Mixed-Integer Programming I

Mixed-Integer Linear Programming

$$
\begin{aligned}
\max \quad & cx + hy \\
\text{s.t.} \quad & Ax + Gy \le b \\
& x \text{ integral}
\end{aligned}
$$

where c, h, A, G, and b are rational vectors and matrices, respectively.

Projections

- Let $P \subseteq \mathbb{R}^{n+p}$, where $(x, y) \in P$ is interpreted as $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$.
- The projection of P onto the x-space $\mathbb{R}^n$ is

$$\text{proj}_x(P) = \{x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^p \text{ with } (x, y) \in P\}.$$

Theorem 1. *Let* $P = \big\{(x, y) : Ax + Gy \le b\big\}$*. Then*

$$\text{proj}_x(P) = \{x : v^t(b - Ax) \ge 0 \text{ for all } t \in T\},$$

where $\{v^t\}_{t\in T}$ *is the set of extreme rays of* $\{v : vG = 0, v \ge 0\}$*.*

The Fundamental Theorem of MILP

Theorem 2 (Meyer 1974)**.** *Given rational matrices* A *and* G *and a rational vector* b*, let* $P = \big\{(x, y) : Ax + Gy \le b\big\}$ *and* $S = \big\{(x, y) \in P : x \text{ integral}\big\}$*. There exist rational matrices* A'*,* G'*, and a rational vector* b' *such that*

$$\text{conv}(S) = \big\{(x, y) : A'x + G'y \le b'\big\}.$$

Proof:

- We may assume that $S \ne \emptyset$.
- By the Minkowski-Weyl Theorem, $P = \text{conv}(V) + \text{cone}(R)$, where $V = (v^1, \dots, v^p)$ and $R = (r^1, \dots, r^q)$.
- We may assume that V is a rational matrix and R is an integral matrix.

- Consider the following truncation of P:

$$T = \Big\{(x,y) : (x,y) = \sum_{i=1}^{p} \lambda_i v^i + \sum_{j=1}^{q} \mu_j r^j, \sum_{i=1}^{p} \lambda_i = 1,$$
$$\lambda \geq 0, 0 \leq \mu \leq 1\Big\}.$$

- T is bounded and is the projection of a rational polyhedron. It therefore is a rational polytope.

- Let $T_I = \big\{(x,y) \in T : x \text{ integral}\big\}$. Claim: $\text{conv}(T_I)$ is a rational polytope.

- Since T is a polytope, $X = \big\{x : \exists \text{y s.th. } (x,y) \in T_I\big\}$ is finite.

- For fixed $\bar{x} \in X$, $T_{\bar{x}} = \big\{(\bar{x},y) : (\bar{x},y) \in T_I\big\}$ is a rational polytope. Hence, $T_{\bar{x}} = \text{conv}(V_{\bar{x}})$ for some rational matrix $V_{\bar{x}}$.

- Since X is finite, there is a rational matrix V_{T_I} which contains all the columns of all matrices $V_{\bar{x}}$, for $\bar{x} \in X$.

- Therefore, $\text{conv}(T_I) = \text{conv}(V_{T_I})$, which proves the claim.

- $(\bar{x},\bar{y}) \in S$ iff $\bar{x}$ is integral and there exist $\lambda \geq 0$, $\sum_{i=1}^{p} \lambda_i = 1$, and $\mu \geq 0$ such that

$$(\bar{x},\bar{y}) = \sum_{i=1}^{p} \lambda_i v^i + \sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j + \sum_{j=1}^{q} \lfloor \mu_j \rfloor r^j.$$

- The point $\sum_{i=1}^{p} \lambda_i v^i + \sum_{j=1}^{q} (\mu_j - \lfloor \mu_j \rfloor) r^j$ belongs to T.

- Since $\bar{x}$ and $\lfloor \mu_j \rfloor r^j$ are integral it also belongs to T_I.

- Thus
$$S = T_I + R_I, \tag{1}$$
where R_I is the set of integral conic combinations of $r^1, \ldots, r^q$.

- (1) implies that
$$\text{conv}(S) = \text{conv}(T_I) + \text{cone}(R).$$

- By the above claim $\text{conv}(T_I)$ is a rational polytope.

- Thus $\text{conv}(S)$ is a rational polyhedron (having the same recession cone as P). $\square$

Union of Polyhedra

- Consider k polyhedra $P_i = \{x \in \mathbb{R}^n : A_i x \leq b^i\}$, $i = 1, \ldots, k$.

- One can show that $\overline{\text{conv}}(\cup_{i=1}^{k} P_i)$ is a polyhedron.

- Furthermore, we will show that this polyhedron can be obtained as the projection onto $\mathbb{R}^n$ of a polyhedron with polynomially many variables and constraints in a higher-dimensional space.

- (The closure is needed: let P_1 be a single point and let P_2 be a line that does not contain P_2.)

