CHAPTER 6

State Observers and State Feedback

Our study of the modal solutions of LTI state-space models made clear in complete analytical detail that the state at any given time summarizes everything about the past that is relevant to future behavior of the model. More specifically, given the value of the state vector at some initial instant, and given the entire input trajectory over some interval of time extending from the initial instant into the future, one can determine the entire future state and output trajectories of the model over that interval. The same general conclusion holds for nonlinear and time-varying state-space models, although they are generally far less tractable analytically. Our focus will be on LTI models.

It is typically the case that we do not have any direct measurement of the initial state of a system, and will have to make some guess or estimate of it. This uncertainty about the initial state generates uncertainty about the future state trajectory, even if our model for the system is perfect, and even if we have accurate knowledge of the inputs to the system.

The first part of this chapter is devoted to addressing the issue of state trajectory estimation, given uncertainty about the initial state of the system. We shall see that the state can actually be asymptotically determined under appropriate conditions, by means of a so-called state observer. The observer uses a model of the system along with past measurements of both the input and output trajectories of the system.

The second part of the chapter examines how the input to the system should be controlled in order to yield desirable system behavior. We shall see that having knowledge of the present state of the system provides a powerful basis for designing feedback control to stabilize or otherwise improve the behavior of the resulting closed-loop system. When direct measurements of the state are not available, the asymptotic state estimate provided by an observer turns out to suffice.

6.1 PLANT AND MODEL

It is important now to make a distinction between the actual, physical (and causal) system we are interested in studying or working with or controlling — what is often termed the plant (as in "physical plant") — and our idealized model for the plant. The plant is usually a complex, highly nonlinear and time-varying object, typically requiring an infinite number (or a continuum) of state variables and parameters to represent it with ultimate fidelity. Our model, on the other hand, is an idealized and simplified (and often LTI) representation, of relatively low order, that aims to capture the behavior of the plant in some limited regime of its operation, while remaining tractable for analysis, computation, simulation and design.

The inputs to the model represent the inputs acting on or driving the actual plant, and the outputs of the model represent signals in the plant that are accessible for measurement. In practice we will typically not know all the driving inputs to the plant exactly. Apart from those driving inputs that we have access to, there will also generally be additional unmeasured disturbance inputs acting on the plant that we are only able to characterize in some general way, perhaps as random processes. Similarly, the measured outputs of the plant will differ from what we might predict on the basis of our limited model, partly because of measurement noise.

6.2 STATE ESTIMATION BY REAL-TIME SIMULATION

Suppose the plant of interest to us is correctly described by the following equations, which constitute an Lth-order LTI state-space representation of the plant:

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] + \mathbf{w}[n] \,, \tag{6.1}$$

$$y[n] = \mathbf{c}^T\mathbf{q}[n] + \mathtt{d}x[n] + \zeta[n] \,. \tag{6.2}$$

Here $x[n]$ denotes the known (scalar) control input, and $\mathbf{w}[n]$ denotes the vector of unknown disturbances that drive the plant, not necessarily through the same channels as the input $x[n]$. For example, we might have $\mathbf{w}[n] = \mathbf{f}v[n]$, where $v[n]$ is a scalar disturbance signal and $\mathbf{f}$ is a vector describing how this scalar disturbance drives the system (just as $\mathbf{b}$ describes how $x[n]$ drives the system). The quantity $y[n]$ denotes the known or measured (scalar) output, and $\zeta[n]$ denotes the unknown noise in this measured output. We refer to $\mathbf{w}[n]$ as plant disturbance or plant noise, and to $\zeta[n]$ as measurement noise. We focus mainly on the DT case now, but essentially everything carries over in a natural way to the CT case.

With the above equations representing the true plant, what sort of model might we use to study or simulate the behavior of the plant, given that we know $x[n]$ and $y[n]$? If nothing further was known about the disturbance variables in $\mathbf{w}[n]$ and the measurement noise $\zeta[n]$, or if we only knew that they could be represented as zero-mean random processes, for instance, then one strategy would be to simply ignore these variables when studying or simulating the plant. If everything else about the plant was known, our representation of the plant's behavior would be embodied in an LTI state-space model of the form

$$\widehat{\mathbf{q}}[n+1] = \mathbf{A}\widehat{\mathbf{q}}[n] + \mathbf{b}x[n] \,, \tag{6.3}$$

$$\widehat{y}[n] = \mathbf{c}^T\widehat{\mathbf{q}}[n] + \mathtt{d}x[n] \,. \tag{6.4}$$

The $x[n]$ that drives our model is the same known $x[n]$ that is an input (along with possibly other inputs) to the plant. However, the state $\widehat{\mathbf{q}}[n]$ and output $\widehat{y}[n]$ of the model will generally differ from the corresponding state $\mathbf{q}[n]$ and output $y[n]$ of the plant, because in our formulation the plant state and output are additionally perturbed by $\mathbf{w}[n]$ and $\zeta[n]$ respectively. The assumption that our model has correctly captured the dynamics of the plant and the relationships among the variables is what allows us to use the same $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$ and $\mathtt{d}$ in our model as occur in the "true" plant.

