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Problem Set 4 - Solutions

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Problem 4.1

A

By Gauss' law, $\nabla \cdot \mathbf{D} = \nabla \cdot (\varepsilon \mathbf{E}) = \rho = 0$ for $a < r < b$. In cylindrical coordinates,

$$\nabla \cdot (\varepsilon \mathbf{E}) = \frac{1}{r}\frac{\partial}{\partial r}(r\varepsilon E_r) = 0,$$

so

$$\frac{1}{r}\frac{\partial}{\partial r}(r\varepsilon E_r) = 0 \implies \frac{\partial}{\partial r}(r\varepsilon E_r) = 0 \implies E_r(r) = \frac{A(t)}{r}.$$

$A(t)$ only depends on time and not on radius.

B

$$v(t) = \int_a^b E_r(r)\, dr = \int_a^b \frac{A(t)}{r} dr = A(t)\ln\left(\frac{b}{a}\right),$$

so

$$A(t) = \frac{v(t)}{\ln\left(\frac{b}{a}\right)} \implies E_r(r) = \frac{v(t)}{r\ln\left(\frac{b}{a}\right)}.$$

C

Resistance $R = V/i$,

$$\mathbf{J} = \sigma\mathbf{E} \implies J_r = \sigma E_r = \frac{\sigma v(t)}{r\ln\left(\frac{b}{a}\right)}$$

$$i = \oint J_r\, da = \frac{\sigma v(t)}{r\ln\left(\frac{b}{a}\right)} 2\pi r l = \frac{2\pi l\sigma v(t)}{\ln\left(\frac{b}{a}\right)} \implies R = \frac{v}{i} = \frac{\ln\left(\frac{b}{a}\right)}{2\pi l\sigma}$$

Capacitance $C = Q/V$. By Gauss' law boundary condition, the surface charge density is

$$\sigma_s = \left[\varepsilon E_r(r=a^+) - \varepsilon E_r(r=a^-)\right] = \varepsilon E_r(r=a^+) = \frac{\varepsilon v(t)}{a\ln\left(\frac{b}{a}\right)}$$

$$Q = 2\pi a l\sigma_s = \frac{2\pi a l\varepsilon v(t)}{a\ln\left(\frac{b}{a}\right)} \implies C = \frac{2\pi l\varepsilon}{\ln\left(\frac{b}{a}\right)}$$

D

By Ampere's law, $\nabla \times \mathbf{H} = \mathbf{0}$. In cylindrical coordinates,

$$\nabla \times \mathbf{H} = \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\hat{\mathbf{e}}_z - \frac{\partial H_\phi}{\partial z}\hat{\mathbf{e}}_r = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = 0,\ \frac{\partial H_\phi}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = 0 \implies \frac{\partial}{\partial r}(rH_\phi) = 0 \implies H_\phi(r) = \frac{A(t)}{r}$$

E

Use Ampere's law:

$$\oint \mathbf{H} \cdot d\mathbf{s} = I \implies 2\pi r \frac{A(t)}{r} = I(t) \implies A(t) = \frac{I(t)}{2\pi} \implies \mathbf{H}(r,t) = \frac{I(t)}{2\pi r}\hat{\mathbf{e}}_\phi$$

F

Inductance $L = \Phi/I$

$$\Phi = \iint_S \mathbf{B} \cdot d\mathbf{S} = l \int_a^b \mu \frac{I(t)}{2\pi r} dr = \frac{\mu I(t) l}{2\pi r} \ln\left(\frac{b}{a}\right) \implies L = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a}\right)$$

G

$$RC = \frac{\ln\left(\frac{b}{a}\right)}{2\pi l \sigma} \frac{2\pi l \varepsilon}{\ln\left(\frac{b}{a}\right)} = \frac{\varepsilon}{\sigma}, \text{ the same RC as parallel plates.}$$

H

$$LC = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a}\right) \frac{2\pi l \varepsilon}{\ln\left(\frac{b}{a}\right)} = \mu \varepsilon l^2, \text{ the same LC as parallel plates of depth } l.$$

The speed of light in the material is $c_m = 1/\sqrt{\mu\varepsilon}$, so $LC = l^2/c_m^2$.

