

Massachusetts Institute of Technology  
Department of Electrical Engineering and Computer Science  
6.685 Electric Machines

Problem Set 9

Solutions

November 19, 2005

**Problem 1:** Variable Reluctance Motor

Much of the solution to this problem is keeping track of where the inductance is varying. Note for some value of flux  $\lambda$ , if  $\frac{\lambda}{L(\theta)} < I_s$ , then:

$$\begin{aligned} I &= \frac{\lambda}{L(\theta)} \\ W'_m &= \frac{1}{2}L(\theta)I^2 \\ T_e &= \frac{I^2}{2} \frac{\partial L}{\partial \theta} \end{aligned}$$

If the flux is greater than that:

$$\begin{aligned} I &= \frac{\lambda}{L_{\min}} + I_s \frac{L(\theta) - L_{\min}}{L_{\min}} \\ W'_m &= \frac{1}{2}I_s^2 L_{\min} + (L(\theta) - L_{\min}) I_s I \\ T_e &= II_s \frac{\partial L}{\partial \theta} \end{aligned}$$

The details are shown in the scripts attached. I do not presume to present these as masterpieces of slick programming, but they seem to work. A subroutine is used to present inductance and rate of change of inductance and then a second subroutine calculates current and torque, given flux, inductance and rate of change of inductance. Here are some pictures. First, Figure 1 shows the (unsaturated) inductance and rate of change of inductance as a function of rotor angle, for one stator pole.

Next, Figure 2 shows current as a function of rotor position for several values of flux, from 0.2 T to 2.0 T in increments of 0.2 T. This is computed easily as:

$$\begin{aligned} i &= \frac{\lambda}{L(\theta)} && \text{if } \frac{\lambda}{L(\theta)} < I_{\text{sat}} \\ &= I_{\text{sat}} \left( 1 - \frac{L(\theta)}{L_{\min}} \right) + \frac{\lambda}{L_{\min}} && \text{otherwise} \end{aligned}$$

To obtain torque, we note that the magnetic coenergy is:

$$\begin{aligned} w'_m &= \frac{1}{2}L(\theta)i^2 && \text{if } \frac{\lambda}{L(\theta)} < I_{\text{sat}} \\ &= \frac{1}{2}L(\theta)I_{\text{sat}}^2 + L(\theta)(i - I_{\text{sat}}) && \text{otherwise} \end{aligned}$$

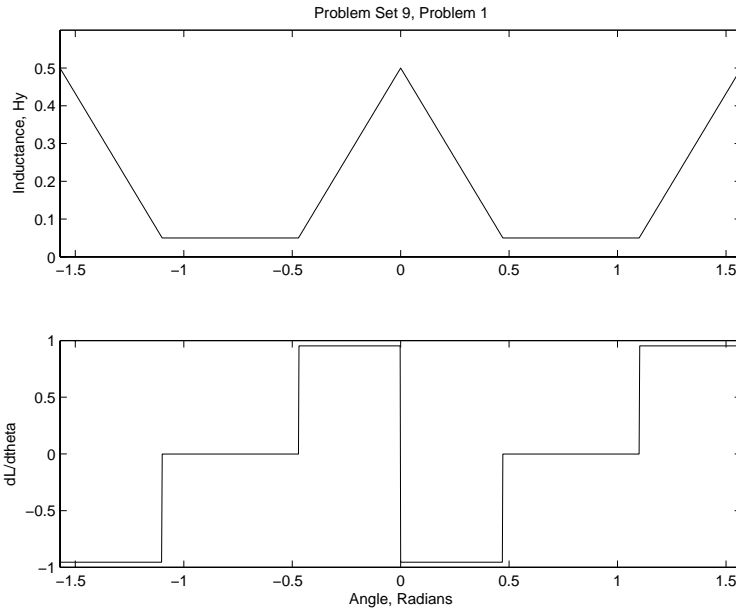


Figure 1: Inductance and rate of change of inductance vs. angle

Then torque is:

$$\begin{aligned}
 T^e &= \frac{i^2}{2} \frac{\partial L}{\partial \theta} && \text{if } \frac{\lambda}{L(\theta)} < I_{\text{sat}} \\
 &= \left( I_{\text{sat}} i - \frac{1}{2} I_{\text{sat}}^2 \right) \frac{\partial L}{\partial \theta} && \text{otherwise}
 \end{aligned}$$

This is shown in Figure 3

To find average torque note there are three phases and four rotor poles, so there are  $3 \times 4 = 12$  cycles per revolution. Thus

$$\langle T_e \rangle = \frac{12}{2\pi} \times W$$

where  $W$  is work done per 'stroke'. We approximate this by:

$$W = \int T_e d\theta \approx \sum T_e \Delta\theta$$

and we have taken  $\Delta\theta$  to be a suitably small value. Note we have used only the duration of one flux pulse to calculate this work.

