason is that it may be impossible to guarantee (with prob. 1) that any given state can be reached
- By contrast, even in stochastic problems, the concept of "optimal cost-to-go" from any state x_k makes clear sense

GENERIC SHORTEST PATH PROBLEMS

- $\{1, 2, \ldots, N, t\}$: nodes of a graph (t: the *destination*)
- a_{ij}: cost of moving from node i to node j
- Find a shortest (minimum cost) path from each node i to node t
- Assumption: All cycles have nonnegative length. Then an optimal path need not take more than N moves
- We formulate the problem as one where we require exactly N moves but allow degenerate moves from a node i to itself with cost $a_{ii} = 0$

$J_k(i)$ = optimal cost of getting from i to t in $N-k$ moves

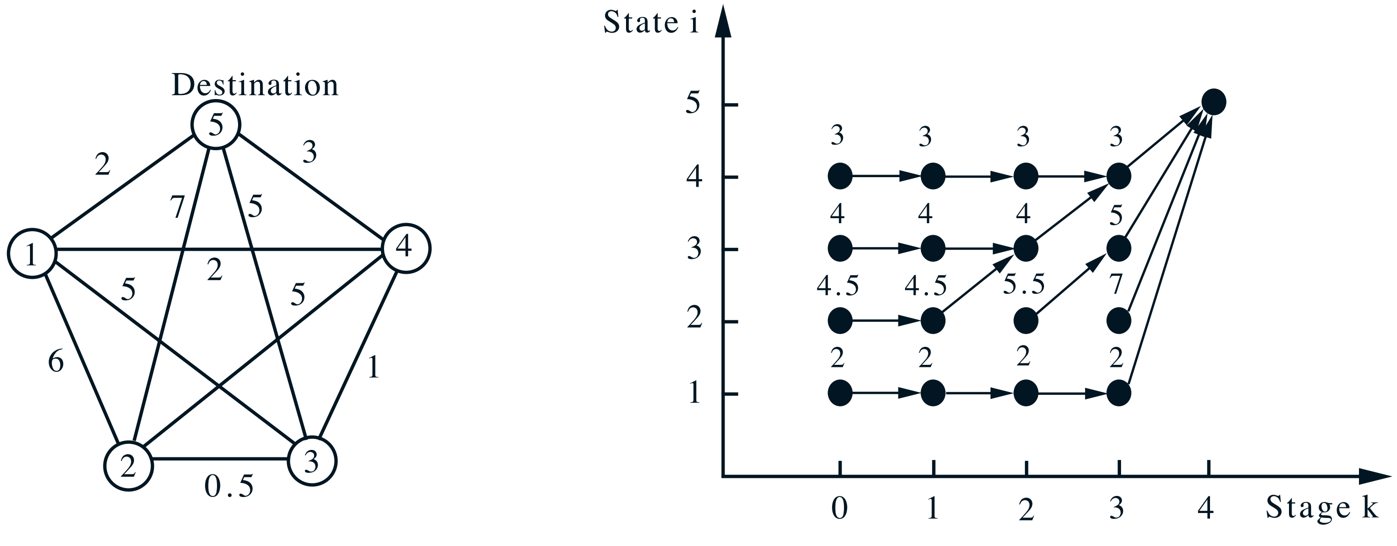
$J_0(i)$: Cost of the optimal path from i to t.

- DP algorithm:

$$J_k(i) = \min_{j=1,\ldots,N} \left[a_{ij} + J_{k+1}(j)\right], \qquad k = 0, 1, \ldots, N-2,$$