Problem 4.2

A

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \implies \nabla \cdot \mathbf{J} = 0 \text{ in DC steady state.}$$

$$\nabla \cdot \mathbf{J} = \frac{\partial J_x}{\partial x} \implies J_x = J_0 \text{ constant.}$$

$$\sigma(x)E(x) = J_0 \implies E(x) = \frac{J_0}{\sigma_0 e^{x/s}}$$

$$\int_0^s E(x)\, dx = \int_0^s \frac{J_0}{\sigma_0} e^{-x/s}\, dx = - \left.\frac{J_0 s}{\sigma_0} e^{-x/s}\right|_0^s = V_0$$

$$\frac{J_0 s}{\sigma_0}(1 - e^{-1}) = V_0 \implies J_0 = \frac{V_0 \sigma_0}{s(1 - e^{-1})}$$

$$E(x) = \frac{V_0}{(1 - e^{-1}) s e^{x/s}}$$

$$R = \frac{V_0}{i} = \frac{V_0}{J_0 l D} = \frac{s(1 - e^{-1})}{l D \sigma_0}$$

B

$$\rho_f = \varepsilon \frac{dE}{dx} = \frac{-\varepsilon V_0 e^{-x/s}}{(1 - e^{-1}) s^2}$$

$$\sigma_f(x = 0) = \varepsilon E(x = 0) = \frac{\varepsilon V_0}{(1 - e^{-1}) s}$$

$$\sigma_f(x = s) = -\varepsilon E(x = s) = \frac{-\varepsilon V_0}{(e - 1) s}$$

C

$$Q_v = ld\int_0^s \frac{-\varepsilon V_0 e^{-x/s}}{(1-e^{-1})s^2}dx = ld\frac{-\varepsilon V_0}{(1-e^{-1})s^2}\int_0^s e^{-x/s}\,dx = \frac{-ld\varepsilon V_0}{s}$$

$$Q_s(x=0) = \frac{\varepsilon V_0}{(1-e^{-1})s}ld$$

$$Q_s(x=s) = \frac{-\varepsilon V_0}{(e-1)s}ld$$

$$Q_{\text{total}} = Q_v + Q_s(x=0) + Q_s(x=s) = \frac{-ld\varepsilon V_0}{s} + \frac{\varepsilon V_0}{(1-e^{-1})s}ld + \frac{-\varepsilon V_0}{(e-1)s}ld = 0$$

Problem 4.3

A

$$\rho_f(t) = \rho_0\frac{x}{s}e^{-t/\tau},\ \tau = \frac{\varepsilon}{\sigma}$$

B

$$\nabla\cdot\mathbf{E} = \frac{\partial E_x}{\partial x} = \frac{\rho_f(t)}{\varepsilon} \implies E_x = \rho_0\frac{x^2}{2\varepsilon s}e^{-t/\tau} + C(t)$$

$$\int_0^s E_x\,dx = \rho_0\frac{s^2}{6\varepsilon}e^{-t/\tau} + C(t)s = 0 \implies C(t) = -\rho_0\frac{s}{6\varepsilon}e^{-t/\tau}$$

$$E_x = \rho_0\frac{x^2}{2\varepsilon s}e^{-t/\tau} + \rho_0\frac{s}{6\varepsilon}e^{-t/\tau} = \rho_0\frac{1}{2\varepsilon s}e^{-t/\tau}\left(x^2 - \frac{s^2}{3}\right)$$

C

$$\sigma_f(x=0) = \varepsilon E(x=0) = -\rho_0\frac{s}{6}e^{-t/\tau}$$

$$\sigma_f(x=s) = -\varepsilon E(x=s) = -\rho_0\frac{s}{3}e^{-t/\tau}$$

D

$$\frac{i(t)}{A} = \sigma E_x(x=s) + \varepsilon\frac{\partial E_x}{\partial t}(x=s) = \rho_0\frac{\sigma s}{3\varepsilon}e^{-t/\tau} - \frac{1}{\tau}\rho_0\frac{s}{3}e^{-t/\tau} = 0$$

Problem 4.4

A

$$\nabla^2\Phi(x,y) = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0, \qquad \Phi = X(x)Y(y)$$

$$\to Y(y)\frac{d^2X(x)}{dx^2} + X(x)\frac{d^2Y(y)}{dy^2} = 0$$

$$\text{Rearranging} \to \underbrace{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{\text{function of } x \text{ only}} + \underbrace{\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}}_{\text{function of } y \text{ only}} = 0$$

