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***High Speed Communication Circuits and Systems***

***Lecture 8***

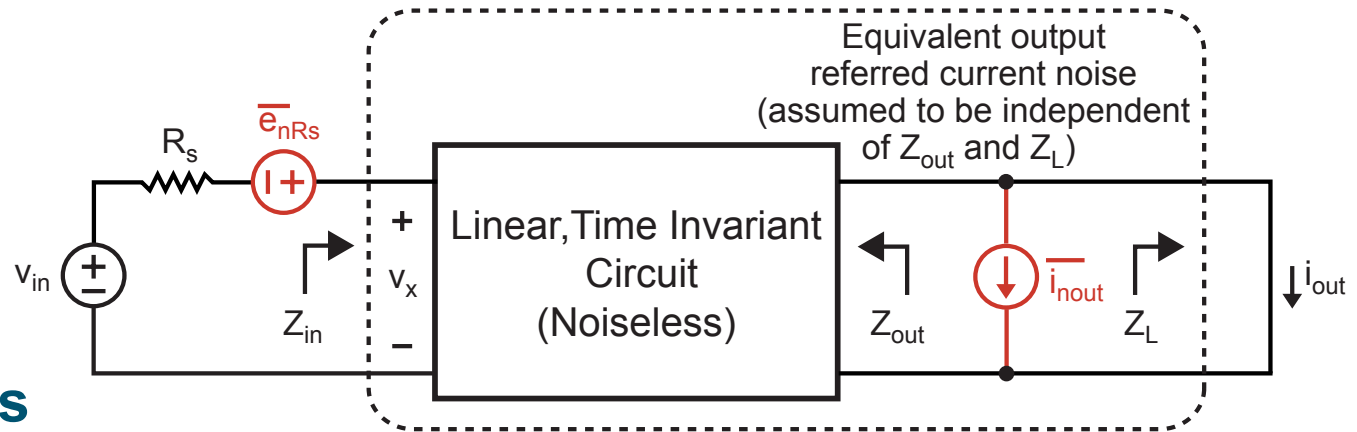
***Noise Figure, Impact of Amplifier Nonlinearities***

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# Noise Factor and Noise Figure (From Lec 7)



## ■ Definitions

$$\text{Noise Factor} = F = \frac{SNR_{in}}{SNR_{out}}$$

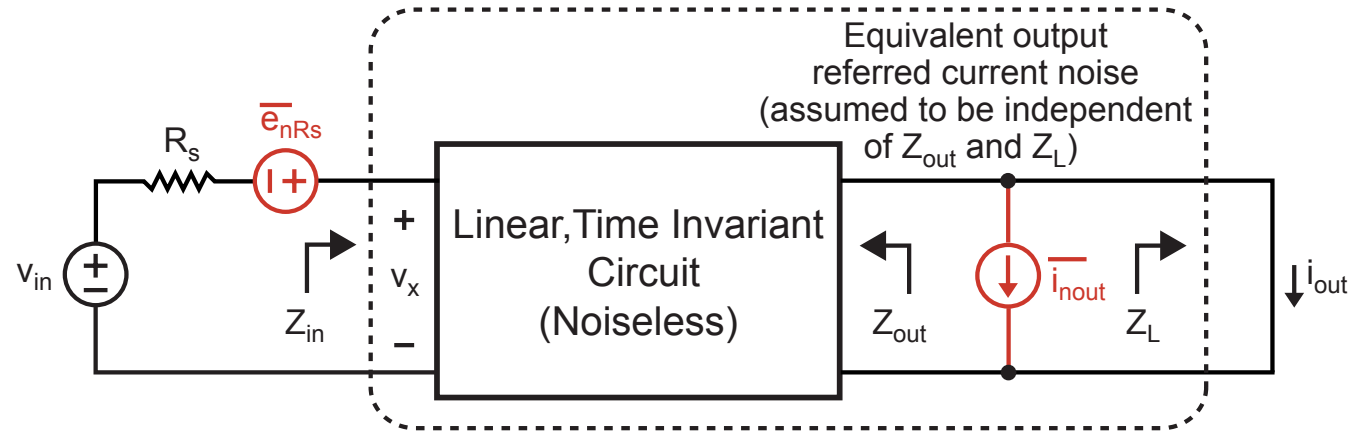
$$\text{Noise Figure} = 10 \log(\text{Noise Factor})$$

## ■ Calculation of $SNR_{in}$ and $SNR_{out}$

$$SNR_{in} = \frac{|\alpha|^2 v_{in}^2}{|\alpha|^2 e_{nRs}^2} = \frac{v_{in}^2}{e_{nRs}^2} \quad \text{where } \alpha = \frac{Z_{in}}{R_s + Z_{in}}$$

$$SNR_{out} = \frac{|\alpha|^2 |G_m|^2 v_{in}^2}{|\alpha|^2 |G_m|^2 e_{nRs}^2 + i_{nout}^2} \quad \text{where } G_m = \frac{i_{out}}{v_x}$$

# Alternative Noise Factor Expression



- From previous slide

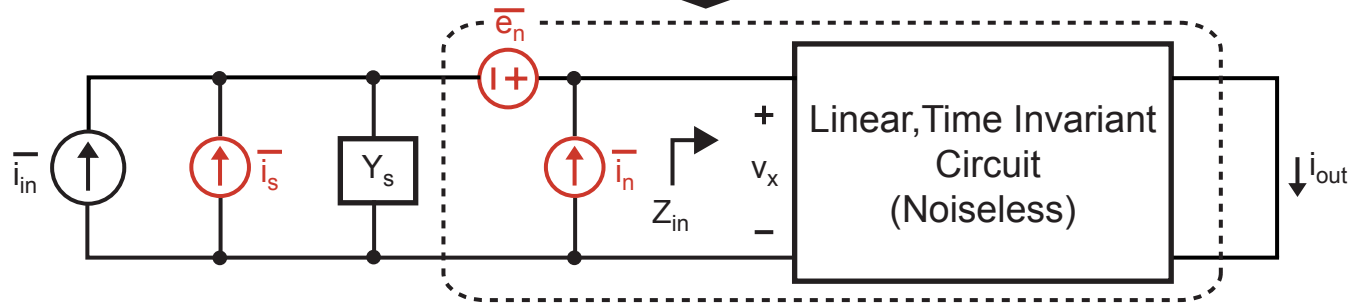
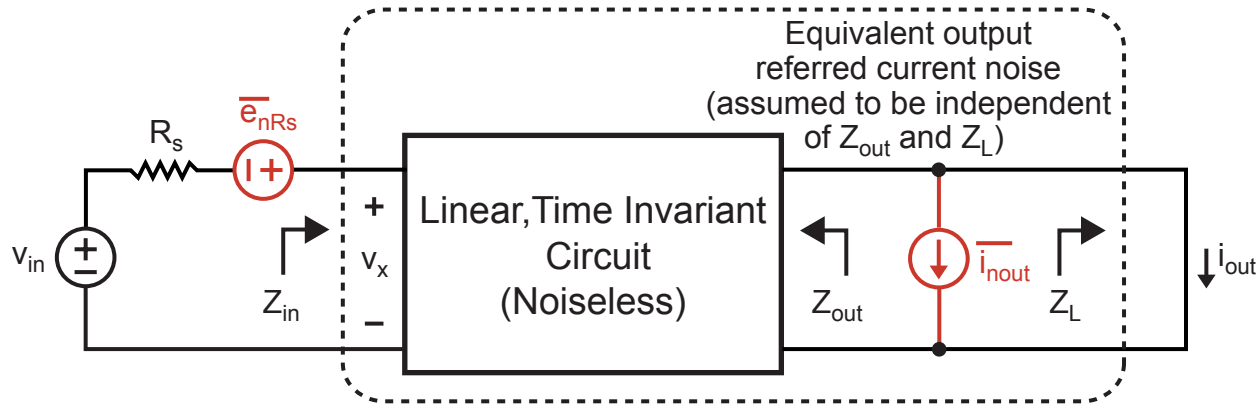
$$SNR_{in} = \frac{|\alpha|^2 v_{in}^2}{|\alpha|^2 e_{nRs}^2} = \frac{v_{in}^2}{e_{nRs}^2} \quad SNR_{out} = \frac{|\alpha|^2 |G_m|^2 v_{in}^2}{|\alpha|^2 |G_m|^2 e_{nRs}^2 + i_{nout}^2}$$

- Calculation of Noise Factor

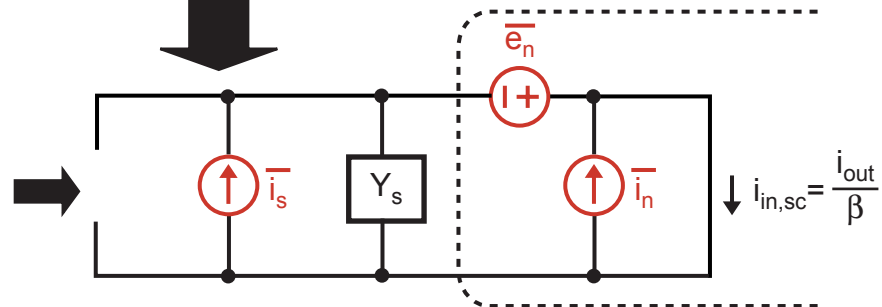
$$F = \frac{SNR_{in}}{SNR_{out}} = \frac{|\alpha|^2 |G_m|^2 e_{nRs}^2 + i_{nout}^2}{|\alpha|^2 |G_m|^2 e_{nRs}^2}$$

$$= \frac{\text{total output noise power}}{\text{output noise due to input source}}$$