It bears repeating that in reality there are several sources of uncertainty we are ignoring here. At the very least, there will be discrepancies between the actual and assumed parameter values — i.e., between the actual entries of $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$ and $\mathtt{d}$ in (6.1), (6.2) and the assumed entries of these matrices in (6.3), (6.4) respectively. Even more troublesome is the fact that the actual system is probably more accurately represented by a nonlinear, time-varying model of much higher order than that of our assumed LTI model, and with various other disturbance signals acting on it. We shall not examine the effects of all these additional sources of uncertainty.

With a model in hand, it is natural to consider obtaining an estimate of the current plant state by running the model forward in real time, as a simulator. For this, we initialize the model (6.3) at some initial time (which we take to be $n = 0$ without loss of generality), picking its initial state $\widehat{\mathbf{q}}[0]$ to be some guess or estimate of the initial state of the plant. We then drive the model with the known input $x[n]$ from time $n = 0$ onwards, generating an estimated or predicted state trajectory $\widehat{\mathbf{q}}[n]$ for $n > 0$. We could then also generate the predicted output $\widehat{y}[n]$ using the prescription in (6.4).

In order to examine how well this real-time simulator performs as a state estimator, we examine the error vector

$$\widetilde{\mathbf{q}}[n] = \mathbf{q}[n] - \widehat{\mathbf{q}}[n] \ . \tag{6.5}$$

Note that $\widetilde{\mathbf{q}}[n]$ is the difference between the actual and estimated (or predicted) state trajectories. By subtracting (6.3) from (6.1), we see that this difference, the estimation error or prediction error $\widetilde{\mathbf{q}}[n]$, is itself governed by an LTI state-space equation:

$$\widetilde{\mathbf{q}}[n+1] = \mathbf{A}\widetilde{\mathbf{q}}[n] + \mathbf{w}[n] \tag{6.6}$$

with initial condition

$$\widetilde{\mathbf{q}}[0] = \mathbf{q}[0] - \widehat{\mathbf{q}}[0] \ . \tag{6.7}$$

This initial condition is our uncertainty about the initial state of the plant.

What (6.6) shows is that, if the original system (6.1) is unstable (i.e., if $\mathbf{A}$ has eigenvalues of magnitude greater than 1), or has otherwise undesirable dynamics, and if either $\widetilde{\mathbf{q}}[0]$ or $\mathbf{w}[n]$ is nonzero, then the error $\widetilde{\mathbf{q}}[n]$ between the actual and estimated state trajectories will grow exponentially, or will have otherwise undesirable behavior, see Figure 6.1. Even if the plant is not unstable, we see from (6.6) that the error dynamics are driven by the disturbance process $\mathbf{w}[n]$, and we have no means to shape the effect of this disturbance on the estimation error. The real-time simulator is thus generally an inadequate way of reconstructing the state.

6.3 THE STATE OBSERVER

To do better than the real-time simulator (6.3), we must use not only the input $x[n]$ but also the measured output $y[n]$. The key idea is to use the discrepancy between

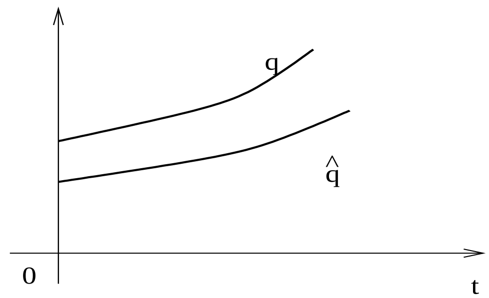

FIGURE 6.1 Schematic representation of the effect of an erroneous initial condition on the state estimate produced by the real-time simulator for an unstable plant.

actual and predicted outputs, $y[n]$ in (6.2) and $\widehat{y}[n]$ in (6.4) respectively — i.e., to use the output prediction error — as a correction term for the real-time simulator. The resulting system is termed a state observer (or state estimator) for the plant, and in our setting takes the form

$$\begin{aligned}\widehat{\mathbf{q}}[n+1] &= \mathbf{A}\widehat{\mathbf{q}}[n] + \mathbf{b}x[n] \\ &\qquad - \ell\Big(y[n] - \widehat{y}[n]\Big) \ .\end{aligned} \tag{6.8}$$

The observer equation above has been written in a way that displays its two constituent parts: a part that simulates as closely as possible the plant whose states we are trying to estimate, and a part that feeds the correction term $y[n]-\widehat{y}[n]$ into this simulation. This correction term is applied through the L-component vector ℓ, termed the observer gain vector, with ith component ℓ_i. (The negative sign in front of ℓ in (6.8) is used only to simplify the appearance of some later expressions). Figure 6.2 is a block-diagram representation of the resulting structure.