Finally, we have picked out two starting angles, one in the motoring range and one in the generating range and have plotted instantaneous flux, current and torque. Note that current in the winding is a positive number, but, particularly in generating operation, current may be freewheeling through the diodes back into the power supply, and so this current can appear to be negative to the supply.

**Problem 2:** Permanent Magnets

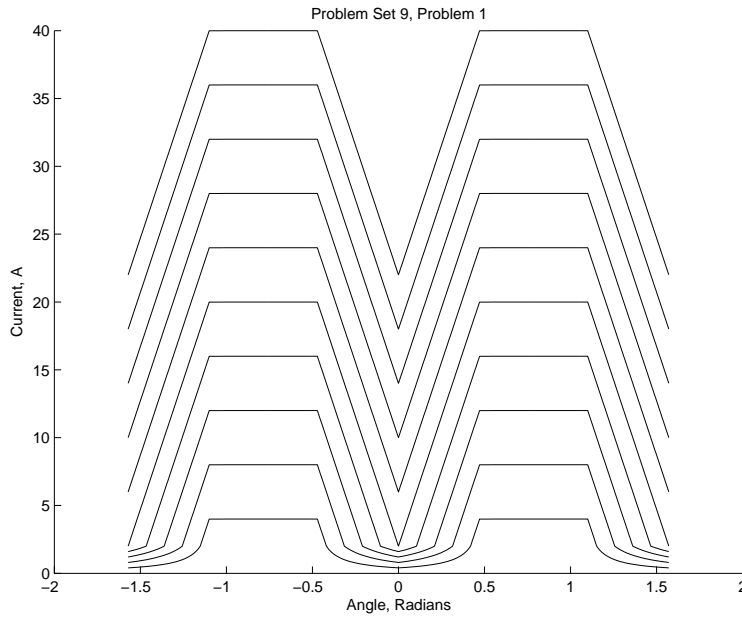


Figure 2: Current vs. position with flux as a parameter

Note we worked this problem in class. The general solution is:

$$B_y = \sum_n B_n \frac{\cosh nk(g+y) \sinh nkh \cos nkx}{\sinh nkh \cosh nkg + \cosh nkh \sinh nkg}$$

$$B_x = -\sum_n B_n \frac{\sinh nk(g+y) \sinh nkh \sin nkx}{\sinh nkh \cosh nkg + \cosh nkh \sinh nkg}$$

where the flux amplitudes are:

$$B_n = B_r \frac{4}{n\pi} \sin \frac{n\theta_m}{2}$$

Here note that

$$\theta_m = 2\pi \frac{2}{\lambda} = kw$$

To carry out the first part of the problem, where  $g \rightarrow \infty$ , we can use:

$$\begin{aligned} \cosh nk(g+y) &\rightarrow \frac{1}{2} e^{nkg} e^{nky} \\ \cosh nkg &\rightarrow \frac{1}{2} e^{nkg} \\ \sinh nkg &\rightarrow \frac{1}{2} e^{nkg} \end{aligned}$$

then the fields become:

$$B_y = \sum_n \frac{1}{2} B_n e^{nky} (1 - e^{2nkh}) \cos nkx$$

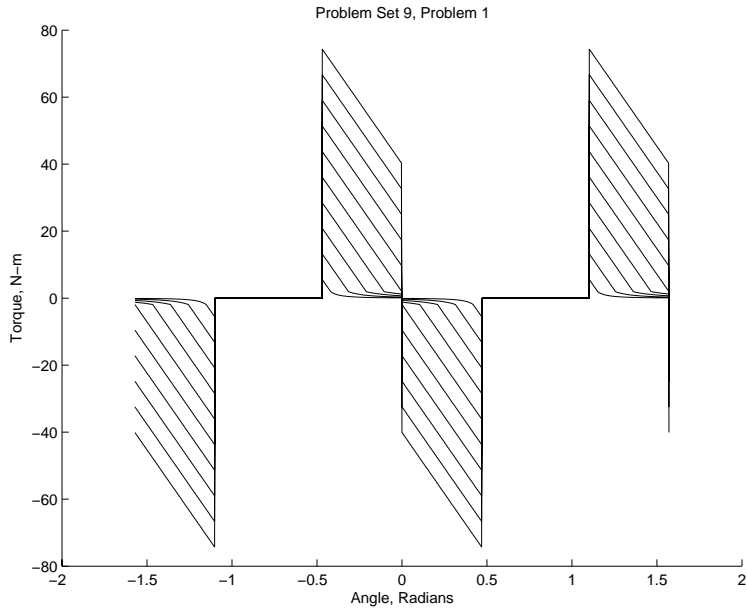


Figure 3: Torque vs. position with flux as a parameter

$$B_x = \sum_n \frac{1}{2} B_n e^{nky} (1 - e^{2nkh}) \sin nkx$$

The results for the infinite gap case are shown in Figure 7. the results for the relatively wide air-gap are shown in Figure 8 and the results for the narrower gap are shown in Figure 9. Note that the narrow air-gap assumption would yield, for a 1/2 cm gap:

$$B_y \approx B_r \frac{h}{h+g} = \frac{1}{1.5} = \frac{2}{3}$$

which is about the value of the field at the peak. It is perhaps not as 'flat' as that approximation would suggest, but then a 5 mm gap is not all that narrow under the circumstances.