Theorem 3. *For $i = 1, \ldots, k$, let $P_i = Q_i + C_i$ be nonempty polyhedra. Then $Q = \operatorname{conv}(\cup_{i=1}^k Q_i)$ is a polytope, $C = \operatorname{conv}(\cup_{i=1}^k C_i)$ is a finitely generated cone, and $\overline{\operatorname{conv}}(\cup_{i=1}^k P_i) = Q + C$.*

- No proof here, but note that the claims on Q and C are straightforward to check.

- One consequence of the proof is that if $P_1, \ldots, P_k$ have identical recession cones, then $\operatorname{conv}(\cup_{i=1}^k P_i)$ is a polyhedron.

Theorem 4 (Balas 1974)**.** *Consider k polyhedra $P_i = Q_i + C_i = \{x \in \mathbb{R}^n : A_i x \leq b^i\}$ and let $Y \subseteq \mathbb{R}^{n+kn+k}$ be the polyhedron defined by*

$$A_i x^i \leq b^i y_i, \sum_{i=1}^k x^i = x, \sum_{i=1}^k y_i = 1, y_i \geq 0 \text{ for } i = 1, \ldots, k.$$

Then

$$\operatorname{proj}_x(Y) = Q + C,$$

where $Q = \operatorname{conv}(\cup_{i=1}^k Q_i)$ and $C = \operatorname{conv}(\cup_{i=1}^k C_i)$.

Proof:

- First, let $x \in Q + C$.

- There exist $w^i \in Q_i$ and $z^i \in C_i$ such that $x = \sum_i y_i w^i + \sum_i z^i$, where $y_i \geq 0$ and $\sum_i y_i = 1$.

- Let $x^i = y_i w^i + z^i$. Then $A_i x^i \leq b^i y_i$ and $x = \sum_i x^i$.

- This shows $x \in \operatorname{proj}_x(Y)$.

- Now, let $x \in \operatorname{proj}_x(Y)$.

- There exist $x^1, \ldots, x^k, y$ such that $x = \sum_i x^i$ where $A_i x^i \leq b^i y_i$, $\sum_i y_i = 1$, $y \geq 0$.

- Let $I = \{i : y_i > 0\}$.

- For $i \in I$, let $z^i = \frac{x^i}{y_i}$. Then $z^i \in P_i$.

- Since $P_i = Q_i + C_i$, we can write $z^i = w^i + \frac{r^i}{y_i}$ where $w^i \in Q_i$ and $r^i \in C_i$.

- For $i \notin I$, we have $A_i x^i \leq 0$, that is $x^i \in C_i$. Let $r^i = x^i$ for $i \notin I$.

- Then,
$$x = \sum_{i \in I} y_i z^i + \sum_{i \notin I} x^i = \sum_{i \in I} y_i w^i + \sum_i r^i \in Q + C.$$

$\square$

Lift-and-Project Revisited

We consider mixed-0/1 linear programs:

$$\begin{array}{lll} \min & cx & \\ \text{s.t.} & Ax \geq b & \\ & x_j \in \{0,1\} & \text{for } j = 1, \ldots, n \\ & x_j \geq 0 & \text{for } j = n+1, \ldots, n+p \end{array}$$

We let $P = \{x \in \mathbb{R}_+^{n+p} : Ax \geq b\}$ and $S = \{x \in \{0,1\}^n \times \mathbb{R}_+^p : Ax \geq b\}$.

We assume that $Ax \geq b$ includes $-x_j \geq -1$ for $j = 1, \ldots, n$, but not $x \geq 0$.

- Given an index $j \in \{1, \ldots, n\}$, let

$$P_j = \operatorname{conv}\big\{(Ax \geq b, x \geq 0, x_j = 0) \cup (Ax \geq b, x \geq 0, x_j = 1)\big\}.$$

- By definition, this is the tightest possible relaxation among all relaxations that ignore the integrality of all variables x_i, $i \neq j$.

- $\bigcap_{j=1}^n P_j$ is called the *lift-and-project closure*:

$$\operatorname{conv}(S) \subseteq \bigcap_{j=1}^{n} P_j \subseteq P.$$

- On 35 mixed-0/1 linear programs from MIPLIB, the lift-and-project closure reduces the integrality gap by 37% on average [Bonami & Minoux 2005].

Lift-and-Project Cuts

P_j is the convex hull of the union of two polyhedra:

$$\begin{array}{r} Ax \geq b \\ x \geq 0 \\ -x_j \geq 0 \end{array} \qquad \text{and} \qquad \begin{array}{r} Ax \geq b \\ x \geq 0 \\ x_j \geq 1 \end{array}$$

By the above theorem:

$$P_j = \operatorname{proj}_x \left(\begin{array}{rcl} Ax^0 & \geq & by_0 \\ -x_j^0 & \geq & 0 \\ Ax^1 & \geq & by_1 \\ x_j^1 & \geq & y_1 \\ x^0 + x^1 & = & x \\ y_0 + y_1 & = & 1 \\ x^0, x^1, y_0, y_1 & \geq & 0 \end{array} \right)$$