Now subtracting (6.8) from (6.1), we find that the state estimation error or observer error satisfies

$$\begin{aligned}\widetilde{\mathbf{q}}[n+1] &= \mathbf{A}\widetilde{\mathbf{q}}[n] + \mathbf{w}[n] + \ell\Big(y[n] - \mathbf{c}^T\widehat{\mathbf{q}}[n] - \mathtt{d}x[n]\Big) \\ &= (\mathbf{A} + \ell\mathbf{c}^T)\widetilde{\mathbf{q}}[n] + \mathbf{w}[n] + \ell\zeta[n] \ .\end{aligned} \tag{6.9}$$

If the observer gain ℓ is 0, then the error dynamics are evidently just the dynamics of the real-time simulator (6.6). More generally, the dynamics are governed by the system's natural frequencies, namely the eigenvalues of $\mathbf{A}+\ell\mathbf{c}^T$ or the roots of the characteristic polynomial

$$\kappa(\lambda) = \det\Big(\lambda\mathbf{I} - (\mathbf{A} + \ell\mathbf{c}^T)\Big) \tag{6.10}$$

$$= \lambda^L + \kappa_{L-1}\lambda^{L-1} + \cdots + \kappa_0 \ . \tag{6.11}$$

(This polynomial, like all the characteristic polynomials we deal with, has real coefficients and is monic, i.e., its highest-degree term is scaled by 1 rather than some non-unit scalar.)

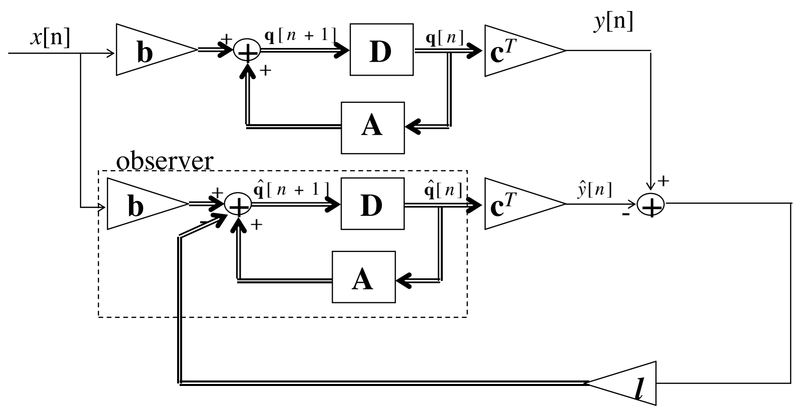

FIGURE 6.2 An observer for the plant in the upper part of the diagram comprises a real-time simulation of the plant, driven by the same input, and corrected by a signal derived from the output prediction error.

Two questions immediately arise:

(i) How much freedom do we have in placing the observer eigenvalues, i.e., the eigenvalues of $\mathbf{A} + \ell\mathbf{c}^T$ or the roots of $\kappa(\lambda)$, by appropriate choice of the observer gain ℓ?

(ii) How does the choice of ℓ shape the effects of the disturbance and noise terms $\mathbf{w}[n]$ and $\zeta[n]$ on the observer error?

Brief answers to these questions are respectively as follows:

(i) At $\ell = 0$ the observer eigenvalues, namely the eigenvalues of $\mathbf{A} + \ell\mathbf{c}^T$, are those of the real-time simulator, which are also those of the given system or plant. By varying the entries of ℓ away from 0, it turns out we can move all the eigenvalues that correspond to observable eigenvalues of the plant (which may number as many as L eigenvalues), and those are the only eigenvalues we can move. Moreover, appropriate choice of ℓ allows us, in principle, to move these observable eigenvalues to any arbitrary set of self-conjugate points in the complex plane. (A self-conjugate set is one that remains unchanged by taking the complex conjugate of the set. This is equivalent to requiring that if a complex point is in such a set, then its complex conjugate is as well.) The self-conjugacy restriction is necessary because we are working with real parameters and gains.

The unobservable eigenvalues of the plant remain eigenvalues of the observer, and cannot be moved. (This claim can be explicitly demonstrated by transformation to modal coordinates, but we omit the details.) The reason for this is that information about these unobservable modes does not make its way into the output prediction error that is used in the observer to correct the real-time simulator.

It follows from the preceding statements that a stable observer can be designed if and only if all unobservable modes of the plant are stable (a property that is termed detectability). Also, the observer can be designed to have an arbitrary characteristic polynomial $\kappa(\lambda)$ if and only if the plant is observable.

We shall not prove the various claims above. Instead, we limit ourselves to proving, later in this chapter, a closely analogous set of results for the case of state feedback control.