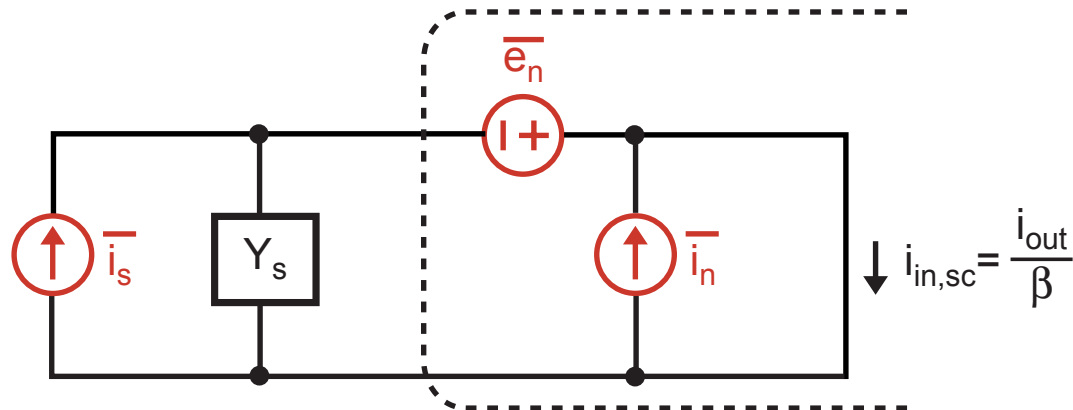
# Input Referred Noise Model



Can remove the signal source since Noise Factor can be expressed as the ratio of total output noise to input noise



# Input-Referred Noise Figure Expression



- We know that

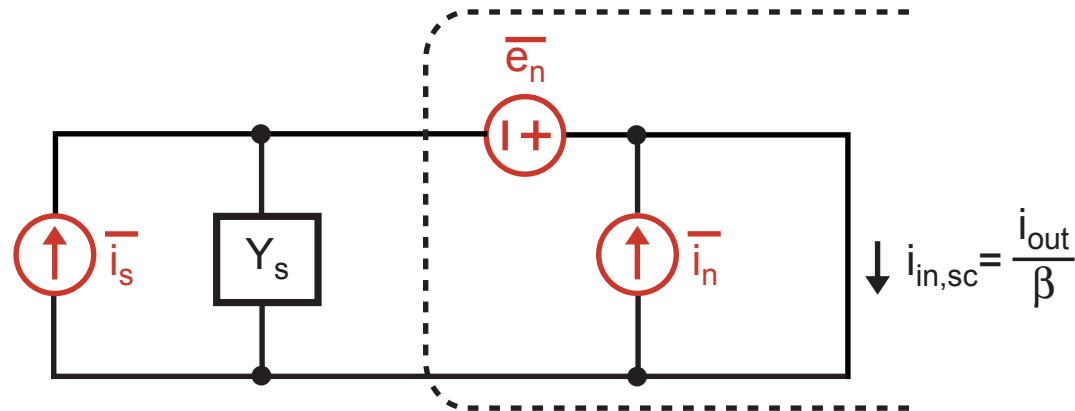
$$F = \frac{\text{total output noise power}}{\text{output noise due to input source}} = \frac{\overline{i_{out}^2(tot)}}{\overline{i_{out}^2(in)}}$$

- Let's express the above in terms of input short circuit current

$$F = \frac{|\beta|^2 \overline{i_{in,sc}^2(tot)}}{|\beta|^2 \overline{i_{in,sc}^2(in)}} = \frac{\overline{i_{in,sc}^2(tot)}}{\overline{i_{in,sc}^2(in)}}$$

$$= \frac{\text{total input short circuit current noise power}}{\text{input short circuit current noise power due to input source}}$$

# Calculation of Noise Factor



- By inspection of above figure

$$F = \frac{\overline{i_{in,sc}^2(tot)}}{\overline{i_{in,sc}^2(in)}} = \frac{\overline{i_s^2} + \overline{|i_n + Y_s e_n|^2}}{\overline{i_s^2}}$$

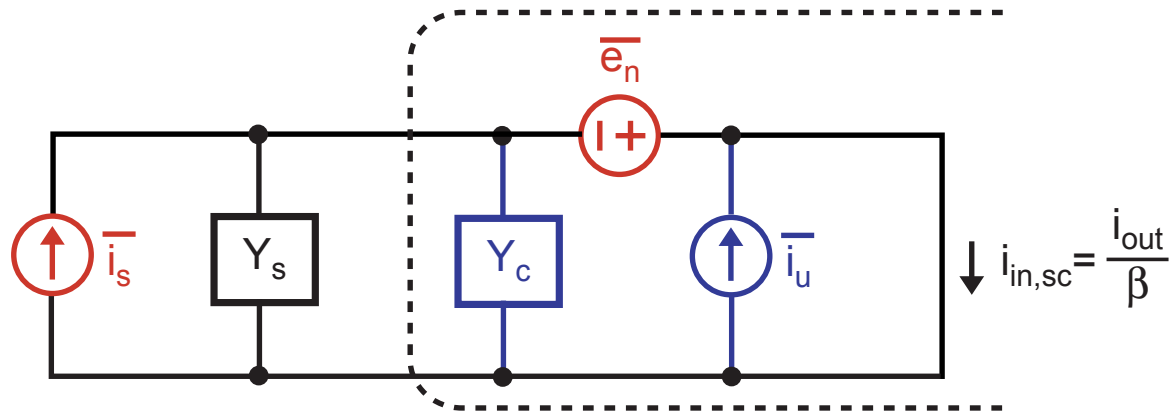
- In general,  $e_n$  and  $i_n$  will be correlated

$$i_n = i_c + i_u, \quad \text{where } i_c = Y_c e_n$$

- $Y_c$  is called the correlation admittance

$$\Rightarrow F = \frac{\overline{i_s^2} + \overline{|i_u + (Y_c + Y_s)e_n|^2}}{\overline{i_s^2}} = 1 + \frac{\overline{i_u^2} + \overline{|Y_c + Y_s|^2 e_n^2}}{\overline{i_s^2}}$$

# Noise Factor Expressed in Terms of Admittances



$$F = 1 + \frac{\bar{i}_u^2 + |Y_c + Y_s|^2 \bar{e}_n^2}{\bar{i}_s^2}$$

- We can replace voltage and current noise currents with impedances and admittances

$$R_n = \frac{\bar{e}_n^2}{4kT\Delta f}, \quad G_u = \frac{\bar{i}_u^2}{4kT\Delta f}, \quad G_s = \frac{\bar{i}_s^2}{4kT\Delta f}$$

$$F = 1 + \frac{G_u + |Y_c + Y_s|^2 R_n}{G_s}$$

## Optimal Source Admittance for Minimum Noise Factor

$$F = 1 + \frac{G_u + |Y_c + Y_s|^2 R_n}{G_s}$$

- Express admittances as the sum of conductance,  $G$ , and susceptance,  $B$

$$F = 1 + \frac{G_u + [(G_c + G_s)^2 + (B_c + B_s)^2] R_n}{G_s}$$

- Take the derivative with respect to source admittance and set to zero (to find minimum  $F$ ), which yields

$$G_s = \sqrt{\frac{G_u}{R_n} + G_c^2} = G_{opt} \quad B_s = -B_c = B_{opt}$$

- Plug these values into expression above to obtain

$$F_{min} = 1 + 2R_n \left( \sqrt{\frac{G_u}{R_n} + G_c^2} + G_c \right) = 1 + 2R_n(G_{opt} + G_c)$$



## Optimal Source Admittance for Minimum Noise Factor

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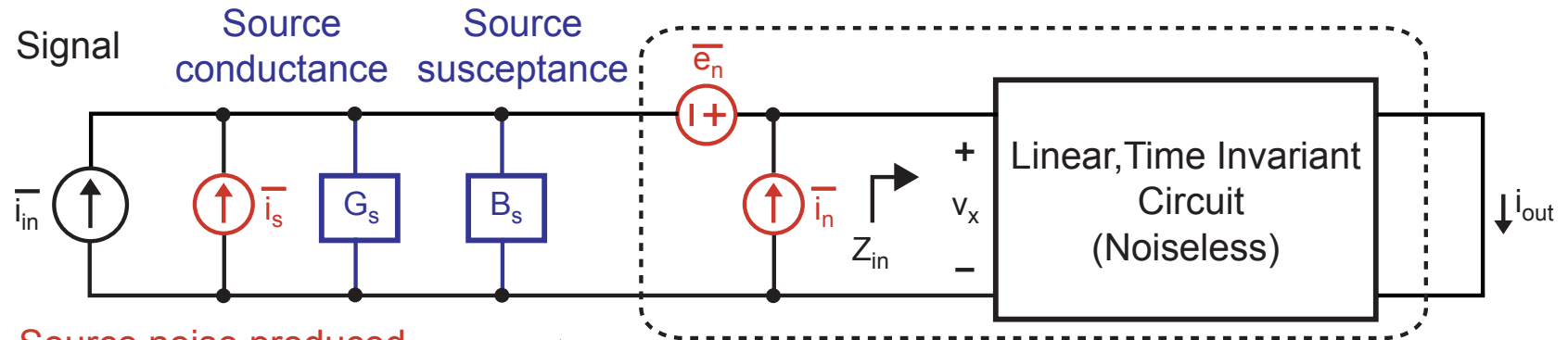
$$F_{min} = 1 + 2R_n \left( \sqrt{\frac{G_u}{R_n} + G_c^2} + G_c \right) \quad \text{at } G_s = G_{opt}, B_s = B_{opt}$$

- After much algebra (see Appendix L of Gonzalez book for derivation), we can derive

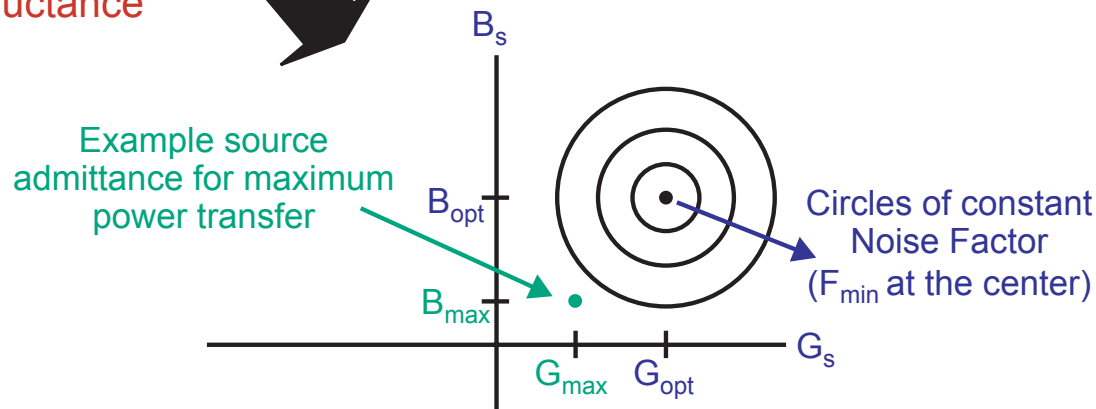
$$F = F_{min} + \frac{R_n}{G_s} \left[ (G_s - G_{opt})^2 + (B_s - B_{opt})^2 \right]$$

- Contours of constant noise factor are circles centered about  $(G_{opt}, B_{opt})$  in the admittance plane
- They are also circles on a Smith Chart (see pp 299-302 of Gonzalez for derivation and examples)
- How does  $(G_{opt}, B_{opt})$  compare to admittance achieving maximum power transfer?

