6.013/ESD.013J — Electromagnetics and Applications	Fall 2005
Problem Set 5 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 5.1

A

$$\rho = \nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \varepsilon E_0 (\hat{\mathbf{x}} x + \hat{\mathbf{y}} y) \sin \omega t = 2 \varepsilon E_0 \sin \omega t$$

$$-\frac{\partial}{\partial t} \mu \mathbf{H} = \nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \mathbf{0} \\ x & y & 0 \end{vmatrix} E_0 \sin \omega t = \hat{\mathbf{z}} \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) E_0 \sin \omega t = 0$$

$$\Leftrightarrow \mathbf{H} = \mathbf{C}(\mathbf{r}) = \mathbf{0}$$

Time-independent magnetic field that could be set to zero, since it is not generated by the time dependent electric field.

$$\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial}{\partial t} \varepsilon \mathbf{E} = \mathbf{0} - \frac{\partial}{\partial t} \varepsilon E_0 (\hat{\mathbf{x}} x + \hat{\mathbf{y}} y) \sin \omega t = -\omega \varepsilon E_0 (\hat{\mathbf{x}} x + \hat{\mathbf{y}} y) \cos \omega t$$

B

$$\rho = \nabla \cdot \varepsilon \mathbf{E} = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \varepsilon E_0 (\hat{\mathbf{x}} y - \hat{\mathbf{y}} x) \cos \omega t = \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) \varepsilon E_0 \cos \omega t = 0$$

$$-\frac{\partial}{\partial t} \mu \mathbf{H} = \nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \mathbf{0} \\ y & -x & 0 \end{vmatrix} E_0 \cos \omega t = \hat{\mathbf{z}} \left(-2 E_0 \cos \omega t \right)$$

$$\Leftrightarrow \mathbf{H} = \hat{\mathbf{z}} \frac{2E_0}{\mu \omega} \sin \omega t + \cancelto{0}{\mathbf{C}(\mathbf{r})} = \hat{\mathbf{z}} \frac{2E_0}{\omega \mu} \sin \omega t$$

$$\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial}{\partial t} \varepsilon \mathbf{E} = \mathbf{0} - \left[-\omega \varepsilon E_0 (\hat{\mathbf{x}} y - \hat{\mathbf{y}} x) \sin \omega t \right] = \omega \varepsilon E_0 (\hat{\mathbf{x}} y - \hat{\mathbf{y}} x) \sin \omega t,$$

where the first term is $\mathbf{0}$ because $\mathbf{H}$ does not depend on position.

Problem 5.2

A

$$\omega = 4\pi \cdot 10^6 \ \frac{\text{rad}}{\text{sec}} \implies f = \frac{\omega}{2\pi} = 2 \cdot 10^6 \text{ Hz} = 2 \text{ MHz}$$

$$k = 4\pi \cdot 10^{-2} \ \frac{1}{\text{m}} \implies \lambda = \frac{2\pi}{k} = \frac{1}{2} \cdot 10^2 \text{ m} = 50 \text{ m}$$

$$c_n = \frac{\omega}{k} = \frac{4\pi \cdot 10^6}{4\pi \cdot 10^{-2}} \ \frac{\text{m}}{\text{sec}} = 10^8 \ \frac{\text{m}}{\text{sec}}$$

B

$$c_n = \frac{c}{n} \Leftrightarrow n = \frac{c}{c_n} = \frac{3 \cdot 10^8 \ \frac{\text{m}}{\text{sec}}}{10^8 \ \frac{\text{m}}{\text{sec}}} = 3$$

$$n = \sqrt{\varepsilon_r \mu_r} = \sqrt{\varepsilon_r \cdot 1} \Leftrightarrow \varepsilon_r = n^2 = 9$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} = \sqrt{\frac{\mu_r}{\varepsilon_r}}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{1}{9}} \cdot 120\pi \ \Omega = \frac{1}{3} \cdot 120\pi \ \Omega = 40\pi \ \Omega$$

$$\boxed{\text{Note:} \begin{cases} n = \text{index of refraction} \\ \eta = \text{impedance} \end{cases}}$$

$$-\frac{\partial}{\partial t}\mu_0 \mathbf{H} = \nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = +\hat{\mathbf{y}} \ \frac{\partial E_x}{\partial z} = \hat{\mathbf{y}} \ \frac{\partial}{\partial z} E_0 \cos(\omega t - kz)$$

