**Massachusetts Institute of Technology**

**Department of Electrical Engineering and Computer Science**

**6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS**

**by A. Megretski**

# Lecture 4: Analysis Based On Continuity [1]

This lecture presents several techniques of qualitative systems analysis based on what is frequently called *topological arguments*, i.e. on the arguments relying on continuity of functions involved.

## 4.1 Analysis using general topology arguments

This section covers results which do not rely specifically on the shape of the state space, and thus remain valid for very general classes of systems. We will start by proving generalizations of theorems from the previous lecture to the case of *discrete-time* autonomous systems.

### 4.1.1 Attractor of an asymptotically stable equilibrium

Consider an autonomous time invariant discrete time system governed by equation

$$x(t+1) = f(x(t)), \quad x(t) \in X, \quad t = 0, 1, 2, \dots, \tag{4.1}$$

where $X$ is a given subset of $\mathbf{R}^n$, $f: X \mapsto X$ is a given function. Remember that $f$ is called *continuous* if $f(x_k) \to f(x_\infty)$ as $k \to \infty$ whenever $x_k, x_\infty \in X$ are such that $x_k \to x_\infty$ as $k \to \infty$). In particular, this means that *every* function defined on a *finite* set $X$ is continuous.

One important source of discrete time models is *discretization* of differential equations. Assume that function $a: \mathbf{R}^n \mapsto \mathbf{R}^n$ is such that solutions of the ODE

$$\dot{x}(t) = a(x(t)), \tag{4.2}$$

[1] Version of September 17, 2003

with $x(0) = \bar{x}$ exist and are unique on the time interval $t \in [0, 1]$ for *all* $\bar{x} \in \mathbf{R}^n$. Then discrete time system (4.1) with $f(\bar{x}) = x(1, \bar{x})$ describes the evolution of continuous time system (4.2) at discrete time samples. In particular, if $a$ is continuous then so is $f$.

Let us call a point in the closure of $X$ *locally attractive* for system (4.1) if there exists $d > 0$ such that $x(t) \to \bar{x}_0$ as $t \to \infty$ for every $x = x(t)$ satisfying (4.1) with $|x(0) - \bar{x}_0| < d$. Note that locally attractive points are not necessarily equilibria, and, even if they are, they are not necessarily asymptotically stable equilibria.

For $\bar{x}_0 \in \mathbf{R}^n$ the set $A = A(\bar{x}_0)$ of all initial conditions $\bar{x} \in X$ in (4.1) which define a solution $x(t)$ converging to $\bar{x}_0$ as $t \to \infty$ is called the *attractor* of $\bar{x}_0$.

**Theorem 4.1** *If $f$ is continuous and $\bar{x}_0$ is locally attractive for (4.1) then the attractor $A = A(\bar{x}_0)$ is a (relatively) open subset of $X$, and its boundary $d(A)$ (in $X$) is $f$-invariant, i.e. $f(\bar{x}) \in d(A)$ whenever $\bar{x} \in d(A)$.*

Remember that a subset $Y \subset X \subset \mathbf{R}^n$ is called *relatively open* in $X$ if for every $y \in Y$ there exists $r > 0$ such that all $x \in X$ satisfying $|x - y| < r$ belong to $Y$. A *boundary* of a subset $Y \subset X \subset \mathbf{R}^n$ in $X$ is he set of all $x \in X$ such that for every $r > 0$ there exist $y \in Y$ and $z \in X/Y$ such that $|y - x| < r$ and $z - x| < r$. For example, the half-open interval $Y = (0, 1]$ is a relatively closed subset of $X = (0, 2)$, and its boundary in $X$ consists of a single point $x = 1$.

**Example 4.1** Assume system (4.1), defined on $X = \mathbf{R}^n$ by a *continuous* function $f : \mathbf{R}^n \mapsto \mathbf{R}^n$, is such that all solutions with $|x(0)| < 1$ converge to zero as $t \to \infty$, and all solutions with $|x(0)| > 100$ converge to infinity as $t \to \infty$. Then, according to Theorem 4.1, the boundary of the attractor $A = A(0)$ is a non-empty $f$-invariant set. By assumptions, $1 \le |\bar{x}| \le 100$ for all $\bar{x} \in A(0)$. Hence we can conclude that there exist solutions of (4.1) which satisfy the constraints $1 \le |x(t)| \le 100$ for all $t$.

**Example 4.2** For system (4.1), defined on $X = \mathbf{R}^n$ by a *continuous* function $f : \mathbf{R}^n \mapsto \mathbf{R}^n$, it is possible to have every trajectory to converge to one of two equilibria. However, it is not possible for *both* equilibria to be locally attractive. Otherwise, according to Theorem 4.1, $\mathbf{R}^n$ would be represented as a union of two disjoint open sets, which contradicts the notion of *connectedness* of $\mathbf{R}^n$.

