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Recitation 4

1 Sums of Random Variables

1.1 The Discrete Case

Let $W = X + Y$, where X and Y are independent integer-valued random variables with pmfs $f_X(x)$ and $f_Y(y)$. Then, for any integer w,

$$\begin{aligned} f_W(w) &= \mathbb{P}(X + Y = w) \\ &= \sum_{x+y=w} \mathbb{P}(X = x, Y = y) \\ &= \sum_x \mathbb{P}(X = x, Y = w - x) \\ &= \sum_x f_X(x) f_Y(w - x). \end{aligned}$$

The resulting pmf f_W is called the **convolution** of the pmfs of X and Y.

1.2 The Continuous Case

Let X and Y be independent continuous random variables with pdfs $f_X(x)$ and $f_Y(y)$. We wish to find the pdf of $W = X + Y$. We have

$$\begin{aligned} F_W(w) &= \mathbb{P}(W \leq w) \\ &= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_X(x) f_Y(y) dy dx \\ &= \int_{x=-\infty}^{\infty} f_X(x) F_Y(w - x) dx. \end{aligned}$$

Differentiating, and assuming everything is 'nice',

$$\begin{aligned} f_W(w) &= \frac{dF_W}{dw}(w) \\ &= \int_{x=-\infty}^{\infty} f_X(x) \frac{d}{dw} F_Y(w - x) dx \\ &= \int_{x=-\infty}^{\infty} f_X(x) f_Y(w - x) dx. \end{aligned}$$

This last formula is again known as the convolution of f_X and f_Y.

Problem. Suppose both X, Y are distributed according to the law $P(V \leq v) = 1 - e^{-\lambda v}$, when $v \geq 0$ and 0 otherwise. Find the pdf of $Z = X + Y$.
Solution: Using the convolution formula, for $z \geq 0$,

$$\begin{aligned} f_Z(z) &= \int_{x=-\infty}^{\infty} f_X(x) f_Y(z-x) dx \\ &= \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} dx \\ &= \lambda^2 e^{-\lambda z} \int_0^z dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

Problem: On the other hand, Gaussianity is preserved under convolution: the convolution of two Gaussians is a Gaussian. Let's work this out for Gaussians of mean zero and variance one. Suppose X and Y are independent and have the distribution

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Let $W = X + Y$. Then,

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{+\infty} e^{-x^2/2} e^{-(w-x)^2/2} dx \\ &= \int_{-\infty}^{+\infty} e^{-x^2} e^{-w^2/2 - wx} dx \\ &= e^{-w^2/4} \int_{-\infty}^{+\infty} e^{-(x-w/2)^2} dx \end{aligned}$$

and now observe that the integral actually does not depend on w, so that

$$f_W(w) = c e^{-w^2/4}.$$

Since the normalizing constant c must be chosen so that $f_W(w)$ integrates to 1, we recognize f_W as the density of a normal with mean 0 and variance 2.