In designing observers analytically for low-order systems, one way to proceed is by specifying a desired set of observer eigenvalues $\epsilon_1, \cdots \epsilon_L$, thus specifying the observer characteristic polynomial $\kappa(\lambda)$ as

$$\kappa(\lambda) = \prod_{i=1}^{L} (\lambda - \epsilon_i) \ . \tag{6.12}$$

Expanding this out and equating it to $\det\Big(\lambda\mathbf{I} - (\mathbf{A} + \ell\mathbf{c}^T)\Big)$, as in (6.10), yields L simultaneous linear equations in the unknown gains $\ell_1, \cdots, \ell_L$. These equations will be consistent and solvable for the observer gains if and only if all the unobservable eigenvalues of the plant are included among the specified observer eigenvalues $\{\epsilon_i\}$.

The preceding results also suggest an alternative way to determine the unobservable eigenvalues of the plant: the roots of $\det\Big(\lambda\mathbf{I} - (\mathbf{A} + \ell\mathbf{c}^T)\Big)$ that cannot be moved, no matter how ℓ is chosen, are precisely the unobservable eigenvalues of the plant. This approach to exposing unobservable modes can be easier in some problems than the approach used in the previous chapter, which required first computing the eigenvectors $\{\mathbf{v}_i\}$ of the system, and then checking for which i we had $\mathbf{c}^T\mathbf{v}_i = 0$.

(ii) We now address how the choice of ℓ shapes the effects of the disturbance and noise terms $\mathbf{w}[n]$ and $\zeta[n]$ on the observer error. The first point to note is that if the error system (6.9) is made asymptotically stable by appropriate choice of observer gain ℓ, then bounded plant disturbance $\mathbf{w}[n]$ and bounded measurement noise $\zeta[n]$ will result in the observer error being bounded. This is most easily proved by transforming to modal coordinates, but we omit the details.

The observer error equation (6.9) shows that the observer gain ℓ enters in two places, first in causing the error dynamics to be governed by the state evolution matrix $\mathbf{A} + \ell\mathbf{c}^T$ rather than $\mathbf{A}$, and again as the input vector for the measurement noise $\zeta[n]$. This highlights a basic tradeoff between error decay and noise immunity. The observer gain can be used to obtain fast error decay, as might be needed in the presence of plant disturbances $\mathbf{w}[n]$ that continually perturb the system state away from where we think it is — but large entries in ℓ may be required to accomplish this (certainly in the CT case, but also in DT if the model is a sampled-data version of some underlying CT system, as in the following example), and these large entries in ℓ will have the undesired result of accentuating the effect of the measurement noise. A large observer gain may also increase the susceptibility of the observer design to mod! eling errors and other discrepancies. In practice, such considerations would lead us design somewhat conservatively, not attempting to obtain overly fast error-decay dynamics.

Some aspects of the tradeoffs above can be captured in a tractable optimization problem. Modeling $\mathbf{w}[n]$ and $\zeta[n]$ as stationary random processes (which are introduced in a later chapter), we can formulate the problem of picking ℓ to minimize some measure of the steady-state variances in the components of the state estimation error $\widetilde{\mathbf{q}}[n]$. The solution to this and a range of related problems is provided by the so-called Kalman filtering framework. We will be in a position to work through some elementary versions of this once we have developed the machinery for dealing with stationary random processes.

EXAMPLE 6.1 Ship Steering

Consider the following simplified sampled-data model for the steering dynamics of a ship traveling at constant speed, with a rudder angle that is controlled in a piecewise-constant fashion by a computer-based controller:

$$\begin{aligned}\mathbf{q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} &= \begin{bmatrix} 1 & \sigma \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} x[n] \\ &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \,. \end{aligned} \tag{6.13}$$

The state vector $\mathbf{q}[n]$ comprises the sampled heading error $q_1[n]$ (which is the direction the ship points in, relative to the desired direction of motion) and the sampled rate of turn $q_2[n]$ of the ship, both sampled at time $t = nT$; $x[n]$ is the constant value of the rudder angle (relative to the direction in which the ship points) in the interval $nT \leq t < nT + T$ (we pick positive rudder angle to be that which would tend to increase the heading error). The positive parameters α, σ and ϵ are determined by the type of ship, its speed, and the sampling interval T. In particular, α is generally smaller than 1, but can be larger than 1 for a large tanker; in any case, the system (6.13) is not asymptotically stable. The constant σ is approximately equal to the sampling interval T.

Suppose we had (noisy) measurements of the rate of turn, so

$$\mathbf{c}^T = \begin{pmatrix} 0 & 1 \end{pmatrix} \,. \tag{6.14}$$

Then

$$\mathbf{A} + \ell\mathbf{c}^T = \begin{pmatrix} 1 & \sigma + \ell_1 \\ 0 & \alpha + \ell_2 \end{pmatrix} \,. \tag{6.15}$$

Evidently one natural frequency of the error equation is fixed at 1, no matter what ℓ is. This natural frequency corresponds to a mode of the original system that is unobservable from rate-of-turn measurements. Moreover, it is not an asymptotically stable mode, so the corresponding observer error will not decay. Physically, the problem is that the rate of turn contains no input from or information about the heading error itself.