### 4.1.2 Proof of Theorem 4.1

According to the definition of local attractiveness, there exists $d > 0$ such that $x(t) \to \bar{x}_0$ as $t \to \infty$ for every $x = x(t)$ satisfying (4.1) with $|x(0) - \bar{x}_0| < d$. Take an arbitrary $\bar{x}_1 \in A(\bar{x}_0)$. Let $x_1 = x_1(t)$ be the solution of (4.1) with $x(0) = \bar{x}_1$. Then $x_1(t) \to \bar{x}_0$ as $t \to \infty$, and hence $|x_1(t_1)| < d/2$ for a sufficiently large $t_1$. Since $f$ is continuous, $x(t)$ is a continuous function of $x(0)$ for every fixed $t \in \{0, 1, 2, \dots\}$. Hence there exists $\delta > 0$ such that $|x(t_1) - x_1(t_1)| < d/2$ whenever $|x(0) - \bar{x}_1| < \delta$. Since this implies $|x(t_1) - \bar{x}_0| < d$, we have $\bar{x} \in A(\bar{x}_0)$ for every $\bar{x} \in X$ such that $|\bar{x} - \bar{x}_1| < \delta$, which proves that $A = A(\bar{x}_0)$ is open.

To show that $d(A)$ is $f$-invariant, note first that $A$ is itself $f$-invariant. Now take an arbitrary $\bar{x} \in d(A)$. By the definition of the boundary, there exists a sequence $\bar{x}_k \in A$ converging to $\bar{x}$. Hence, by the continuity of $f$, the sequence $f(\bar{x}_k)$ converges to $f(\bar{x})$. If $f(\bar{x}) \notin A$, this implies $f(\bar{x}) \in d(A)$. Let us show that the opposite is impossible. Indeed, if $f(\bar{x}) \in A$ then, since $A$ is proven open, there exists $\epsilon > 0$ such that $z \in A$ for every $z \in X$ such that $|z - f(\bar{x})| < \epsilon$. Since $f$ is continuous, there exists $\delta > 0$ such that $|f(y) - f(\bar{x})| < \epsilon$ whenever $y \in X$ is such that $|y - \bar{x}| < \delta$. Hence $f(y) \in A$ whenever $|y - \bar{x}| < \delta$. Since, by the definition of attractor, $f(y) \in A$ imlies $y \in A$, $y \in A$ whenever $|y - \bar{x}| < \delta$, which contradicts the assumption that $\bar{x} \in d(A)$.

### 4.1.3 Limit points of planar trajectories

For a given solution $x = x(t)$ of (4.2), the set $\lim(x) \subset \mathbf{R}^n$ of all possible limits $x(t_k) \to \tilde{x}$ as $k \to \infty$, where $\{t_k\}$ converges to infinity, is called the *limit set* of $x$.

**Theorem 4.2** *Assume that $a: \mathbf{R}^n \mapsto \mathbf{R}^n$ is a locally Lipschitz function. If $x: [0,\infty) \mapsto \mathbf{R}^n$ is a solution of (4.2) then the set $\lim(x)$ of its limit points is a closed subset of $\mathbf{R}^n$, and every solution of (4.2) with initial conditions in $\lim(x)$ lies completely in $\lim(x)$.*

**Proof** First, if $t_{k,q} \to \infty$ and $x(t_{k,q}, x(0)) \to \bar{x}_q$ as $k \to \infty$ for every $q$, and $\bar{x}_q \to \bar{x}_\infty$ as $q \to \infty$ then one can select $q = q(k)$ such that $t_{k,q(k)} \to \infty$ and $x(t_{k,q(k)}, x(0) \to \bar{x}_\infty$ as $k \to \infty$. This proves the closedness (continuity of solutions was not used yet).

Second, by assumption

$$\bar{x}_0 = \lim_{k\to\infty} x(t_k, x(0)).$$

Hence, by the continuous dependence of solutions on initial conditions,

$$x(t, \bar{x}_0) = \lim_{k\to\infty} x(t, x(t_k, x(0))) = \lim_{k\to\infty} x(t + t_k, x(0)).$$

■

In general, limit sets of ODE solutions can be very complicated. However, in the case when $n = 2$, a relatively simple classification exists.

**Theorem 4.3** *Assume that $a: \mathbf{R}^2 \mapsto \mathbf{R}^2$ is a locally Lipschitz function. Let $x_0: [0,\infty) \mapsto \mathbf{R}^2$ be a solution of (4.2). Then one of the following is true:*

*(a) $|x_0(t)| \to \infty$ as $t \to \infty$;*

*(b) there exists $T > 0$ and a non-constant solution $x_p: (-\infty, +\infty) \mapsto \mathbf{R}^2$ such that $x_p(t + T) = x_p(t)$ for all $t$, and the set of limit points of $x$ is the trajectory (the range) of $x_p$;*

*(c) the limit set is a union of trajectories of maximal solutions $x: \ (t_1, t_2) \mapsto \mathbf{R}^2$ of (4.2), each of which has a limit (possibly infinite) as $t \to t_1$ or $t \to t_2$.*

The proof of Theorem 4.3 is based on the more specific topological arguments, to be discussed in the next section.