# Optimizing For Noise Figure versus Power Transfer



Source noise produced by source conductance



- One cannot generally achieve minimum noise figure if maximum power transfer is desired

# Optimal Noise Factor for MOS Transistor Amp

- Consider the common source MOS amp (no degeneration) considered in Lecture 7
  - In Tom Lee's book (pp. 272-276), the noise impedances are derived as

$$G_c = 0 \quad B_c = wC_{gs} \left( 1 - \alpha|c| \sqrt{\frac{\delta}{5\gamma}} \right) \quad (\text{corrected})$$

$$R_n = \frac{\gamma g_{d0}}{g_m^2} = \left( \frac{\gamma}{\alpha} \right) \frac{1}{g_m} \quad G_u = \frac{\delta w^2 C_{gs}^2 (1 - |c|^2)}{5g_{d0}}$$

- The optimal source admittance values to minimize noise factor are therefore

$$B_{opt} = -B_c = -wC_{gs} \left( 1 - \alpha|c| \sqrt{\frac{\delta}{5\gamma}} \right)$$

$$G_{opt} = \sqrt{\frac{G_u}{R_n} + G_c^2} = \alpha w C_{gs} \sqrt{\frac{\delta}{5\gamma} (1 - |c|^2)}$$

## Optimal Noise Factor for MOS Transistor Amp (Cont.)

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- Optimal admittance consists of a resistor and inductor (wrong frequency behavior – broadband match fundamentally difficult)

$$B_{opt} = -B_c = -wC_{gs} \left( 1 - \alpha|c| \sqrt{\frac{\delta}{5\gamma}} \right)$$
$$G_{opt} = \sqrt{\frac{G_u}{R_n} + G_c^2} = \alpha w C_{gs} \sqrt{\frac{\delta}{5\gamma} (1 - |c|^2)}$$

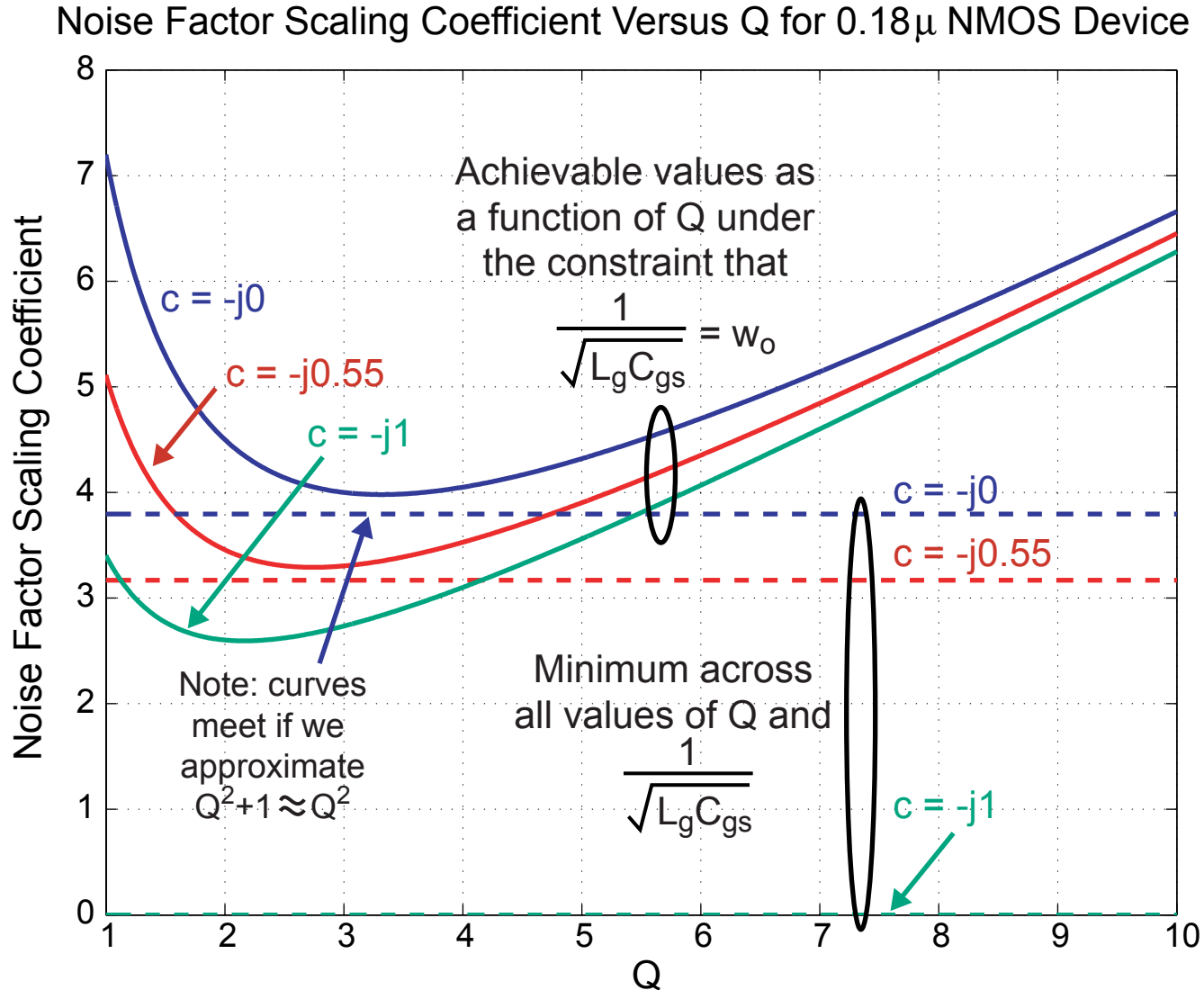
- If there is zero correlation, inductor value should be set to resonate with  $C_{gs}$  at frequency of operation

- Minimum noise figure

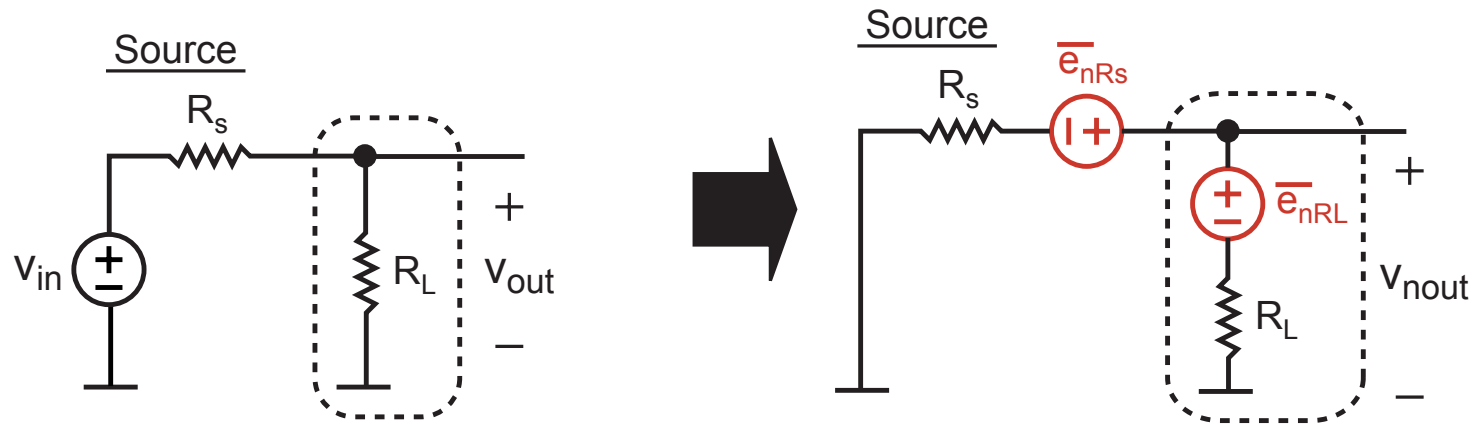
$$F_{min} = 1 + 2R_n(G_{opt} + G_c) \approx 1 + \frac{2}{\sqrt{5}} \frac{w}{w_t} \sqrt{\gamma\delta(1 - |c|^2)}$$

- Exact if one defines  $w_t = g_m/C_{gs}$

# Recall Noise Factor Comparison Plot From Lecture 7



## Example: Noise Factor Calculation for Resistor Load



### ■ Total output noise

$$\overline{v_{nout}^2} = \left( \frac{R_L}{R_s + R_L} \right)^2 \overline{e_{nRs}^2} + \left( \frac{R_s}{R_s + R_L} \right)^2 \overline{e_{nRL}^2}$$