The only way the two terms can add to zero for every x and y value is if

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = k^2, \qquad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k^2$$

B

For $k^2 = 0$, we have

$$\frac{d^2X(x)}{dx^2} = 0, \text{ and } \frac{d^2Y(y)}{dy^2} = 0$$

Therefore,

$$\begin{cases} X(x) & = ax + b \\ Y(y) & = cy + d \end{cases} \implies \Phi = Axy + Bx + Cy + D,$$

where we have used $\boxed{\Phi = X(x)Y(y)}$, and a, b, c, d, A, B, C, D are arbitrary constants.

C

Boundary Conditions:

$$\Phi(x,y) = \begin{cases} 0; & x = 0 & (1) \\ 0; & y = 0 & (2) \\ V_0; & xy = ab & (3) \end{cases}$$

$\Phi(x,y) = Axy + Bx + Cy + D$ (we know $\Phi(x,y)$ is of this form)

Boundary condition (1) $\Phi(x = 0, y) = 0 \implies Cy + D = 0$. This has to hold for *every* value of y. This means that $\boxed{C = 0}$ and $\boxed{D = 0}$.
Boundary condition (2) $\Phi(x, y = 0) = 0 \implies Bx + D = 0$. We already know that $D = 0$, so $Bx = 0$. This has to hold for *every* value of x, so $\boxed{B = 0}$.
Boundary condition (3) $\Phi(x,y)$ such that $xy = ab = V_0$. We know $D = 0,\ C = 0,\ B = 0$, so $\Phi(x,y) = Axy$ on the boundary $xy = ab$.

$$\Phi(x,y) = Aab = V_0 \implies A = \frac{V_0}{ab} \to \boxed{\Phi(x,y) = \frac{V_0}{ab}xy}$$

D

$$\mathbf{E}_1 = -\nabla\Phi = -\frac{\partial\Phi}{\partial x}\,\hat{\mathbf{x}} - \frac{\partial\Phi}{\partial y}\,\hat{\mathbf{y}} - \underbrace{\frac{\partial\Phi}{\partial z}}_{0}\,\hat{\mathbf{z}} = -\frac{V_0}{ab}y\,\hat{\mathbf{x}} - \frac{V_0}{ab}x\,\hat{\mathbf{y}}$$

We use the boundary condition $\hat{\mathbf{n}} \cdot [\mathbf{E}_1 - \mathbf{E}_2] = \sigma_s$ on the $x = 0$ plane and the normal $\hat{\mathbf{n}} = \hat{\mathbf{x}}$.

$$\hat{\mathbf{x}} \cdot \underbrace{[\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2^{\,\nearrow 0}]}_{\text{b/c perfect conductor}} = \sigma_s \implies \sigma_s = +\varepsilon_1 E_{1,x} = -\frac{\varepsilon_0 V_0}{ab}y$$

On the $y = 0$ plane, $\hat{\mathbf{n}} = \hat{\mathbf{y}}$ and

$$\hat{\mathbf{y}} \cdot [\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2^{\,\nearrow 0}] = -\frac{\varepsilon_0 V_0}{ab}x = \sigma_s$$

E

$$\frac{dy}{dx} = \frac{x}{y} \implies y\,dy = x\,dx \implies \frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$C = \frac{1}{2}(y_0^2 - x_0^2) \implies y^2 - x^2 = (y_0^2 - x_0^2)$$

F

$$\frac{1}{X(x)\frac{d^2X(x)}{dx^2}} = +k^2 \text{ and } \frac{1}{Y(y)\frac{d^2Y(y)}{dy^2}} = -k^2$$

The solution is

$$X(x) = Ae^{kx} + Be^{-kx}$$
$$Y(y) = C\sin(ky) + D\cos(ky)$$

where $A,\ ,B,\ C$, and D are arbitrary constants.

$$\Phi(x,y) = X(x)Y(y) = [a\sin(ky) + b\cos(ky)]e^{kx} + [c\sin(ky) + d\cos(ky)]e^{-kx}$$

where $a,\ b,\ c$, and d are arbitrary constants.