$$\Longrightarrow \ -\frac{\partial}{\partial t}\mu_0 \mathbf{H} = \hat{\mathbf{y}} \ E_0 k \sin(\omega t - kz) \ \Longrightarrow \ \mathbf{H} = \hat{\mathbf{y}} \ \frac{E_0 k}{\omega \mu_0} \cos(\omega t - kz) + \cancelto{0}{\mathbf{C}(r)}$$

$$\Longrightarrow \ \mathbf{H} = \hat{\mathbf{y}} \ \frac{E_0}{c_n \mu_0} \cos(\omega t - kz) = \hat{\mathbf{y}} \ \frac{E_0}{\eta} \cos(\omega t - kz)$$

$$\Longrightarrow \ \mathbf{H} = \hat{\mathbf{y}} \ \frac{1}{4\pi} \cos(4\pi \cdot 10^6 t - 4\pi \cdot 10^{-2} z) \ \text{Ampères/m}$$

C

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ E_x & 0 & 0 \\ 0 & H_y & 0 \end{vmatrix} = \hat{\mathbf{z}} \ E_x H_y$$

$$\Longrightarrow \ \mathbf{S} = \hat{\mathbf{z}} \ E_0 \frac{E_0}{\eta} \cos^2(\omega t - kz) = \hat{\mathbf{z}} \ 10 \cdot \frac{1}{4\pi} \cos^2(4\pi \cdot 10^6 t - 4\pi \cdot 10^{-2} z) \ \frac{\text{W}}{\text{m}^2}$$

$$\Longrightarrow \ \mathbf{S} = \hat{\mathbf{z}} \ \frac{2.5}{\pi} \cos^2(4\pi \cdot 10^6 t - 4\pi \cdot 10^{-2} z) \ \frac{\text{W}}{\text{m}^2}$$

Problem 5.3

A

$$\frac{\partial}{\partial t}\mathbf{J} = {\omega_p}^2 \varepsilon \mathbf{E} \ \Longrightarrow \ \frac{\partial}{\partial t}\text{Re}[\hat{\mathbf{J}} e^{j\omega t}] = {\omega_p}^2 \varepsilon \text{Re}[\hat{\mathbf{E}} e^{j\omega t}]$$

$$\Longrightarrow \ \text{Re}[j\omega \hat{\mathbf{J}} e^{j\omega t}] = \text{Re}[{\omega_p}^2 \varepsilon \hat{\mathbf{E}} e^{j\omega t}] \ \Longrightarrow \ j\omega \hat{\mathbf{J}} = {\omega_p}^2 \varepsilon \hat{\mathbf{E}}$$

$$\Longrightarrow \ \sigma(\omega) = \frac{\hat{\mathbf{J}}}{\hat{\mathbf{E}}} = \frac{{\omega_p}^2 \varepsilon}{j\omega} = -j\varepsilon \frac{{\omega_p}^2}{\omega}$$

B

$$j\omega\varepsilon(\omega)\hat{\mathbf{E}} = \hat{\mathbf{J}} + j\omega\varepsilon\hat{\mathbf{E}} = \sigma(\omega)\hat{\mathbf{E}} + j\omega\varepsilon\hat{\mathbf{E}} \ \Longrightarrow \ j\omega\varepsilon(\omega) = \sigma(\omega) + j\omega\varepsilon$$

$$j\omega\varepsilon(\omega) = -j\varepsilon\frac{{\omega_p}^2}{\omega} + j\omega\varepsilon \ \Longrightarrow \ \varepsilon(\omega) = \varepsilon\left(1 - \frac{{\omega_p}^2}{\omega^2}\right)$$

C

$$k = \omega\sqrt{\varepsilon(\omega)\mu_0} \implies$$

- $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$
- $k_p = \omega\sqrt{\varepsilon\left(1 - \frac{\omega_p^{\,2}}{\omega^2}\right)\mu_0} = \omega\sqrt{\varepsilon\mu_0}\sqrt{1 - \frac{\omega_p^{\,2}}{\omega^2}} = \begin{cases} \omega\sqrt{\varepsilon\mu_0}\sqrt{1 - \frac{\omega_p^{\,2}}{\omega^2}}, & \omega \geq \omega_p \\ j\omega\sqrt{\varepsilon\mu_0}\sqrt{\frac{\omega_p^{\,2}}{\omega^2} - 1}, & \omega < \omega_p \end{cases}$