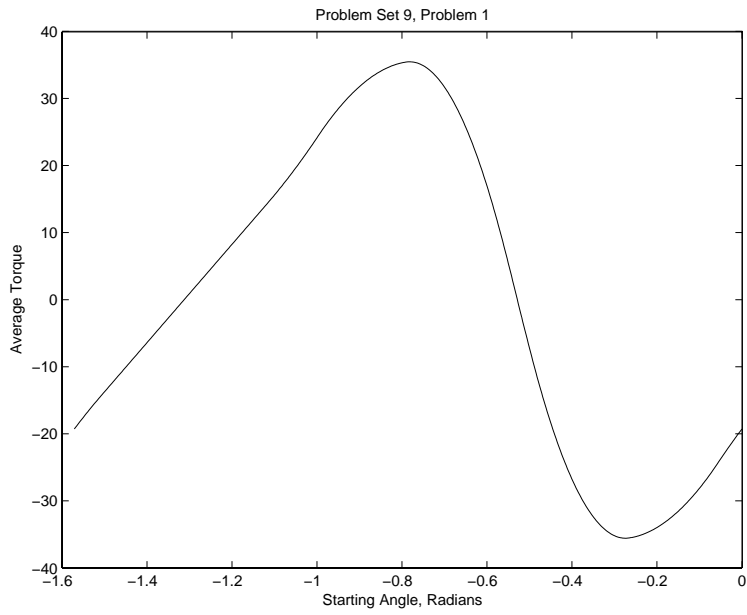


Figure 4: Average torque as a function of pulse start angle

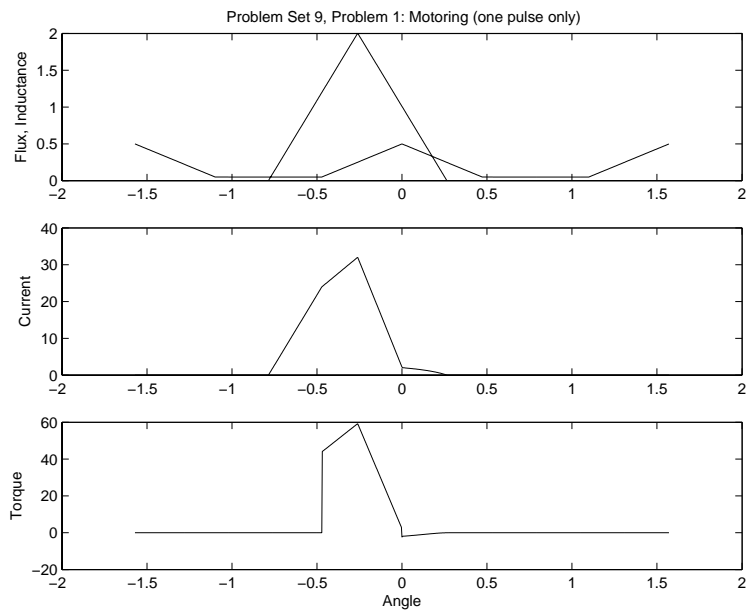


Figure 5: Motoring Operation

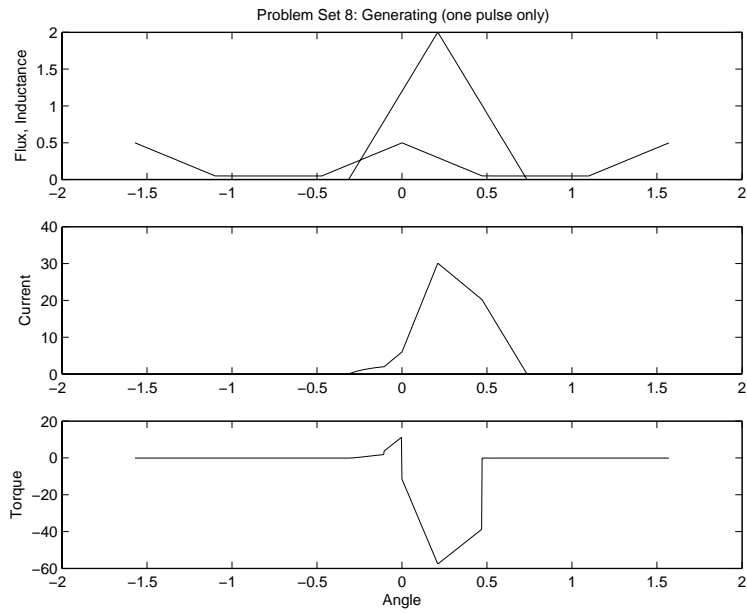


Figure 6: Generating Operation

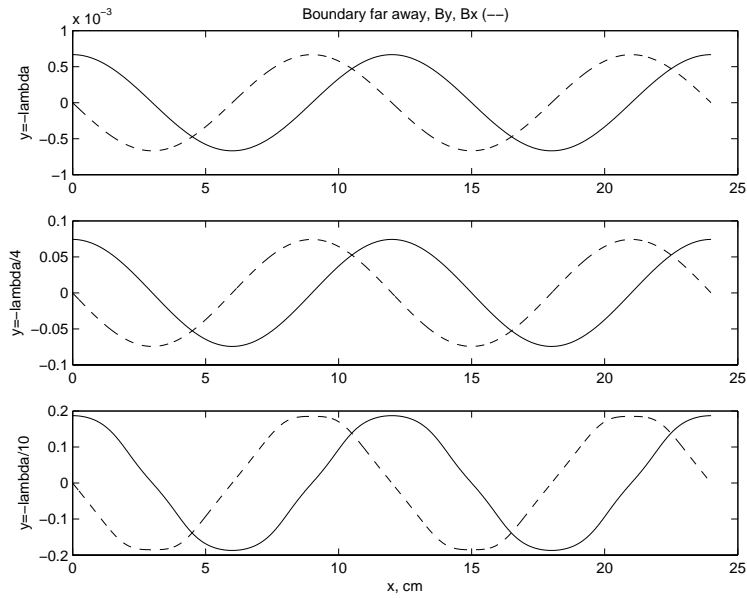


Figure 7: Fields with infinite gap.  $B_y$  is solid,  $B_x$  is dashed

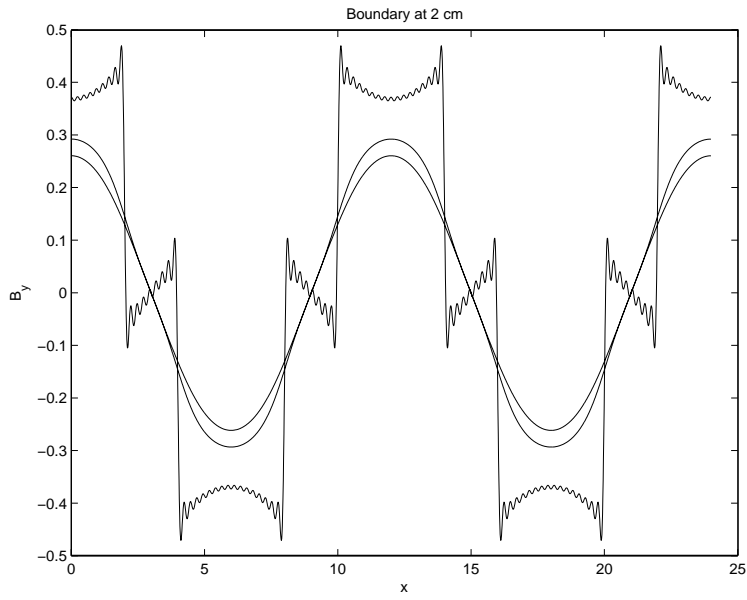


Figure 8: Fields with 2 cm gap: Fields at Stator Surface, Magnet Surface and Mid Gap

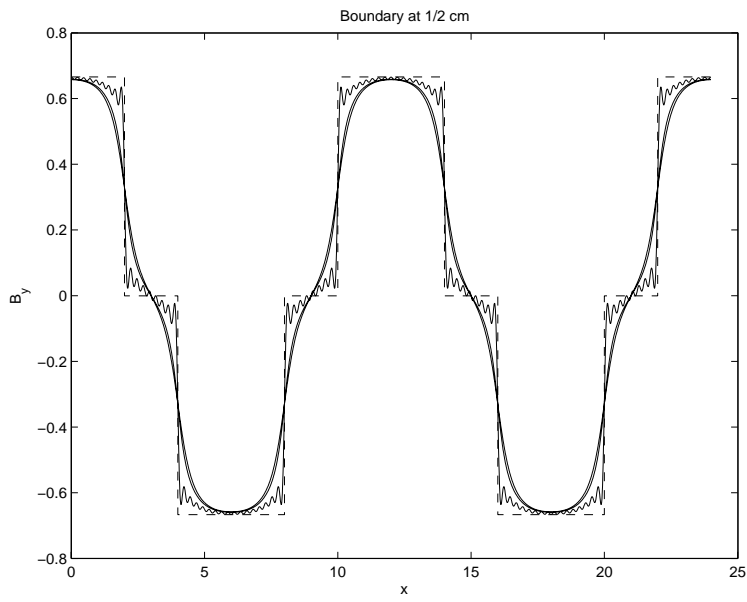


Figure 9: Fields with 1/2 cm gap: Stator Surface, Magnet Surface and halfway in between are solid. Narrow gap assumption field is dashed

```

% 6.685 2003 Problem Set 9, Problem 1
Isat = 2;

% first, test inductance procedure
dth = pi/1200;
theta = -pi/2:dth:pi/2;
L = zeros(size(theta));
dL = zeros(size(theta));
for i=1:length(theta),
    [L(i), dL(i)] = srmind(theta(i));
end

figure(1)
subplot 211
plot(theta, L)
title('Problem Set 9, Problem 1')
ylabel('Inductance, Hy')
axis([-pi/2 pi/2 0 0.6])
subplot 212
plot(theta, dL)
ylabel('dL/dtheta')
xlabel('Angle, Radians')
axis([-pi/2 pi/2 -1 1])

% now we compute current over a range of fluxes
lambda = 0.2:0.2:2;

I = zeros(size(theta));
T = zeros(size(theta));

figure(2)
clf
hold on
figure (3)
clf
hold on

for k = 1:length(lambda),
    lam = lambda(k);
    for i = 1:length(theta),
        [I(i), T(i)] = cur(L(i), dL(i), lam);
    end
    figure(2)
    plot(theta, I)
    figure(3)
    plot(theta, T)
end
figure(2)
hold off
title('Problem Set 9, Problem 1')
ylabel('Current, A')

```



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xlabel('Angle, Radians')

figure(3)
title('Problem Set 9, Problem 1')
ylabel('Torque, N-m')
xlabel('Angle, Radians')