If, instead, we have (noisy) measurements of the heading error, so

$$\mathbf{c}^T = \begin{pmatrix} 1 & 0 \end{pmatrix} . \tag{6.16}$$

In this case

$$\mathbf{A} + \ell\mathbf{c}^T = \begin{pmatrix} 1+\ell_1 & \sigma \\ \ell_2 & \alpha \end{pmatrix} . \tag{6.17}$$

The characteristic polynomial of this matrix is

$$\kappa(\lambda) = \lambda^2 - \lambda(1+\ell_1+\alpha) + \alpha(1+\ell_1) - \ell_2\sigma . \tag{6.18}$$

This can be made into an arbitrary monic polynomial of degree 2 by choice of the gains ℓ_1 and ℓ_2, which also establishes the observability of our plant model.

One interesting choice of observer gains in this case is $\ell_1 = -1-\alpha$ and $\ell_2 = -\alpha^2/\sigma$ (which, for typical parameter values, results in ℓ_2 being large). With this choice,

$$\mathbf{A} + \ell\mathbf{c}^T = \begin{pmatrix} -\alpha & \sigma \\ -\alpha^2/\sigma & \alpha \end{pmatrix} . \tag{6.19}$$

The characteristic polynomial of this matrix is $\kappa(\lambda) = \lambda^2$, so the natural frequencies of the observer error equation are both at 0.

A DT LTI system with all natural frequencies at 0 is referred to as deadbeat, because its zero-input response settles exactly to the origin in finite time. (This finite-time settling is possible for the zero-input response of an LTI DT system, but not for an LTI CT system, though of course it is possible for an LTI CT system to have an arbitrarily small zero-input response after any specified positive time.) We have not discussed how to analyze LTI state-space models with non-distinct eigenvalues, but to verify the above claim of finite settling for our observer, it suffices to confirm from (6.19) that $(\mathbf{A} + \ell\mathbf{c}^T)^2 = 0$ when the gains ℓ_i are chosen to yield $\kappa(\lambda) = \lambda^2$. This implies that in the absence of plant disturbance and measurement noise, the observer error goes to 0 in at most two steps.

In the presence of measurement noise, one may want to choose a slower error decay, so as to keep the observer gain ℓ — and ℓ_2 in particular — smaller than in the deadbeat case, and thereby not accentuate the effects of measurement noise on the estimation error.

6.4 STATE FEEDBACK CONTROL

For a causal system or plant with inputs that we are able to manipulate, it is natural to ask how the inputs should be chosen in order to cause the system to behave in some desirable fashion. Feedback control of such a system is based on sensing its present or past behavior, and using the measurements of the sensed variables to generate control signals to apply to it. Feedback control is also referred to as closed-loop control.

Open-loop control, by contrast, is not based on continuous monitoring of the plant, but rather on using only information available at the time that one starts interacting with the system. The trouble with open-loop control is that errors, even if recognized, are not corrected or compensated for. If the plant is poorly behaved or unstable, then uncorrected errors can lead to bad or catastrophic consequences.

Feedforward control refers to schemes incorporating measurements of signals that currently or in the future will affect the plant, but that are not themselves affected by the control. For example, in generating electrical control signals for the positioning motor of a steerable radar antenna, the use of measurements of wind velocity would correspond to feedforward control, whereas the use of measurements of antenna position would correspond to feedback control. Controls can have both feedback and feedforward components.

Our focus in this section is on feedback control. To keep our development streamlined, we assume the plant is well modeled by the following Lth-order LTI state-space description:

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \tag{6.20}$$

$$y[n] = \mathbf{c}^T\mathbf{q}[n] \tag{6.21}$$

rather than the more elaborate description (6.1), (6.2). As always, $x[n]$ denotes the control input and $y[n]$ denotes the measured output, both taken to be scalar functions of time. We shall also refer to this as the open-loop system. Again, we treat the DT case, but essentially everything carries over naturally to CT. Also, for notational simplicity, we omit from (6.21) the direct feedthrough term $\mathtt{d}x[n]$ that has appeared in our system descriptions until now, because this term can complicate the appearance of some of the expressions we derive, without being of much significance in itself; it is easily accounted for if necessary.

Denote the characteristic polynomial of the matrix $\mathbf{A}$ in (6.20) by

$$a(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \prod_{i=1}^{L}(\lambda - \lambda_i) \ . \tag{6.22}$$

The transfer function $H(z)$ of the system (6.20), (6.21) is given by

$$H(z) = \mathbf{c}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \tag{6.23}$$

$$= \frac{\eta(z)}{a(z)} \ . \tag{6.24}$$

(The absence of the direct feedthrough term in (6.21) causes the degree of the polynomial $\eta(z)$ to be strictly less than L. If the feedthrough term was present, the transfer function would simply have $\mathtt{d}$ added to the $H(z)$ above.) Note that there may be pole-zero cancelations involving common roots of $a(z)$ and $\eta(z)$ in (6.24), corresponding to the presence of unreachable and/or unobservable modes of the system. Only the uncanceled roots of $a(z)$ survive as poles of $H(z)$, and similarly only the uncanceled roots of $\eta(z)$ survive as zeros of the transfer function.