D

$$\nabla \times \hat{\mathbf{E}} = -j\omega\mu_0\hat{\mathbf{H}} \implies \hat{\mathbf{H}} = -\frac{1}{j\omega\mu_0}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \frac{\partial}{\partial z} \\ \hat{E}_x & 0 & 0 \end{vmatrix} = -\hat{\mathbf{y}}\,\frac{1}{j\omega\mu_0}\frac{\partial \hat{E}_x}{\partial z}$$

- $\hat{\mathbf{H}}_0 = -\hat{\mathbf{y}}\,\frac{jk_0}{j\omega\mu_0}\hat{E}_0 e^{jk_0 z} = -\hat{\mathbf{y}}\,\frac{k_0}{\omega\mu_0}\hat{E}_0 e^{jk_0 z},\ z < 0$
- $\hat{\mathbf{H}}_0 = -\hat{\mathbf{y}}\,\frac{-jk_p}{j\omega\mu_0}\hat{E}_p e^{-jk_p z} = \hat{\mathbf{y}}\,\frac{k_p}{\omega\mu_0}\hat{E}_p e^{-jk_p z},\ z > 0$

Note: The hat ˆ above x, y, z denotes a unit vector, but above field components denotes a complex phasor.

E

- $\hat{\mathbf{z}} \times (\hat{\mathbf{E}}_p - \hat{\mathbf{E}}_0) = \mathbf{0} \implies \hat{\mathbf{y}}(\hat{E}_p - \hat{E}_0) = \mathbf{0} \implies \hat{E}_p = \hat{E}_0 \equiv \hat{E}$
- $\hat{\mathbf{z}} \times (\hat{\mathbf{H}}_p - \hat{\mathbf{H}}_0) = \hat{\mathbf{K}} \implies -\hat{\mathbf{x}}(\hat{H}_p - \hat{H}_0) = K_0\hat{\mathbf{x}} \implies \hat{H}_p - \hat{H}_0 = -K_0$

F

$$\hat{H}_0 - \hat{H}_p = K_0 \overset{\text{D,E}}{\implies} -\frac{k_0}{\omega\mu_0}\hat{E} - \frac{k_p}{\omega\mu_0}\hat{E} = K_0 \implies \hat{E} = -\frac{\omega\mu_0}{k_0 + k_p}K_0$$

Therefore:

$$\hat{\mathbf{E}}(z) = \begin{cases} -\hat{\mathbf{x}}\,\frac{\omega\mu_0}{k_0+k_p}K_0 e^{-jk_p z}, & z > 0 \\ -\hat{\mathbf{x}}\,\frac{\omega\mu_0}{k_0+k_p}K_0 e^{jk_0 z}, & z < 0 \end{cases}$$

and

$$\hat{\mathbf{H}}(z) = \begin{cases} -\hat{\mathbf{y}}\,\frac{k_p}{k_0+k_p}K_0 e^{-jk_p z}, & z > 0 \\ \hat{\mathbf{y}}\,\frac{k_0}{k_0+k_p}K_0 e^{jk_0 z}, & z < 0 \end{cases}$$

G

In the field expressions above μ_0, ω, k_0, K_0 are real, therefore:

$$\langle \mathbf{S} \rangle = \frac{1}{2}\text{Re}[\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*] = \frac{1}{2}\text{Re}[\hat{E}_x\hat{H}_y^*]\hat{\mathbf{z}} \implies$$

- $\langle \mathbf{S}_0 \rangle = -\hat{\mathbf{z}}\,\frac{1}{2}\text{Re}\left[\frac{\omega\mu_0}{k_0 + k_p}K_0 e^{jk_0 z} \cdot \frac{k_0}{k_0 + k_p^*}K_0 e^{-jk_0 z}\right]$

$$= -\hat{\mathbf{z}}\,\frac{1}{2}\text{Re}\left[\frac{\omega\mu_0 k_0}{|k_0 + k_p|^2}K_0^2\right] = -\hat{\mathbf{z}}\,\frac{\omega\mu_0 k_0}{2|k_0 + k_p|^2}K_0^2$$