% now that we know what is up, we construct a steady state waveform
% we use only one pulse of flux
theta_s = -pi/2:dth:0;           % voltage pulse starts at these locations
theta_1 = -pi/6:dth:0;
theta_2 = 0:dth:pi/6;
theta_v = [theta_1 theta_2];
lambda_1 = 2*(6/pi) .* (theta_1 + pi/6);
lambda_2 = 2*(6/pi) .* (pi/6 - theta_2);
lambda_v = [lambda_1 lambda_2];
Tav = zeros(size(theta_s));
for it = 1:length(theta_s);
    Lambda = zeros(size(theta));
    I_l = zeros(size(theta));
    T_l = zeros(size(theta));
    for il = 1:length(lambda_v),
        Lambda(it+il) = lambda_v(il);
        [I_l(it+il), T_l(it+il)] = cur(L(it+il), dL(it+il), lambda_v(il));
    end
    Tav(it) = (6/pi)*dth*sum(T_l);
end
figure(4)
plot(theta_s, Tav)
title('Problem Set 9, Problem 1')
ylabel('Average Torque')
xlabel('Starting Angle, Radians')

% then we pick out a motoring and generating case
im = 300;
Lambda = zeros(size(theta));
I_l = zeros(size(theta));
T_l = zeros(size(theta));
for il = 1:length(lambda_v),
    Lambda(im+il) = lambda_v(il);
    [I_l(im+il), T_l(im+il)] = cur(L(im+il), dL(im+il), lambda_v(il));
end
figure(5)
subplot 311
plot(theta, L, theta, Lambda)
title('Problem Set 9, Problem 1: Motoring (one pulse only)')
ylabel('Flux, Inductance')
subplot 312
plot(theta, I_l)
ylabel('Current')
subplot 313

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plot(theta, T_1)
ylabel('Torque')
xlabel('Angle')

ig = 480;
Lambda = zeros(size(theta));
I_1 = zeros(size(theta));
T_1 = zeros(size(theta));
for il = 1:length(lambda_v),
    Lambda(ig+il) = lambda_v(il);
    [I_1(ig+il), T_1(ig+il)] = cur(L(ig+il), dL(ig+il), lambda_v(il));
end
figure(6)
subplot 311
plot(theta, L, theta, Lambda)
title('Problem Set 9, Problem 1: Generating (one pulse only)')
ylabel('Flux, Inductance')
subplot 312
plot(theta, I_1)
ylabel('Current')
subplot 313
plot(theta, T_1)
ylabel('Torque')
xlabel('Angle')

```

```

function [L, dL] = srmind(th)
% returns VRM inductance per problem set 9, problem 1
tho = pi/3;           % overlap angle
thr = pi/2;           % repetition angle: 2*pi/Nr
Lmax = .5;            % these are max and min inductance
Lmin = .05;

th1 = tho/2;          % Overlap angle
thd = thr/2-th1*(1-Lmin/Lmax); % angle over which L does not vary!
thz = thr/2-th1;      % position of zero projection of L
thm = mod(th, thr);
thp = abs(thr/2-thm); % variation angle
if thp<thd,           % is the angle where L does not vary?
    L = Lmin;         % then we are at mininum inductance
else
    L = Lmax*(thp-thz)/th1; % this works because there is no region of max inductance
end

% now to get derivative
thc1 = th1*(1-Lmin/Lmax); % lower corner
thc2 = thr-thc1;         % upper corner

dLdt = Lmax/th1;
if thm < thc1,
    dL = -dLdt;
elseif thm > thc2,
    dL = dLdt;
else dL = 0;
end

function [I,T] = cur(L, dL, lam)
Isat = 2;
Lmin = 0.05;

if lam/L < Isat,
    I = lam/L;
    T = .5*I^2 * dL;
else
    I = lam/Lmin - Isat*(L/Lmin-1);
    T = (Isat*I-.5*Isat^2)* dL;
end

```

```

% Problem 9.2
lam = 12;           % carry dimensions as centimeters: wavelength
h = 1;             % magnet height
w = 4;             % magnet width
Br=1;              % flux density
thm = 2*pi*w/lam; % magnet angle

n = 1:2:51;        % vector of space harmonics
Bn = (4*Br/pi) .* sin(n .* thm/2) ./ n; % amplitudes
k = 2*pi/lam;

x = 0:lam/1000:2*lam;

figure(7)
subplot 311
y = -lam;
By = zeros(size(x));
Bx = zeros(size(x));

for i = 1:length(n)
    By = By+.5*Bn(i) * (1 - exp(-n(i) * 2*k*h)) *exp(n(i)*k*y) .* cos(n(i)*k .* x);
    Bx = Bx-.5*Bn(i) * (1 - exp(-n(i) * 2*k*h)) *exp(n(i)*k*y) .* sin(n(i)*k .* x);
end
plot(x, By, x, Bx, '--')
title('Boundary far away, By, Bx (--')
ylabel('y=-lambda')