- Using the projection theorem, we get that P_j is defined by the inequalities $\alpha x \geq \beta$ such that

$$\begin{array}{cccccccc} \alpha & & -uA & +u_0 e_j & & & \geq & 0 \\ \alpha & & & & -vA & -v_0 e_j & \geq & 0 \\ & \beta & -ub & & & & \leq & 0 \\ & \beta & & & -vb & -v_0 & \leq & 0 \\ & & u, & u_0, & v, & v_0 & \geq & 0 \end{array} \tag{2}$$

- Such an inequality $\alpha x \geq \beta$ is called a lift-and-project inequality.
- Given a fractional point $\bar{x}$, we can determine if there exists a lift-and-project inequality $\alpha x \geq \beta$ valid for P_j that cuts off $\bar{x}$.
- This problem amounts to finding $(\alpha, \beta, u, u_0, v, v_o)$ satisfying (2) such that $\alpha\bar{x} - \beta < 0$.
- In order to find a "best" cut in cone (2), we solve the *cut-generating LP*:

$$\begin{array}{ccccccccc} \min & \alpha\bar{x} & -\beta & & & & & & \\ & \alpha & & -uA & +u_0 e_j & & & \geq & 0 \\ & \alpha & & & & -vA & -v_0 e_j & \geq & 0 \\ & & \beta & -ub & & & & \leq & 0 \\ & & \beta & & & -vb & -v_0 & \leq & 0 \\ & & & \sum_i u_i & +u_0 & +\sum_i v_i & +v_0 & = & 1 \\ & & & u, & u_0, & v, & v_0 & \geq & 0 \end{array}$$

Mixed Integer Inequalities

- Consider $S = \big\{(x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^p_+ : \sum_{j=1}^{n} a_j x_j + \sum_{j=1}^{p} g_j y_j = b\big\}$.
- Let $b = \lfloor b \rfloor + f_0$ where $0 < f_0 < 1$.
- Let $a_j = \lfloor a_j \rfloor + f_j$ where $0 \leq f_j < 1$.
- Then $\sum_{f_j \leq f_0} f_j x_j + \sum_{f_j > f_0} (f_j - 1) x_j + \sum_{j=1}^{p} g_j y_j = k + f_0$, where k is some integer.
- Since $k \leq -1$ or $k \geq 0$, any $x \in S$ satisfies

$$\sum_{f_j \leq f_0} \frac{f_j}{f_0} x_j - \sum_{f_j > f_0} \frac{1 - f_j}{f_0} x_j + \sum_{j=1}^{p} \frac{g_j}{f_0} y_j \geq 1 \tag{3}$$

  OR

$$-\sum_{f_j \leq f_0} \frac{f_j}{1 - f_0} x_j + \sum_{f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j - \sum_{j=1}^{p} \frac{g_j}{1 - f_0} y_j \geq 1. \tag{4}$$

- This is of the form $\sum_j a^1_j x_j \geq 1$ or $\sum_j a^2_j x_j \geq 1$, which implies $\sum_j \max\{a^1_j, a^2_j\} x_j \geq 1$ for any $x \geq 0$.

- For each variable, what is the max coefficient in (3) and (4)?
- We get
$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{g_j > 0} \frac{g_j}{f_0} y_j - \sum_{g_j < 0} \frac{g_j}{1-f_0} y_j \ge 1.$$
- This is the *Gomory mixed integer (GMI) inequality*.
- In the pure integer programming case, the GMI inequality reduces to
$$\sum_{f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{f_j > f_0} \frac{1-f_j}{1-f_0} x_j \ge 1.$$
- Since $\frac{1-f_j}{1-f_0} < \frac{f_j}{f_0}$ when $f_j > f_0$, the GMI inequality dominates
$$\sum_{j=1}^{n} f_j x_j \ge f_0,$$
which is known as the *fractional cut*.
- Consider now $S = \{(x,y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p : Ax + Gy \le b\}$.
- Let $P = \{(x,y) \in \mathbb{R}_+^n \times \mathbb{R}_+^p : Ax + Gy \le b\}$ be the underlying polyhedron.
- Let $\alpha x + \gamma y \le \beta$ be any valid for P.
- Add a nonnegative slack variable s, use $\alpha x + \gamma y + s = \beta$ to derive a GMI inequality, and eliminate $s = \beta - \alpha x - \gamma y$ from it.
- The result is a valid inequality for S.
- These inequalities are called the GMI inequalities for S.
- In contrast to lift-and-project cuts, it is in general NP-hard to find a GMI inequality that cuts off a point $(\bar{x}, \bar{y}) \in P \setminus S$, or show that none exists.
- However, one can easily find a GMI inequality that cuts off a basic feasible solution.
- On 41 MIPLIB instances, adding the GMI cuts generated from the optimal simplex tableaux reduces the integrality gap by 24% on average [Bonami et al. 2008]
- GMI cuts are widely used in commercial codes today.
- Numerical issues need to be addressed, however.