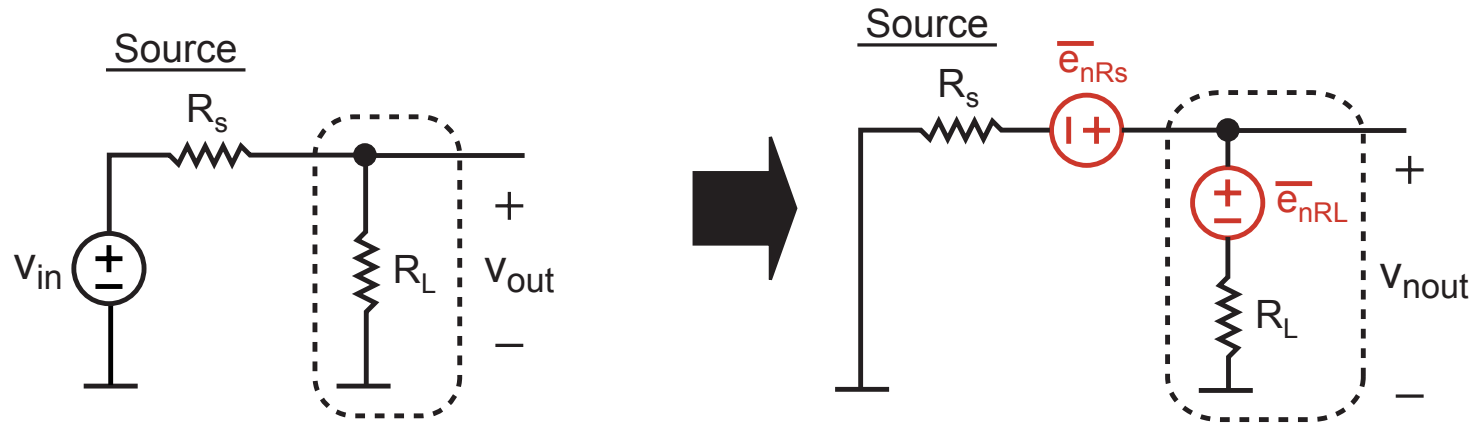
### ■ Total output noise due to source

$$\overline{v_{nout}^2} = \left( \frac{R_L}{R_s + R_L} \right)^2 \overline{e_{nRs}^2}$$

### ■ Noise Factor

$$F = 1 + \left( \frac{R_s}{R_L} \right)^2 \frac{\overline{e_{nRL}^2}}{\overline{e_{nRs}^2}} = 1 + \left( \frac{R_s}{R_L} \right)^2 \frac{4kTR_L}{4kTR_s} = \boxed{1 + \frac{R_s}{R_L}}$$

# Comparison of Noise Figure and Power Match



$$F = 1 + \frac{R_s}{R_L}$$

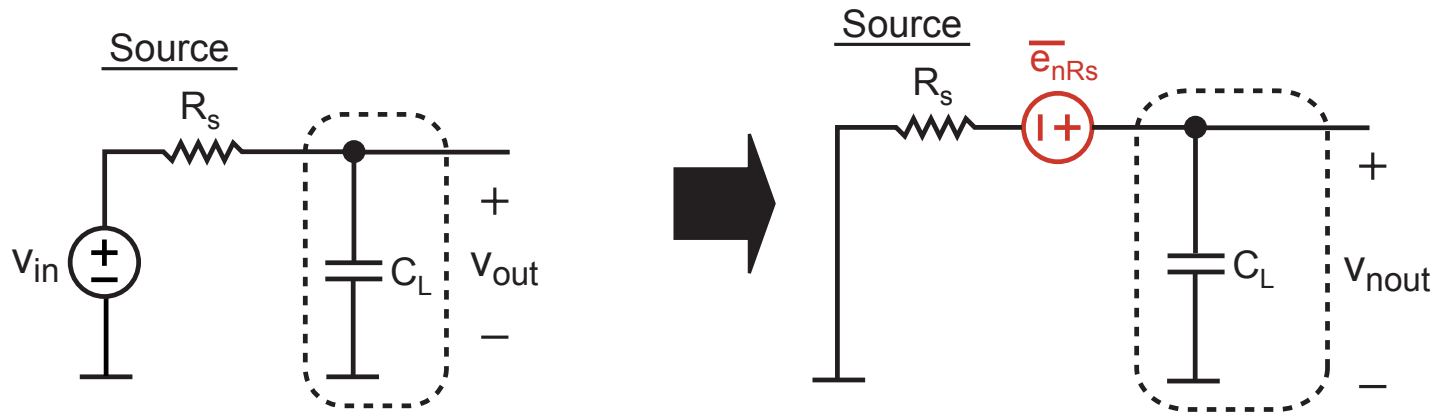
- To achieve minimum Noise Factor

$$R_L \rightarrow \infty$$

- To achieve maximum power transfer

$$R_L = R_s$$

## Example: Noise Factor Calculation for Capacitor Load



- **Total output noise**

$$\overline{v_{nout(tot)}^2} = \left( \frac{1/(sC_L)}{R_s + 1/(sC_L)} \right)^2 \overline{e_{nRs}^2}$$

- **Total output noise due to source**

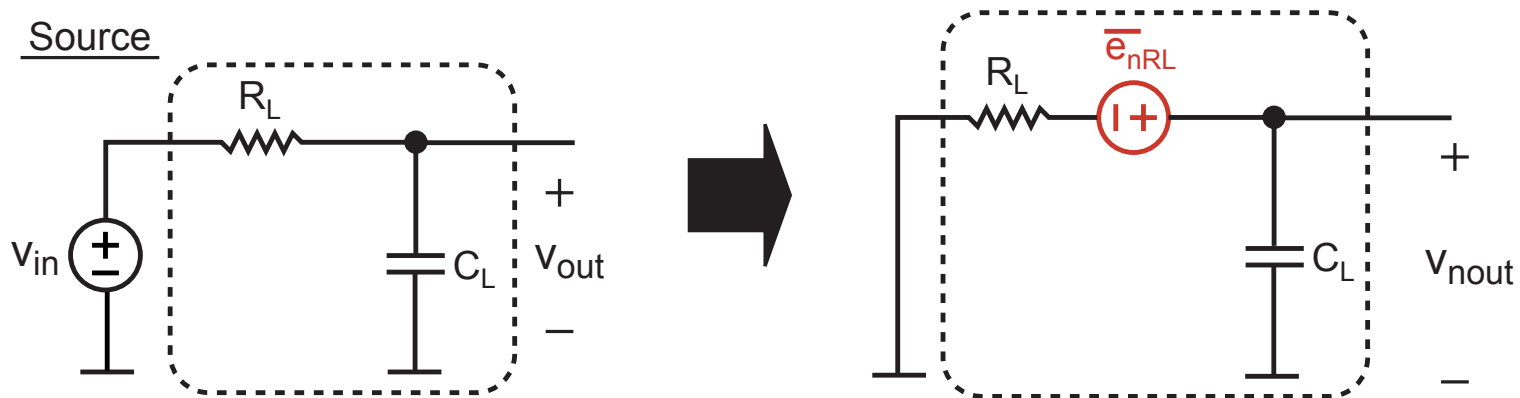
$$\overline{v_{nout(in)}^2} = \left( \frac{1/(sC_L)}{R_s + 1/(sC_L)} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = \overline{v_{nout(tot)}^2} / \overline{v_{nout(in)}^2} = 1$$



## Example: Noise Factor with Zero Source Resistance



- **Total output noise**

$$\overline{v_{nout}^2} = \left( \frac{1/(sC_L)}{R_s + 1/(sC_L)} \right)^2 \overline{e_{nRs}^2}$$

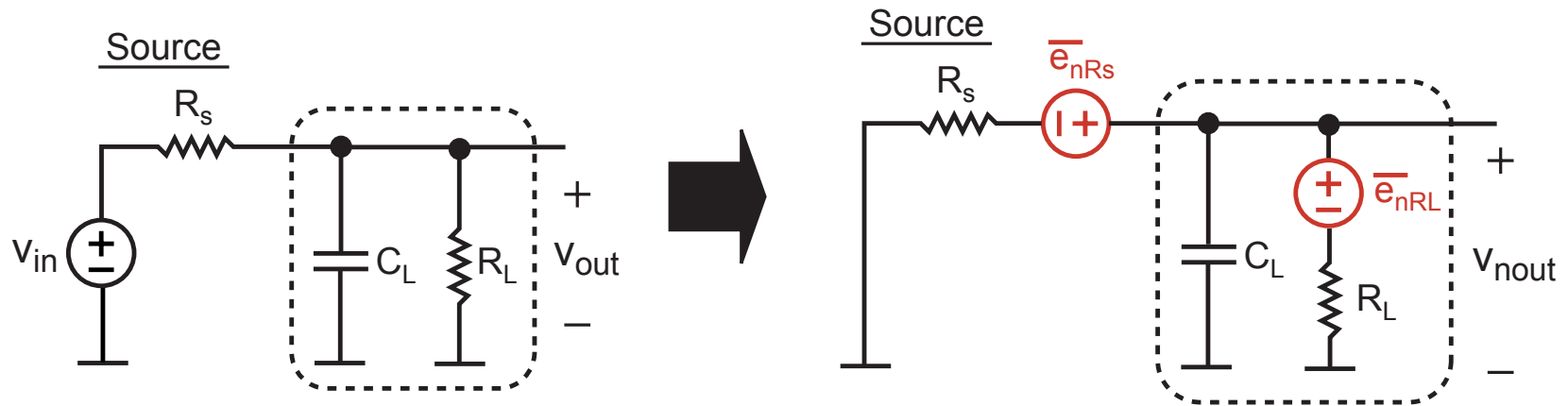
- **Total output noise due to source**

$$\overline{v_{nout(in)}^2} = 0$$

- **Noise Factor**

$$F = \overline{v_{nout(tot)}^2} / \overline{v_{nout(in)}^2} = \infty$$

## Example: Noise Factor Calculation for RC Load



- **Total output noise**

$$= \left( \frac{1/(sC_L) \parallel R_L}{R_s + 1/(sC_L) \parallel R_L} \right)^2 \overline{e_{nRs}^2} + \left( \frac{1/(sC_L) \parallel R_s}{R_L + 1/(sC_L) \parallel R_s} \right)^2 \overline{e_{nRL}^2}$$