We reiterate that the model undoubtedly differs from the plant in many ways, but we shall not examine the effects of various possible sources of discrepancy and uncertainty. A proper treatment of such issues constitutes the field of robust control, which continues to be an active area of research.

Since the state of a system completely summarizes the relevant past of the system, we should expect that knowledge of the state at every instant gives us a powerful basis for designing feedback control signals. In this section we consider the use of state feedback for the system (6.20), assuming that we have access to the entire state vector at each time. Though this assumption is unrealistic in general, it will allow us to develop some preliminary results as a benchmark. We shall later consider what happens when we treat the more realistic situation, where the state cannot be measured but has to be estimated instead. It will turn out in the LTI case that the state estimate provided by an observer will actually suffice to accomplish much of what can be achieved when the actual state is used for feedback.

The particular case of LTI state feedback is represented in Figure 6.3, in which the feedback part of the input $x[n]$ is a constant linear function of the state $\mathbf{q}[n]$ at that instant:

$$x[n] = p[n] + \mathbf{g}^T \mathbf{q}[n] \tag{6.25}$$

where the L-component row vector $\mathbf{g}^T$ is the state feedback gain vector (with ith component g_i), and $p[n]$ is some external input signal that can be used to augment the feedback signal. Thus $x[n]$ is $p[n]$ plus a weighted linear combination of the state variables $q_i[n]$, with constant weights g_i.

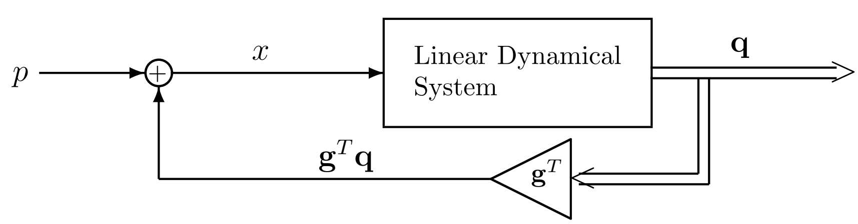

FIGURE 6.3 Linear dynamical system with LTI state feedback. The single lines denote scalar signals and the double lines denote vector signals.

With this choice for $x[n]$, the system (6.20) becomes

$$\begin{aligned} \mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}\Big(p[n] + \mathbf{g}^T\mathbf{q}[n]\Big) \\ &= \Big(\mathbf{A} + \mathbf{b}\mathbf{g}^T\Big)\mathbf{q}[n] + \mathbf{b}p[n] \ . \end{aligned} \tag{6.26}$$

The behavior of this closed-loop system, and in particular its stability, is governed by its natural frequencies, namely by the L eigenvalues of the matrix $\mathbf{A}+\mathbf{b}\mathbf{g}^T$ or the roots of the characteristic polynomial

$$\nu(\lambda) = \det\Big(\lambda\mathbf{I} - (\mathbf{A}+\mathbf{b}\mathbf{g}^T)\Big) \tag{6.27}$$

$$= \lambda^L + \nu_{L-1}\lambda^{L-1} + \cdots + \nu_0 \ . \tag{6.28}$$

Some questions immediately arise:

(i) How much freedom do we have in placing the closed-loop eigenvalues, i.e., the eigenvalues of $\mathbf{A}+\mathbf{b}\mathbf{g}^T$ or the roots of $\nu(\lambda)$, by appropriate choice of the state feedback gain $\mathbf{g}^T$?

(ii) How does state feedback affect reachability, observability and the transfer function of the system?

(iii) How does the choice of $\mathbf{g}^T$ affect the state behavior and the control effort that is required?

Brief answers to these (inter-related) questions are respectively as follows:

(i) By varying the entries of $\mathbf{g}^T$ away from 0, we can move all the reachable eigenvalues of the system (which may number as many as L), and only those eigenvalues. Moreover, appropriate choice of $\mathbf{g}^T$ allows us, in principle, to move the reachable eigenvalues to any arbitrary set of self-conjugate points in the complex plane.

The unreachable eigenvalues of the open-loop system remain eigenvalues of the closed-loop system, and cannot be moved. (This can be explicitly demonstrated by transformation to modal coordinates, but we omit the details.) The reason for this is that the control input cannot access these unreachable modes.

It follows from the preceding claims that a stable closed-loop system can be designed if and only if all unreachable modes of the open-loop system are stable (a property that is termed stabilizability). Also, state feedback can yield an arbitrary closed-loop characteristic polynomial $\nu(\lambda)$ if and only if the open-loop system (6.20) is reachable.

The proof for the above claims is presented in Section 6.4.1.