with $J_{N-1}(i) = a_{it}$, $i = 1, 2, \ldots, N$

EXAMPLE



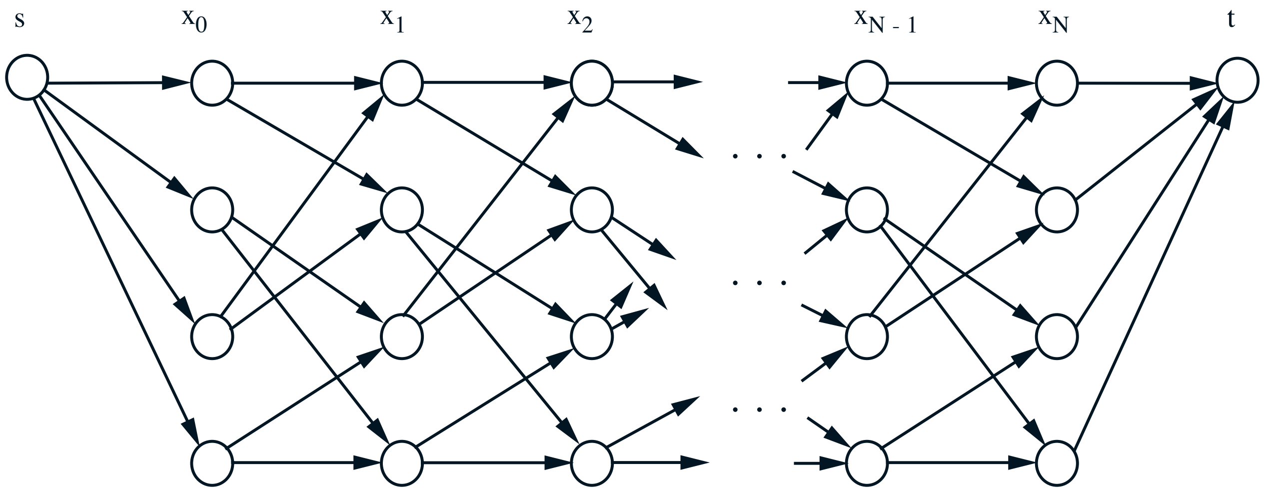
(a) (b)

$$J_{N-1}(i) = a_{it}, \qquad i = 1, 2, \ldots, N,$$

$$J_k(i) = \min_{j=1,\ldots,N} \bigl[a_{ij} + J_{k+1}(j)\bigr], \qquad k = 0, 1, \ldots, N-2.$$

ESTIMATION / HIDDEN MARKOV MODELS

- Markov chain with transition probabilities p_{ij}
- State transitions are hidden from view
- For each transition, we get an (independent) observation
- $r(z;i,j)$: Prob. the observation takes value z when the state transition is from i to j
- Trajectory estimation problem: Given the observation sequence $Z_N = \{z_1, z_2, \ldots, z_N\}$, what is the "most likely" state transition sequence $\hat{X}_N = \{\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N\}$ [one that maximizes $p(X_N \mid Z_N)$ over all $X_N = \{x_0, x_1, \ldots, x_N\}$].



VITERBI ALGORITHM

- We have

$$p(X_N \mid Z_N) = \frac{p(X_N, Z_N)}{p(Z_N)}$$

where $p(X_N, Z_N)$ and $p(Z_N)$ are the unconditional probabilities of occurrence of (X_N, Z_N) and Z_N

- Maximizing $p(X_N \mid Z_N)$ is equivalent with maximizing $\ln(p(X_N, Z_N))$
- We have

$$p(X_N, Z_N) = \pi_{x_0} \prod_{k=1}^{N} p_{x_{k-1}x_k} r(z_k; x_{k-1}, x_k)$$

so the problem is equivalent to

$$\text{minimize} - \ln(\pi_{x_0}) - \sum_{k=1}^{N} \ln\bigl(p_{x_{k-1}x_k} r(z_k; x_{k-1}, x_k)\bigr)$$

over all possible sequences $\{x_0, x_1, \ldots, x_N\}$.

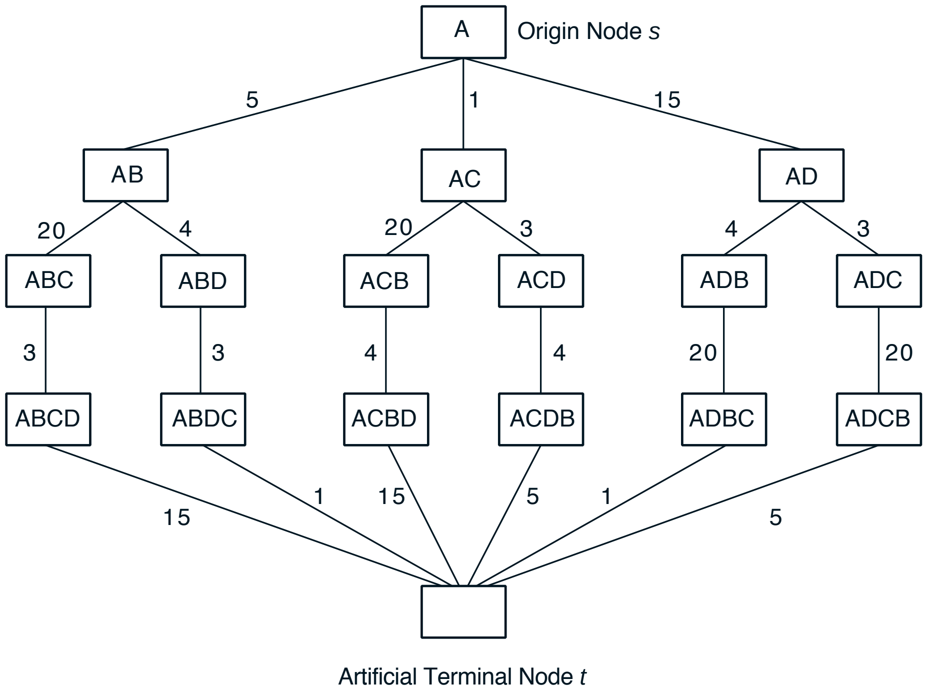
- This is a shortest path problem.