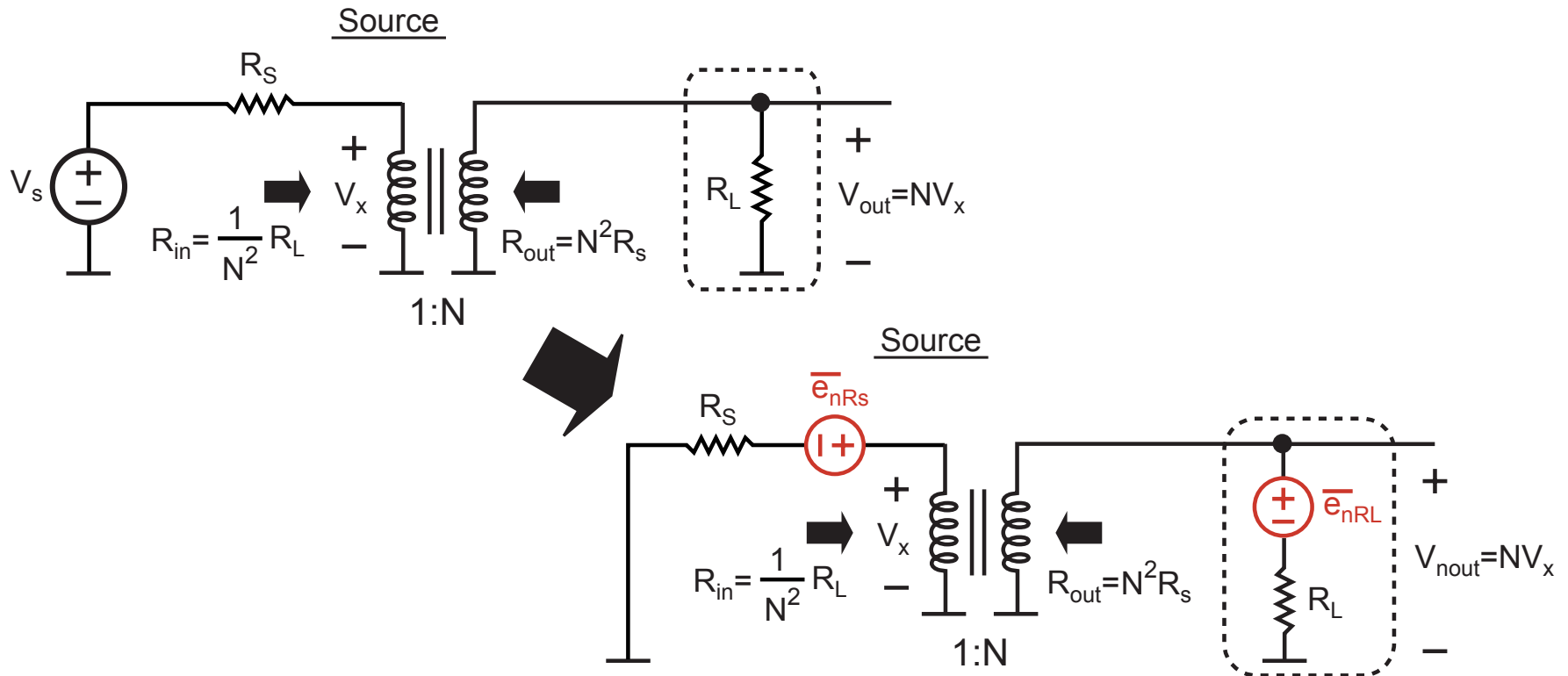
- **Total output noise due to source**

$$= \left( \frac{1/(sC_L) \parallel R_L}{R_s + 1/(sC_L) \parallel R_L} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = 1 + \left( \frac{R_s}{R_L} \right)^2 \frac{\overline{e_{nRL}^2}}{\overline{e_{nRs}^2}} = 1 + \left( \frac{R_s}{R_L} \right)^2 \frac{4kTR_L}{4kTR_s} = \boxed{1 + \frac{R_s}{R_L}}$$

## Example: Resistive Load with Source Transformer

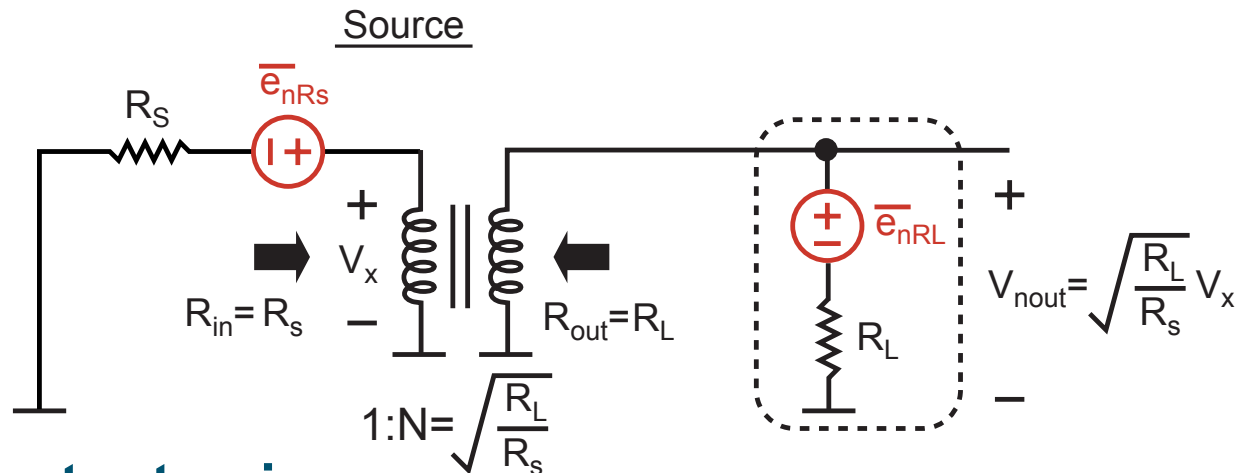


- For maximum power transfer (as derived in Lecture 3)

$$N^2 = \frac{R_L}{R_s}$$

$$\Rightarrow R_{in} = R_s, \quad R_{out} = R_L, \quad V_{out} = \sqrt{\frac{R_L}{R_s}} V_x$$

# Noise Factor with Transformer Set for Max Power Transfer



- **Total output noise**

$$\overline{v_{nout}^2} = \left( \frac{R_s}{R_s + R_s} \sqrt{\frac{R_L}{R_s}} \right)^2 \overline{e_{nRs}^2} + \left( \frac{R_L}{R_L + R_L} \right)^2 \overline{e_{nRL}^2}$$

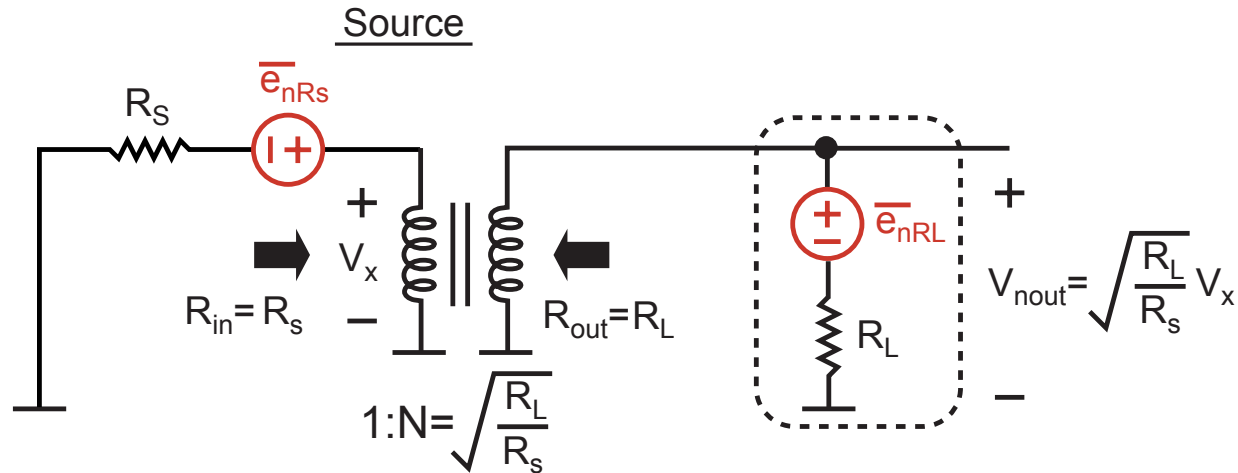
- **Total output noise due to source**

$$\overline{v_{nout}^2} = \left( \frac{R_s}{R_s + R_s} \sqrt{\frac{R_L}{R_s}} \right)^2 \overline{e_{nRs}^2}$$

- **Noise Factor**

$$F = 1 + \frac{R_s}{R_L} \frac{\overline{e_{nRL}^2}}{\overline{e_{nRs}^2}} = 1 + \left( \frac{R_s}{R_L} \right) \frac{4kTR_L}{4kTR_s} = 2 \quad (= 3dB)$$

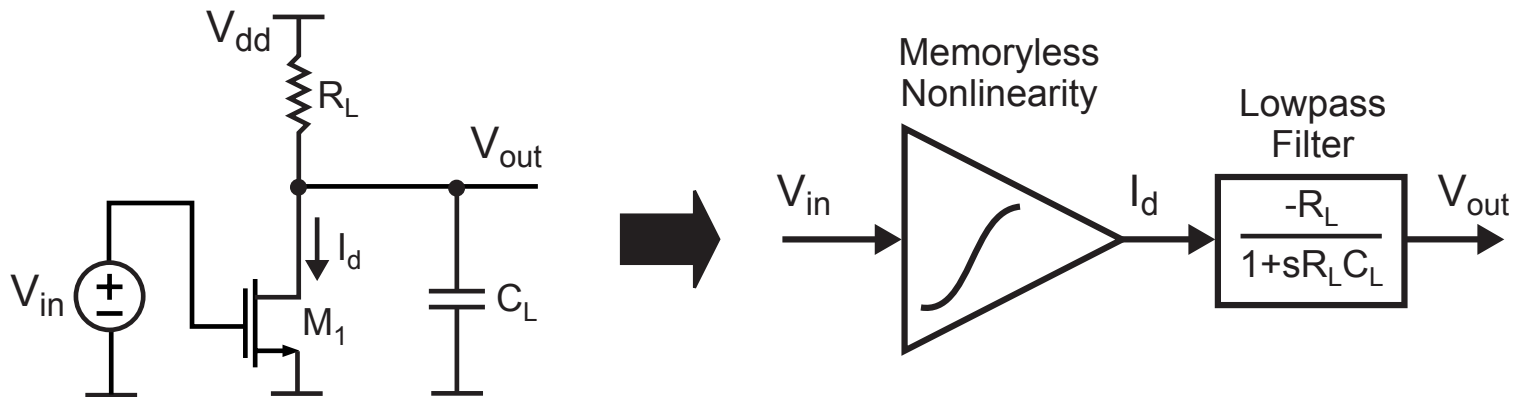
# Observations



- If you need to power match to a resistive load, you must pay a 3 dB penalty in Noise Figure
  - A transformer does not alleviate this issue
- What value does a transformer provide?
  - Almost-true answer: maximizes voltage gain given the power match constraint, thereby reducing effect of noise of following amplifiers
  - Accurate answer: we need to wait until we talk about cascaded noise factor calculations