## 4.2 Map index in system analysis

The notion of *index* of a continuous function is a remarkably powerful tool for proving existence of mathematical objects with certain properties, and, as such, is very useful in qualitative system analysis.

### 4.2.1 Definition and fundamental properties of index

For $n = 1, 2, \ldots$ let

$$S^n = \{x \in \mathbf{R}^{n+1} : \ |x| = 1\}$$

denote the unit sphere in $\mathbf{R}^{n+1}$. Note the use of $n$, not $n+1$, in the $S$-notation: it indicates that locally the sphere in $\mathbf{R}^{n+1}$ looks like $\mathbf{R}^n$. There exists a way to define the *index* $\mathrm{ind}(F)$ of every continuous map $F: \ S^n \mapsto S^n$ in such a way that the following conditions will be satisfied:

(a) if $H: \ S^n \times [0,1] \mapsto S^n$ is continuous then

$$\mathrm{ind}(H(\cdot, 0)) = \mathrm{ind}(H(\cdot, 1))$$

(such maps $H$ is called a *homotopy* between $H(\cdot, 0)$ and $H(\cdot, 1)$);

(b) if the map $\hat{F}: \ \mathbf{R}^{n+1} \mapsto \mathbf{R}^{n+1}$ defined by

$$\hat{F}(z) = |z| F(z/|z|)$$

is continuously differentiable in a neigborhood of $S^n$ then

$$\mathrm{ind}(F) = \int_{x \in S^n} \det(J_x(\hat{F})) dm(x),$$

where $J_x(\hat{F})$ is the Jacobian of $\hat{F}$ at $x$, and $m(x)$ is the normalized Lebesque measure on $S^n$ (i.e. $m$ is invariant with respect to unitary coordinate transformations, and the total measure of $S^n$ equals 1).

Once it is proven that the integral in (b) is always an integer (uses standard volume/surface integration relations), it is easy to see that conditions (a),(b) define $\mathrm{ind}(F)$ correctly and uniquely. For $n = 1$, the index of a continuous map $F: \ S^1 \mapsto S^1$ turns out to be simply the *winding number* of $F$, i.e. the number of rotations around zero the trajectory of $F$ makes.

It is also easy to see that $\mathrm{ind}(F_I) = 1$ for the identity map $F_I(x) = x$,and $\mathrm{ind}(F_c) = 0$ for every *constant* map $F_c(x) = x_0 = \mathrm{const}$.

### 4.2.2 The Brower's fixed point theorem

One of the classical mathematical results that follow from the very *existence* of the index function is the famous Brower's fixed point theorem, which states that for every continuous function $G:\ B^n \mapsto B^n$, where

$$B^n = \{x \in \mathbf{R}^{n+1}:\ |x| \leq 1\},$$

equation $F(x) = x$ has at least one solution.

The statement is obvious (though still very useful) when $n = 1$. Let us prove it for $n > 1$, starting with assume the contrary. Then the map $\hat{G}:\ B^n \mapsto B^n$ which maps $x \in B^n$ to the point of $S^{n-1}$ which is the (unique) intersection of the open ray starting from $G(x)$ and passing through $x$ with $S^{n-1}$. Then $H:\ S^{n-1} \times [0,1] \mapsto S^{n-1}$ defined by

$$H(x,t) = \hat{G}(tx)$$

is a homotopy between the identity map $H(\cdot,1)$ and the constant map $H(\cdot,0)$. Due to existence of the index function, such a homotopy does not exist, which proves the theorem.

### 4.2.3 Existence of periodic solutions

Let $a:\ \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ be locally Lipschitz and $T$-periodic with respect to the second argument, i.e.

$$a(\bar{x}, t+T) = a(\bar{x}, t) \quad \forall\ x, t$$

where $T > 0$ is a given number. Assume that solutions of the ODE

$$\dot{x}(t) = a(x(t), t) \tag{4.3}$$

with initial conditions $x(0) \in B^n$ remain in $B^n$ for all times. Then (4.3) has a $T$-periodic solution $x = x(t) = x(t+T)$ for all $t \in \mathbf{R}$.

Indeed, the map $\bar{x} \mapsto x(T, 0, \bar{x})$ is a continuous function $G:\ B^n \mapsto B^n$. The solution of $\bar{x} = G(\bar{x})$ defines the initial conditions for the periodic trajectory.