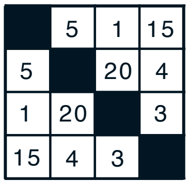
GENERAL SHORTEST PATH ALGORITHMS

- There are many nonDP shortest path algorithms. They can all be used to solve deterministic finite-state problems

- They may be preferable than DP if they avoid calculating the optimal cost-to-go of EVERY state

- This is essential for problems with HUGE state spaces. Such problems arise for example in combinatorial optimization





■	5	1	15
5	■	20	4
1	20	■	3
15	4	3	■

LABEL CORRECTING METHODS

- Given: Origin s, destination t, lengths $a_{ij} \geq 0$.
- Idea is to progressively discover shorter paths from the origin s to every other node i
- **Notation:**
  - d_i (label of i): Length of the shortest path found (initially $d_s = 0$, $d_i = \infty$ for $i \neq s$)
  - UPPER: The label d_t of the destination
  - OPEN list: Contains nodes that are currently active in the sense that they are candidates for further examination (initially OPEN=$\{s\}$)

Label Correcting Algorithm

Step 1 (Node Removal): Remove a node i from OPEN and for each child j of i, do step 2

Step 2 (Node Insertion Test): If $d_i + a_{ij} < \min\{d_j, \text{UPPER}\}$, set $d_j = d_i + a_{ij}$ and set i to be the parent of j. In addition, if $j \neq t$, place j in OPEN if it is not already in OPEN, while if $j = t$, set UPPER to the new value $d_i + a_{it}$ of d_t