- $\langle \mathbf{S}_p \rangle = \hat{\mathbf{z}}\,\frac{1}{2}\text{Re}\left[\frac{\omega\mu_0}{k_0+k_p}K_0 e^{-jk_p z}\cdot\frac{k_p^*}{k_0+k_p^*}K_0 e^{+jk_p^* z}\right]$

$$= \hat{\mathbf{z}}\,\frac{1}{2}\text{Re}\left[\frac{\omega\mu_0 k_p^*}{|k_0+k_p|^2}K_0^2 e^{2\text{Im}\{k_p\}z}\right] = \hat{\mathbf{z}}\,\frac{\omega\mu_0}{2|k_0+k_p|^2}K_0^2\text{Re}\{k_p\}e^{2\text{Im}\{k_p\}z}$$

$$\Longrightarrow \langle \hat{\mathbf{S}}_p \rangle = \begin{cases} \hat{\mathbf{z}}\frac{\omega\mu_0}{2|k_0+k_p|^2}K_0^2\omega\sqrt{\varepsilon\mu_0}\sqrt{1-\frac{\omega_p{}^2}{\omega^2}}, & \omega \geq \omega_p \\ 0, & \omega < \omega_p \text{ (because Re}\{k_p\}=0) \end{cases}$$

When the wavevector is purely imaginary inside a medium, the fields decay exponentially (they are called "evanescent") and no power is carried by them.

Problem 5.4

A

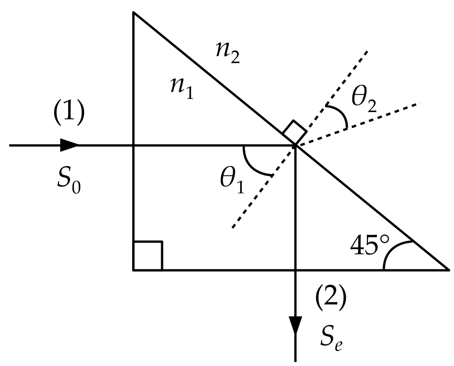

Figure 1: Diagram for Problem 5.4 Part A. (Image by MIT OpenCourseWare.)

For no power to be transmitted across the prism hypotenuse, the angle of incidence must be above the critical angle. In general, from Snell's Law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

and for total internal reflection:

$$\sin\theta_2 \geq 1 \implies \frac{n_1}{n_2}\sin\theta_1 \geq 1 \overset{\theta_1=45^\circ}{\Longrightarrow} n_1 \geq n_2\sqrt{2}$$

So for free space $(n_2 = 1):\ n_{1,\text{min}} = \sqrt{2} \approx 1.414$
and for water $(n_2 = 1.33):\ n_{1,\text{min}} = 1.33\sqrt{2} \approx 1.88$

B

The reflection coefficient at the input surface (1) is

$$r_{(1)} = \frac{n_1 - n_2}{n_1 + n_2} = r$$

and at the output surface (2) is

$$r_{(2)} = \frac{n_2 - n_1}{n_2 + n_1} = -r.$$

Therefore the reflectivity is

$$R = |r|^2 = \left|\frac{n_1 - n_2}{n_1 + n_2}\right|^2 = \left|\frac{\frac{n_1}{n_2} - 1}{\frac{n_1}{n_2} + 1}\right|^2.$$

For $n_1 \geq n_2\sqrt{2}$ no power is lost at the hypotenuse, so the power transmitted over the input is:

$$S_{(2)} = (1-R)(1-R)S_{(1)} \implies \frac{S_{(2)}}{S_{(1)}} = (1-R)^2 = \left[1 - \left(\frac{\frac{n_1}{n_2} - 1}{\frac{n_1}{n_2} + 1}\right)^2\right]^2$$

$$\implies \frac{S_{(2)}}{S_{(1)}} = \left\{\frac{\left[\left(\frac{n_1}{n_2}\right)^2 + 2\frac{n_1}{n_2} + 1\right] - \left[\left(\frac{n_1}{n_2}\right)^2 - 2\frac{n_1}{n_2} + 1\right]}{\left(\frac{n_1}{n_2} + 1\right)^2}\right\}^2 = \left[\frac{4\frac{n_1}{n_2}}{\left(\frac{n_1}{n_2} + 1\right)^2}\right]^2$$

For the values calculated in part A $n_1 = n_2\sqrt{2}$ for both cases, therefore:

$$\frac{S_{(2)}}{S_{(1)}} = \left[\frac{4\sqrt{2}}{(\sqrt{2}+1)^2}\right]^2 \approx 0.943$$

for both cases.