In designing state feedback control analytically for low-order examples, one way to proceed is by specifying a desired set of closed-loop eigenvalues $\mu_1, \cdots \mu_L$, thus specifying $\nu(\lambda)$ as

$$\nu(\lambda) = \prod_{i=1}^{L}(\lambda - \nu_i) \ . \tag{6.29}$$

Expanding this out and equating it to $\det\Big(\lambda\mathbf{I} - (\mathbf{A}+\mathbf{b}\mathbf{g}^T)\Big)$, as in (6.27), yields L simultaneous linear equations in the unknown gains $g_1, \cdots, g_L$. These equations will be consistent and solvable for the state feedback gains if and only if all the unreachable eigenvalues of the plant are included among the specified closed-loop eigenvalues $\{\mu_i\}$.

The preceding results also suggest an alternative way to determine the unreachable eigenvalues of the given plant: the roots of $\det\Big(\lambda\mathbf{I}-(\mathbf{A}+\mathbf{b}\mathbf{g}^T)\Big)$ that cannot be moved, no matter how $\mathbf{g}^T$ is chosen, are precisely the unreachable eigenvalues of the plant. This approach to exposing unreachable modes can be easier in some problems than the approach used in the previous chapter, which required first computing the eigenvectors $\{\mathbf{v}_i\}$ of the plant, and then checking which of these eigenvectors were not needed in writing $\mathbf{b}$ as a linear combination of the eigenvectors.

[The above discussion has closely paralleled our discussion of observers, except that observability statements have been replaced by reachability statements throughout. The underlying reason for this "duality" is that the eigenvalues of $\mathbf{A}+\mathbf{b}\mathbf{g}^T$ are the same as those of its transpose, namely $\mathbf{A}^T+\mathbf{g}\mathbf{b}^T$. The latter matrix has exactly the structure of the matrix $\mathbf{A}+\ell\mathbf{c}^T$ that was the focus of our discussion of observers, except that $\mathbf{A}$ is now replaced by $\mathbf{A}^T$, and $\mathbf{c}^T$ is replaced by $\mathbf{b}^T$. It is not hard to see that the structure of observable and unobservable modes determined by the pair $\mathbf{A}^T$ and $\mathbf{b}^T$ is the same as the structure of reachable and unreachable modes determined by the pair $\mathbf{A}$ and $\mathbf{b}$.]

(ii) The results in part (i) above already suggest the following fact: that whether or not an eigenvalue is reachable from the external input — i.e., from $x[n]$ for the open-loop system and $p[n]$ for the closed-loop system — is unaffected by state feedback. An unreachable eigenvalue of the open-loop system cannot be excited from the input $x[n]$, no matter how the input is generated, and therefore cannot be excited even in closed loop (which also explains why it cannot be moved by state feedback). Similarly, a reachable eigenvalue of the open-loop system can also be excited in the closed-loop system, because any $x[n]$ that excites it in the open-loop system may be generated in the closed-loop system by choosing $p[n]=x[n]-\mathbf{g}^T\mathbf{q}[n]$.

The proof in Section 6.4.1 of the claims in (i) will also establish that the transfer function of the closed-loop system, from $p[n]$ to $y[n]$, is now

$$H_{cl}(z) = \mathbf{c}^T\Big(z\mathbf{I}-(\mathbf{A}+\mathbf{b}\mathbf{g}^T)\Big)^{-1}\mathbf{b} \tag{6.30}$$

$$= \frac{\eta(z)}{\nu(z)} \,. \tag{6.31}$$

Thus the zeros of the closed-loop transfer function are still drawn from the roots of the same numerator polynomial $\eta(z)$ in (6.24) that contains the zeros of the open-loop system; state feedback does not change $\eta(z)$. However, the actual zeros of the closed-loop system are those roots of $\eta(z)$ that are not canceled by roots of the new closed-loop characteristic polynomial $\nu(z)$, and may therefore differ from the zeros of the open-loop system.

We know from the previous chapter that hidden modes in a transfer function are the result of the modes being unreachable and/or unobservable. Because state feedback cannot alter reachability properties, it follows that any changes in cancelations of roots of $\eta(z)$, in going from the original open-loop system to the closed-loop one, must be the result of state feedback altering the observability properties of the original modes. If an unobservable (but reachable) eigenvalue of the open-loop system is moved by state feedback and becomes observable, then a previously canceled root of $\eta(z)$ is no longer canceled and now appears as a zero of the closed-loop system. Similarly, if an observable (and reachable) eigenvalue of the open-loop system is moved by state feedback to a location where it now cancels a root of $\eta(z)$, then this root is no longer a zero of the closed-loop system, and this hidden mode corresponds to a mode that has been made unobservable by state feed! back.

(iii) We turn now to the question of how the choice of $\mathbf{g}^T$ affects the state behavior and the control effort that is required. Note first that if $\mathbf{g}^T$ is chosen such that the closed-loop system is asymptotically stable, then a bounded external signal $p[n]$ in (6.26) will lead to a bounded state trajectory in the closed-loop system. This is easily seen by considering the transformation of (6.26) to modal coordinates, but we omit the details.