G

$$\Phi_1(x,y) = [a_1\sin(ky) + b_1\cos(ky)]e^{kx} + [c_1\sin(ky) + d_1\cos(ky)]e^{-kx}$$
$$\Phi_2(x,y) = [a_2\sin(ky) + b_2\cos(ky)]e^{-kx} + [c_2\sin(ky) + d_2\cos(ky)]e^{kx}$$

Region (1) is for $x \geq 0$ and Region (2) is for $x \leq 0$.
Boundary Conditions:

(1) $\Phi_1(x,y) = 0; x \to \infty$

(2) $\Phi_2(x,y) = 0; x \to -\infty$

(3) $\Phi_1(x,y)|_{x=0} = \Phi_2(x,y)|_{x=0} = V_0\sin(ay)$

Boundary condition (1) $\implies$ no e^{kx} terms for $\Phi_1(x,y)$ because they blow up as $x \to \infty$, so

$$\Phi_1(x,y) = [c_1\sin(ky) + d_1\cos(ky)]e^{-kx}.$$

Boundary condition (2) $\implies$ no e^{-kx} terms for $\Phi_2(x,y)$ because they blow up as $x \to -\infty$, so

$$\Phi_2(x,y) = [c_2\sin(ky) + d_2\cos(ky)]e^{kx}.$$

Boundary condition (3) $\implies$

$$c_1\sin(ky) + d_1\cos(ky) = c_2\sin(ky) + d_2\cos(ky) = V_0\sin(ay).$$

Clearly $c_1 = c_2$ and $d_1 = d_2$ because sine and cosine are independent (you can't make a sine equal a cosine for all y). That said,

$$c_1 = c_2 = V_0, \qquad d_1 = d_2 = 0, \qquad k = a$$

$$\Phi_1 = V_0\sin(ay)e^{-ax}; \quad x \geq 0$$
$$\Phi_2 = V_0\sin(ay)e^{ax}; \quad x \leq 0$$

H

$$\mathbf{E}_1 = -\nabla\Phi = \left(-\frac{\partial\Phi}{\partial x}\right)\hat{\mathbf{x}} + \left(-\frac{\partial\Phi}{\partial y}\right)\hat{\mathbf{y}} + \cancelto{0}{\left(-\frac{\partial\Phi}{\partial x}\right)}\hat{\mathbf{z}}$$

$$\mathbf{E}_1 = aV_0\sin(ay)e^{-ax}\,\hat{\mathbf{x}} - aV_0\cos(ay)e^{-ax}\,\hat{\mathbf{y}}$$
$$\mathbf{E}_2 = -aV_0\sin(ay)e^{ax}\,\hat{\mathbf{x}} - aV_0\cos(ay)e^{ax}\,\hat{\mathbf{y}}$$

To find the surface charge we need to use the condition $\hat{\mathbf{n}}\cdot[\varepsilon_1\mathbf{E}_1 - \varepsilon_2\mathbf{E}_2] = \sigma_s$ at $x = 0$, where $\hat{\mathbf{n}} = \hat{\mathbf{x}} \implies \varepsilon_1 E_{1,x}|_{x=0} - \varepsilon_2 E_{2,x}|_{x=0} = \sigma_s$.

$$E_{1,x}|_{x=0} = aV_0\sin(ay), \qquad E_{2,x}|_{x=0} = -aV_0\sin(ay)$$

$$\boxed{\sigma_s = aV_0\sin(ay)(\varepsilon_1+\varepsilon_2)}$$

For $x > 0$,

$$\mathbf{E}_1 = aV_0e^{-ax}[\sin(ay)\,\hat{\mathbf{x}} - \cos(ay)\,\hat{\mathbf{y}}]$$

$$\frac{dy}{dx} = -\frac{\cos(ay)}{\sin(ay)} = -\cot(ay) \implies dx = -\frac{\sin(ay)}{\cos(ay)}dy$$

Let $u = \cos(ay)$ so that $du = -a\sin(ay)\,dy$:

$$dx = +\frac{1}{a}\frac{du}{u} \implies x = +\frac{1}{a}\ln(u) + C = +\frac{1}{a}\ln(\cos(ay)) + C$$

$$C = x_0 - \frac{1}{a}\ln(\cos(ay_0))$$

$$\boxed{(x - x_0) = +\frac{1}{a}\ln\left(\frac{\cos(ay)}{\cos(ay_0)}\right)}$$