subplot 312
y = -lam/4;
By = zeros(size(x));
Bx = zeros(size(x));

for i = 1:length(n)
    By = By+.5*Bn(i) * (1 - exp(-n(i) * 2*k*h)) *exp(n(i)*k*y) .* cos(n(i)*k .* x);
    Bx = Bx-.5*Bn(i) * (1 - exp(-n(i) * 2*k*h)) *exp(n(i)*k*y) .* sin(n(i)*k .* x);
end
plot(x, By, x, Bx, '--')
ylabel('y=-lambda/4')

subplot 313
y = -lam/10;
By = zeros(size(x));
Bx = zeros(size(x));

for i = 1:length(n)
    By = By+.5*Bn(i) * (1 - exp(-n(i) * 2*k*h)) *exp(n(i)*k*y) .* cos(n(i)*k .* x);
    Bx = Bx-.5*Bn(i) * (1 - exp(-n(i) * 2*k*h)) *exp(n(i)*k*y) .* sin(n(i)*k .* x);
end
plot(x, By, x, Bx, '--')
ylabel('y=-lambda/10')
xlabel('x, cm')

```

```

% now bring the boundary in

g = 2;
Bya = zeros(size(x));
Byb = zeros(size(x));
Byc = zeros(size(x));
ya = -g;
yb = -g/2;
yc = 0;

for i = 1:length(n)
    Bya = Bya + Bn(i) * (cosh(n(i)*k*(g+ya))*sinh(n(i)*k*h)/(sinh(n(i)*k*h)*cosh(n(i)*k*g)...
        +cosh(n(i)*k*h)*sinh(n(i)*k*g))) .* cos(n(i)*k*x);
    Byb = Byb + Bn(i) * (cosh(n(i)*k*(g+yb))*sinh(n(i)*k*h)/(sinh(n(i)*k*h)*cosh(n(i)*k*g)...
        +cosh(n(i)*k*h)*sinh(n(i)*k*g))) .* cos(n(i)*k*x);
    Byc = Byc + Bn(i) * (cosh(n(i)*k*(g+yc))*sinh(n(i)*k*h)/(sinh(n(i)*k*h)*cosh(n(i)*k*g)...
        +cosh(n(i)*k*h)*sinh(n(i)*k*g))) .* cos(n(i)*k*x);
end
figure(8)
plot(x, Bya, x, Byb, x, Byc)
title('Boundary at 2 cm')
ylabel('B_y')
xlabel('x')

g = 1/2;
Bya = zeros(size(x));
Byb = zeros(size(x));
Byc = zeros(size(x));
ya = -g;
yb = -g/2;
yc = 0;

for i = 1:length(n)
    Bya = Bya + Bn(i) * (cosh(n(i)*k*(g+ya))*sinh(n(i)*k*h)/(sinh(n(i)*k*h)*cosh(n(i)*k*g)...
        +cosh(n(i)*k*h)*sinh(n(i)*k*g))) .* cos(n(i)*k*x);
    Byb = Byb + Bn(i) * (cosh(n(i)*k*(g+yb))*sinh(n(i)*k*h)/(sinh(n(i)*k*h)*cosh(n(i)*k*g)...
        +cosh(n(i)*k*h)*sinh(n(i)*k*g))) .* cos(n(i)*k*x);
    Byc = Byc + Bn(i) * (cosh(n(i)*k*(g+yc))*sinh(n(i)*k*h)/(sinh(n(i)*k*h)*cosh(n(i)*k*g)...
        +cosh(n(i)*k*h)*sinh(n(i)*k*g))) .* cos(n(i)*k*x);
end

% now narrow gap assumption
hm = h/(h+g);
xs = [0 2 2 4 4 8 8 10 10 14 14 16 16 20 20 22 22 24];
Bs = [hm hm 0 0 -hm -hm 0 0 hm hm 0 0 -hm -hm 0 0 hm hm];
figure(9)
plot(x, Bya, x, Byb, x, Byc, xs, Bs, '--')
title('Boundary at 1/2 cm')
ylabel('B_y')
xlabel('x')

```