# Nonlinearities in Amplifiers

- We can generally break up an amplifier into the cascade of a memoryless nonlinearity and an input and/or output transfer function

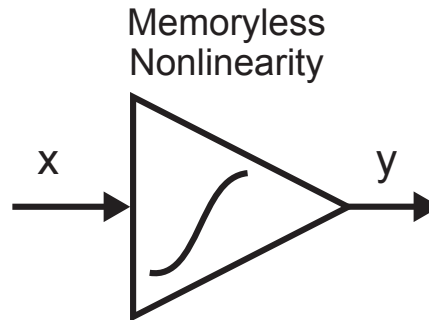


- Impact of nonlinearities with sine wave input
  - Causes harmonic distortion (i.e., creation of harmonics)
- Impact of nonlinearities with several sine wave inputs
  - Causes harmonic distortion for each input AND intermodulation products

# Analysis of Amplifier Nonlinearities

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- **Focus on memoryless nonlinearity block**
  - The impact of filtering can be added later



- **Model nonlinearity as a Taylor series expansion up to its third order term (assumes small signal variation)**

$$y(t) \approx c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

- **For harmonic distortion, consider**

$$x(t) = A \cos(\omega t)$$

- **For intermodulation, consider**

$$x(t) = A(\cos(\omega_1 t) + \cos(\omega_2 t))$$

## Harmonic Distortion

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$$y(t) = c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

$$\text{where } x(t) = A \cos wt$$

---

- **Substitute x(t) into polynomial expression**

$$y(t) - c_0 = c_1A \cos wt + c_2A^2 \cos^2 wt + c_3A^3 \cos^3 wt$$

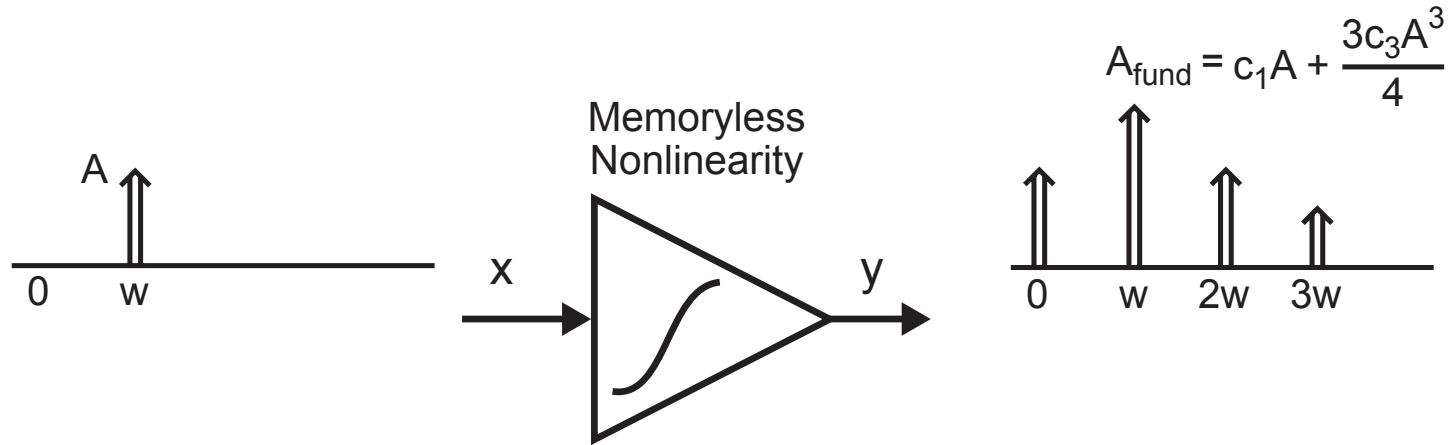
$$= c_1A \cos wt + \frac{c_2A^2}{2}(1 + \cos 2wt) + \frac{c_3A^3}{4}(3 \cos wt + \cos 3wt)$$

$$= \frac{c_2A^2}{2} + \underbrace{\left( c_1A + \frac{3c_3A^3}{4} \right) \cos wt}_{\text{Fundamental}} + \underbrace{\frac{c_2A^2}{2} \cos 2wt + \frac{c_3A^3}{4} \cos 3wt}_{\text{Harmonics}}$$

- **Notice that each harmonic term,  $\cos(nwt)$ , has an amplitude that grows in proportion to  $A^n$** 
  - **Very small for small A, very large for large A**

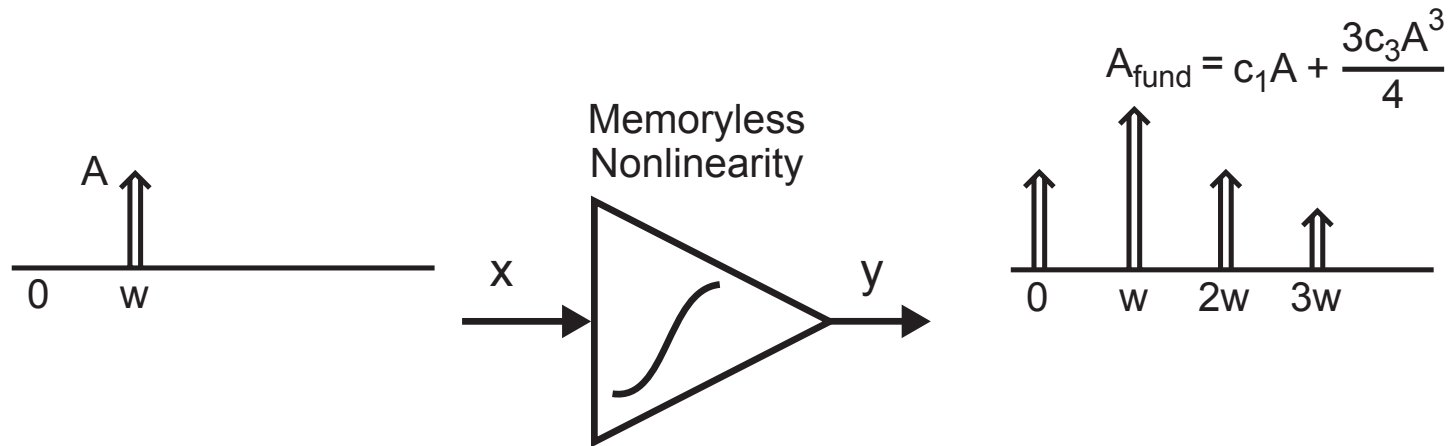


# Frequency Domain View of Harmonic Distortion



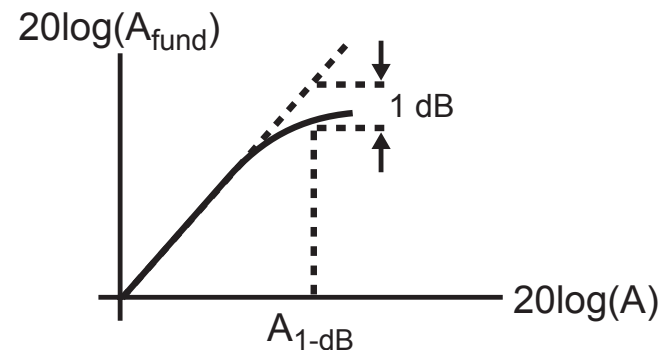
- **Harmonics cause “noise”**
  - Their impact depends highly on application
    - LNA – typically not of consequence
    - Power amp – can degrade spectral mask
    - Audio amp – depends on your listening preference!
- **Gain for fundamental component depends on input amplitude!**

# 1 dB Compression Point



- **Definition: input signal level such that the small-signal gain drops by 1 dB**

- Input signal level is high!



- **Typically calculated from simulation or measurement rather than analytically**
  - Analytical model must include many more terms in Taylor series to be accurate in this context

# Harmonic Products with An Input of Two Sine Waves

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$$y(t) = c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

where  $x(t) = A(\cos w_1t + \cos w_2t)$

---

- **DC and fundamental components**

$$\left(c_0 + c_2A^2\right) + \left(\left(c_1A + \frac{9}{4}c_3A^3\right)(\cos w_1t + \cos w_2t)\right)$$

- **Second and third harmonic terms**

$$\left(\frac{c_2A^2}{2}(\cos 2w_1t + \cos 2w_2t)\right) + \left(\frac{c_3A^3}{4}(\cos 3w_1t + \cos 3w_2t)\right)$$

- **Similar result as having an input with one sine wave**
  - **But, we haven't yet considered cross terms!**

## Intermodulation Products

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$$y(t) = c_0 + c_1x(t) + c_2x(t)^2 + c_3x(t)^3$$

$$\text{where } x(t) = A(\cos w_1t + \cos w_2t)$$

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- **Second-order intermodulation (IM2) products**