Step 3 (Termination Test): If OPEN is empty, terminate; else go to step 1

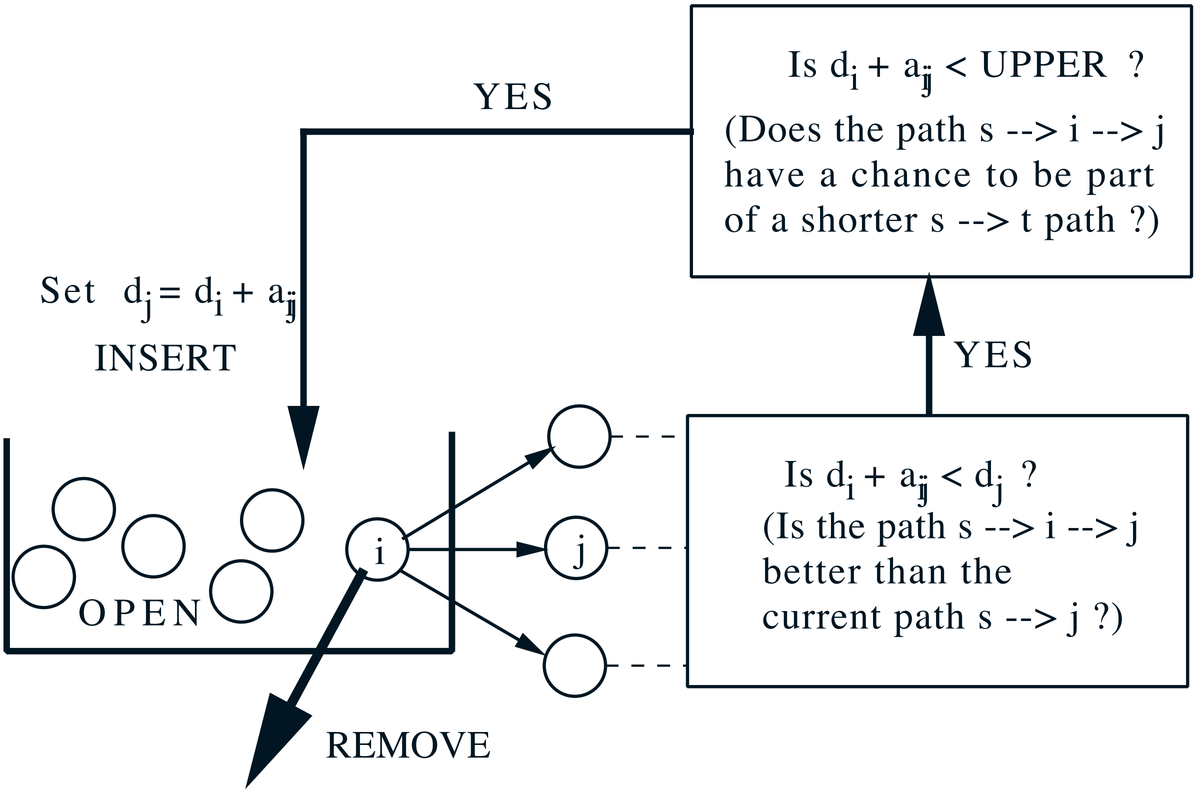
VISUALIZATION/EXPLANATION

- Given: Origin s, destination t, lengths $a_{ij} \geq 0$

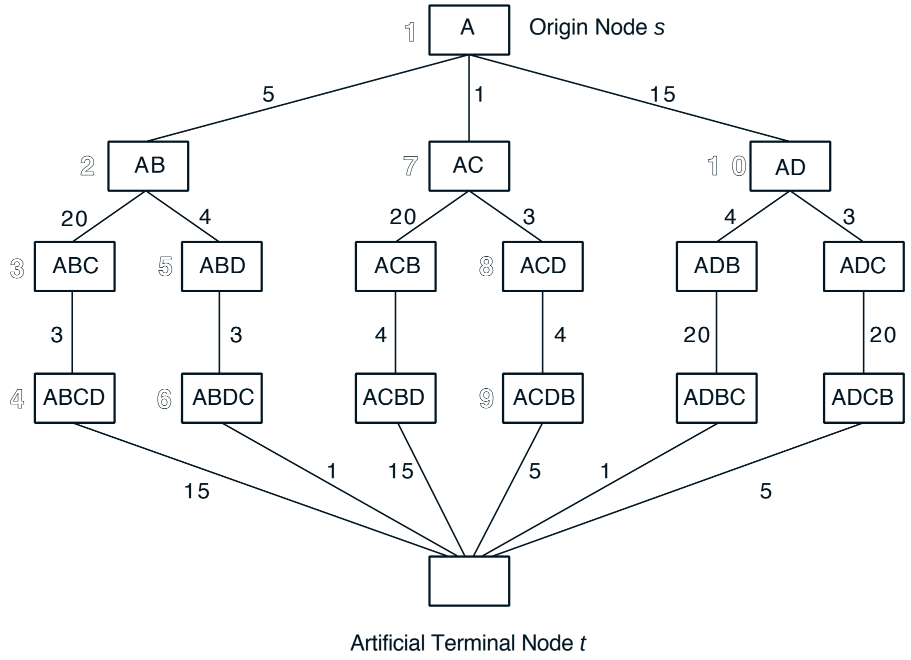
- d_i (label of i): Length of the shortest path found thus far (initially $d_s = 0$, $d_i = \infty$ for $i \neq s$). The label d_i is implicitly associated with an $s \to i$ path

- UPPER: The label d_t of the destination

- OPEN list: Contains "active" nodes (initially OPEN=$\{s\}$)



EXAMPLE



Iter. No.	Node Exiting OPEN	OPEN after Iteration	UPPER
0	-	1	∞
1	1	2, 7,10	∞
2	2	3, 5, 7, 10	∞
3	3	4, 5, 7, 10	∞
4	4	5, 7, 10	43
5	5	6, 7, 10	43
6	6	7, 10	13
7	7	8, 10	13
8	8	9, 10	13
9	9	10	13
10	10	Empty	13

- Note that some nodes never entered OPEN