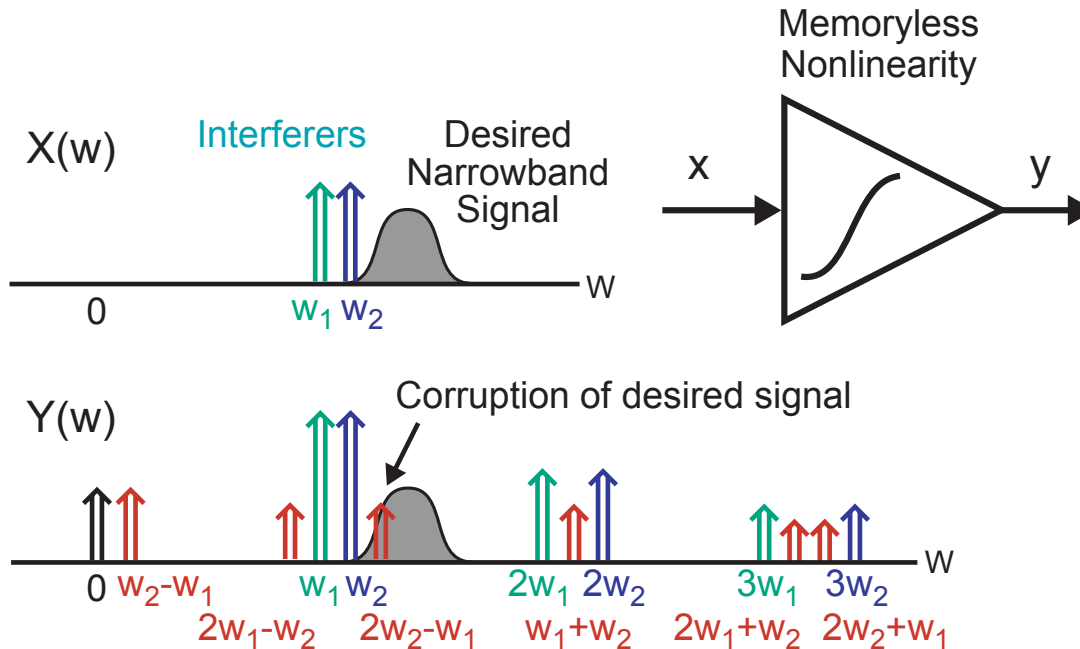
$$c_2A^2(\cos(w_1 + w_2)t + \cos(w_2 - w_1)t)$$

- **Third-order intermodulation (IM3) products**

$$\frac{3}{4}c_3A^3 \left( \cos(2w_1 + w_2)t + \cos(2w_1 - w_2)t \right. \\ \left. + \cos(2w_2 + w_1)t + \cos(2w_2 - w_1)t \right)$$

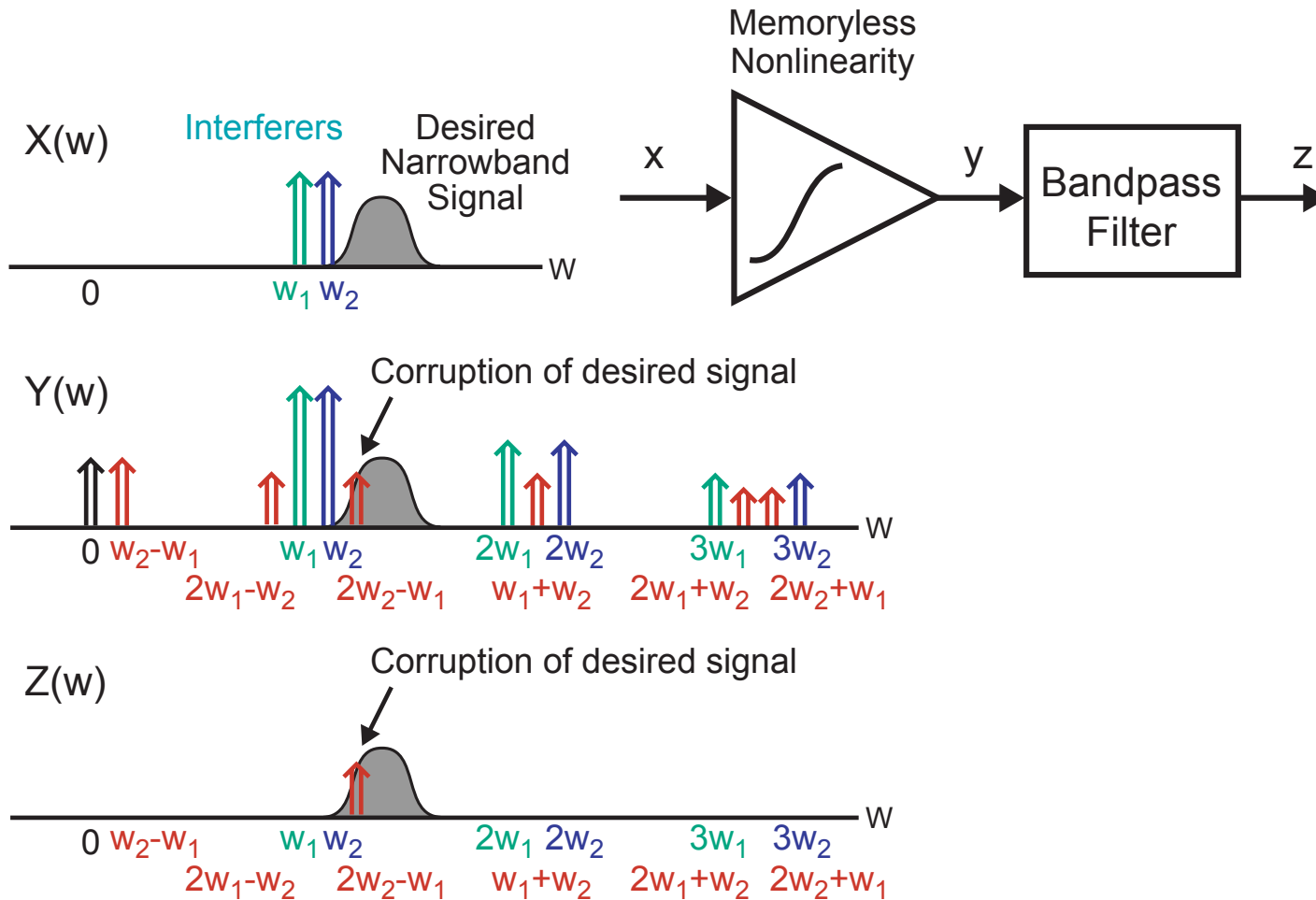
- **These are the troublesome ones for narrowband systems**

# Corruption of Narrowband Signals by Interferers



- **Wireless receivers must select a desired signal that is accompanied by interferers that are often much larger**
  - LNA nonlinearity causes the creation of harmonic and intermodulation products
  - Must remove interference and its products to retrieve desired signal

# Use Filtering to Remove Undesired Interference



- Ineffective for IM3 term that falls in the desired signal frequency band

## Characterization of Intermodulation

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- **Magnitude of third order products is set by  $c_3$  and input signal amplitude (for small A)**

$$\frac{3}{4}c_3A^3 \left( \cos(2w_1 + w_2)t + \cos(2w_1 - w_2)t \right. \\ \left. + \cos(2w_2 + w_1)t + \cos(2w_2 - w_1)t \right)$$

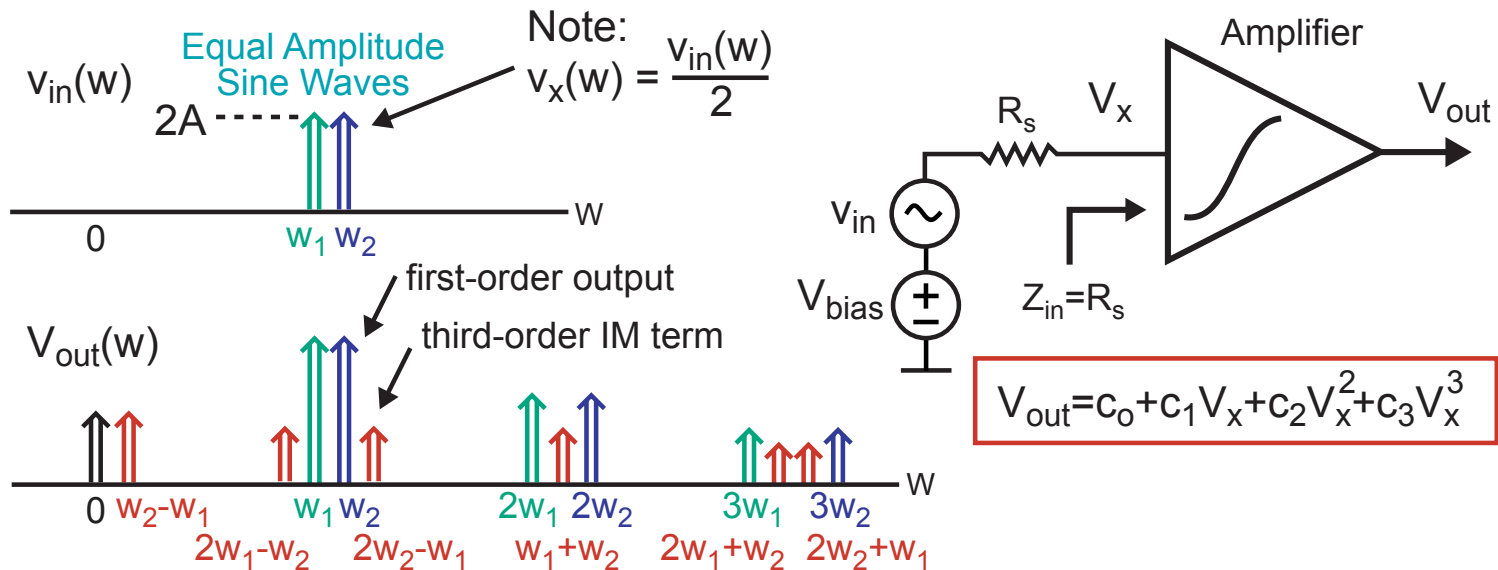
- **Magnitude of first order term is set by  $c_1$  and A (for small A)**

$$\left( c_1A + \frac{9}{4}c_3A^3 \right) (\cos w_1t + \cos w_2t) \approx c_1A (\cos w_1t + \cos w_2t)$$

- **Relative impact of intermodulation products can be calculated once we know A and the ratio of  $c_3$  to  $c_1$** 
  - **Problem: it's often hard to extract the polynomial coefficients through direct DC measurements**
    - **Need an indirect way to measure the ratio of  $c_3$  to  $c_1$**

# Two Tone Test

- Input the sum of two equal amplitude sine waves into the amplifier (assume  $Z_{in}$  of amplifier =  $R_s$  of source)

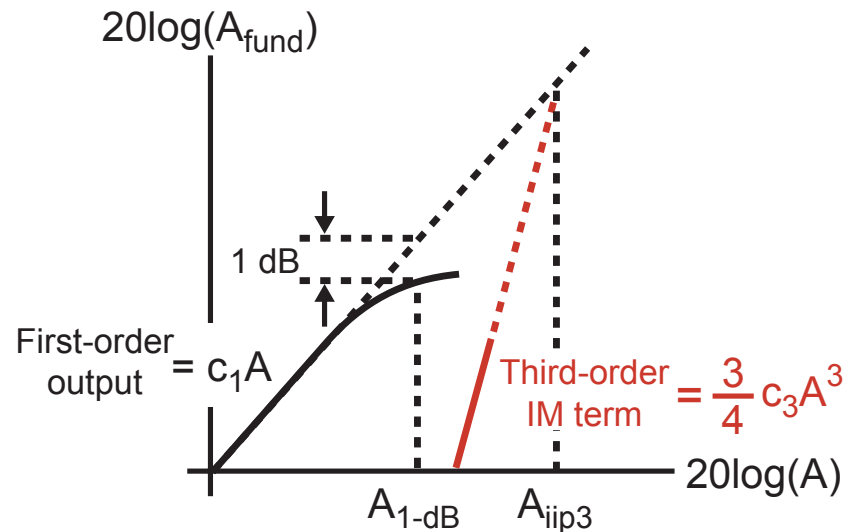


- On a spectrum analyzer, measure first order and third order terms as  $A$  is varied ( $A$  must remain small)
  - First order term will increase linearly
  - Third order IM term will increase as the cube of  $A$



# Input-Referred Third Order Intercept Point (IIP3)

- Plot the results of the two-tone test over a range of  $A$  (where  $A$  remains small) on a log scale (i.e., dB)
  - Extrapolate the results to find the intersection of the first and third order terms



- IIP3 defined as the input power at which the extrapolated lines intersect (higher value is better)
  - Note that IIP3 is a small signal parameter based on extrapolation, in contrast to the 1-dB compression point

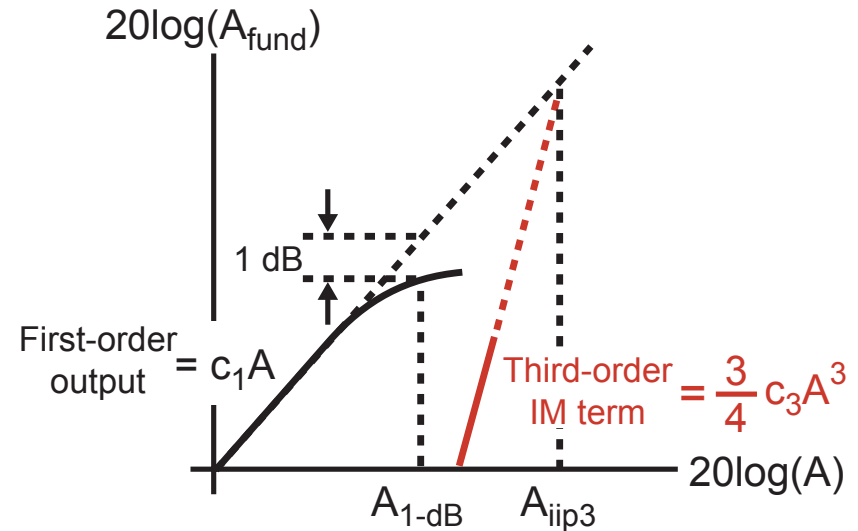
## Relationship between IIP3, $c_1$ and $c_3$

- **Intersection point**

$$|c_1 A| = \left| \frac{3}{4} c_3 A^3 \right|$$

- **Solve for A (gives  $A_{iip3}$ )**

$$\Rightarrow A^2 = \frac{4}{3} \left| \frac{c_1}{c_3} \right| (V_p^2)$$



- **Note that A corresponds to the peak value of the two cosine waves coming into the amplifier input node ( $V_x$ )**
  - **Would like to instead like to express IIP3 in terms of power**

# IIP3 Expressed in Terms of Power at Source

- IIP3 referenced to  $V_x$  (peak voltage)

$$A^2 = \frac{4}{3} \left| \frac{c_1}{c_3} \right| (V_p^2)$$

- IIP3 referenced to  $V_x$  (rms voltage)

$$A_{rms}^2 = \left( \frac{A}{\sqrt{2}} \right)^2 = \frac{2}{3} \left| \frac{c_1}{c_3} \right| (V_{rms}^2)$$

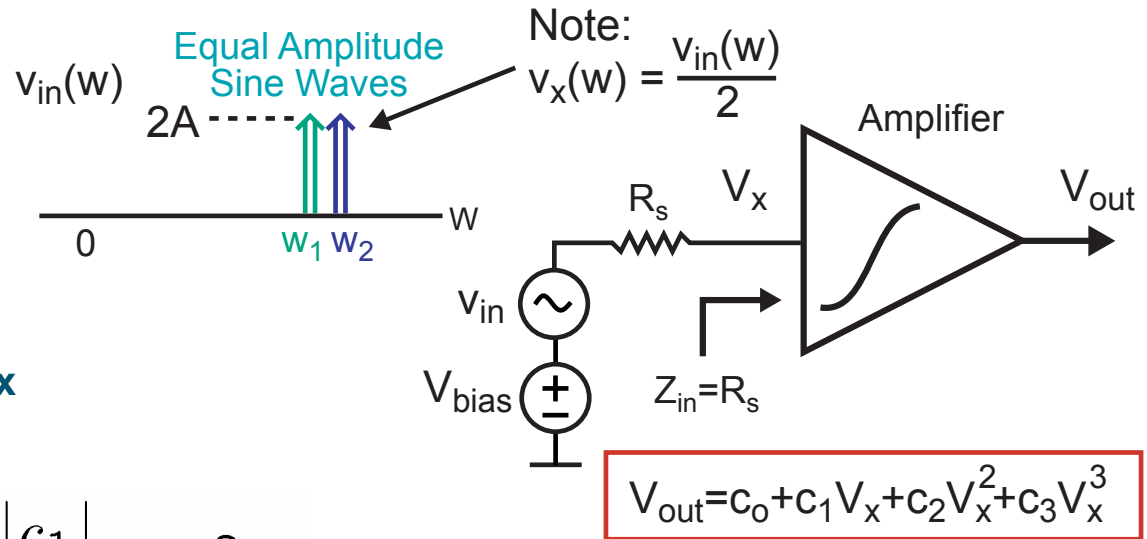
- Power across  $Z_{in} = R_s$

$$\frac{A_{rms}^2}{R_s} = \frac{2}{3} \left| \frac{c_1}{c_3} \right| \frac{1}{R_s} \text{ (Watts)}$$

- Note: Power from  $v_{in}$

$$2 \frac{A_{rms}^2}{R_s} = \frac{4}{3} \left| \frac{c_1}{c_3} \right| \frac{1}{R_s} \text{ (Watts)}$$

$$\Rightarrow IIP3(dBm) = 10 \log \left( (10^3) \frac{2}{3} \left| \frac{c_1}{c_3} \right| \frac{1}{R_s} \right) \text{ (dBm)}$$

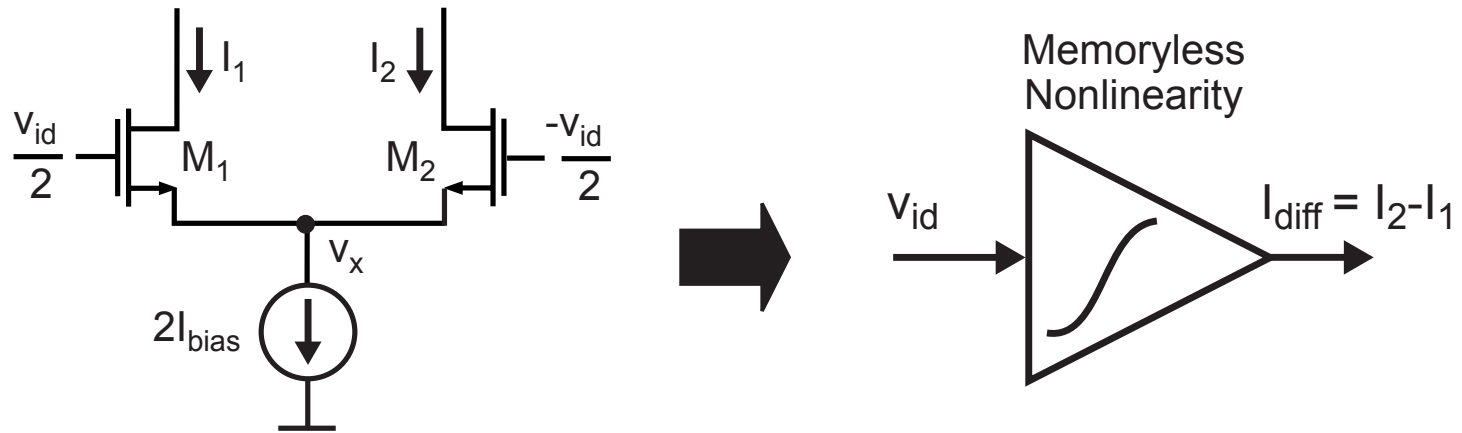


## ***IIP3 as a Benchmark Specification***

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- **Since IIP3 is a convenient parameter to describe the level of third order nonlinearity in an amplifier, it is often quoted as a benchmark spec**
- **Measurement of IIP3 on a discrete amplifier would be done using the two-tone method described earlier**
  - This is rarely done on integrated amplifiers due to poor access to the key nodes
  - Instead, for a radio receiver for instance, one would simply put in interferers and see how the receiver does
    - Note: performance in the presence of interferers is not just a function of the amplifier nonlinearity
- **Calculation of IIP3 is most easily done using a simulator such as Hspice or Spectre**
  - Two-tone method is not necessary – simply curve fit to a third order polynomial
  - Note: two-tone can be done in CppSim

# Impact of Differential Amplifiers on Nonlinearity



- Assume  $v_x$  is approximately incremental ground

$$I_{diff} = c_0 + c_1 \frac{v_{id}}{2} + c_2 \left( \frac{v_{id}}{2} \right)^2 + c_3 \left( \frac{v_{id}}{2} \right)^3 - \left( c_0 + c_1 \frac{-v_{id}}{2} + c_2 \left( \frac{-v_{id}}{2} \right)^2 + c_3 \left( \frac{-v_{id}}{2} \right)^3 \right)$$

$$\Rightarrow I_{diff} = c_1 v_{id} + \frac{c_3}{4} v_{id}^3$$

- Second order term removed and IIP3 increased!