The state feedback gain $\mathbf{g}^T$ affects the closed-loop system in two key ways, first by causing the dynamics to be governed by the eigenvalues of $\mathbf{A} + \mathbf{b}\mathbf{g}^T$ rather than those of $\mathbf{A}$, and second by determining the scaling of the control input $x[n]$ via the relationship in (6.25). This highlights a basic tradeoff between the response rate and the control effort. The state feedback gain can be used to obtain a fast response, to bring the system state from its initially disturbed value rapidly back to the origin — but large entries in $\mathbf{g}^T$ may be needed to do this (certainly in the CT case, but also in DT if the model is a sampled-data version of some underlying CT system), and these large entries in $\mathbf{g}^T$ result in large control effort being expended. Furthermore, the effects of any errors in measuring or estimating the state vector, or of modeling errors and other discrepancies, are likely to be accentuated with large feedback gains. In practice, these considerations would lead us design somewhat conservatively, not attempting to obtain overly fast closed-loop dynamics. Again, some aspects of the tradeoffs involved can be captured in tractable optimization problems, but these are left to more advanced courses.

We work through a CT example first, partly to make clear that our development carries over directly from the DT to the CT case.

EXAMPLE 6.2 Inverted Pendulum with Torque Control

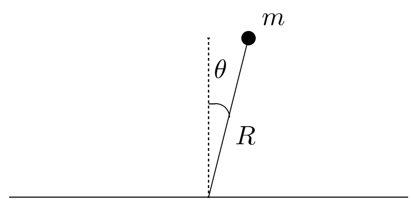

FIGURE 6.4 Inverted pendulum.

Consider the inverted pendulum shown in Figure 6.4, comprising a mass m at the end of a light, hinged rod of length R. For small deviations $\theta(t)$ from the vertical,

$$\frac{d^2\theta(t)}{dt^2} = K\theta(t) + \sigma x(t) \ , \tag{6.32}$$

where $K = g/R$ (g being the acceleration due to gravity), $\sigma = 1/(mR^2)$, and a torque input $x(t)$ is applied at the point of support of the pendulum. Define $q_1(t) = \theta(t)$, $q_2(t) = \dot{\theta}(t)$; then

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ K & 0 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} x(t) \ . \tag{6.33}$$

We could now determine the system eigenvalues and eigenvectors to decide whether the system is reachable. However, this step is actually not necessary in order to assess reachability and compute a state feedback. Instead, considering directly the effect of the state feedback, we find

$$x(t) = \mathbf{g}^T \mathbf{q}(t) \tag{6.34}$$

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ K & 0 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} \begin{bmatrix} g_1 & g_2 \end{bmatrix} \mathbf{q}(t) \tag{6.35}$$

$$= \begin{bmatrix} 0 & 1 \\ K + \sigma g_1 & \sigma g_2 \end{bmatrix} \mathbf{q}(t) \ . \tag{6.36}$$

The corresponding characteristic polynomial is

$$\nu(\lambda) = \lambda^2 - \lambda\sigma g_2 - (K + \sigma g_1) \ . \tag{6.37}$$

Inspection of this expression shows that by appropriate choice of the real gains g_1 and g_2 we can make this polynomial into any desired monic second-degree polynomial. In other words, we can obtain any self-conjugate set of closed-loop eigenvalues. This also establishes that the original system is reachable.

Suppose we want the closed-loop eigenvalues at particular numbers μ_1, μ_2, which is equivalent to specifying the closed-loop characteristic polynomial to be

$$\nu(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2) = \lambda^2 - \lambda(\mu_1 + \mu_2) + \mu_1\mu_2 \ . \tag{6.38}$$

Equating this to the polynomial in (6.37) shows that

$$g_1 = -\frac{\mu_1\mu_2 + K}{\sigma} \qquad \text{and} \qquad g_2 = \frac{\mu_1 + \mu_2}{\sigma} \ . \tag{6.39}$$

Both gains are negative when μ_1 and μ_2 form a self-conjugate set in the open left-half plane.

We return now to the ship steering example introduced earlier.

EXAMPLE 6.3 Ship Steering (continued)

Consider again the DT state-space model in Example 6.1, repeated here for convenience:

$$\begin{aligned}\mathbf{q}[n+1] = \begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} &= \begin{bmatrix} 1 & \sigma \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} \epsilon \\ \sigma \end{bmatrix} x[n] \\ &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \ . \end{aligned} \tag{6.40}$$

(A model of this form is also obtained for other systems of interest, for instance the motion of a DC motor whose input is a voltage that is held constant over intervals of length T by a computer-based controller. In that case, for $x[n]$ in appropriate units, we have $\alpha = 1$, $\sigma = T$, and $\epsilon = T^2/2$.)

For the purposes of this example, take

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{bmatrix} , \quad \mathbf{b} = \begin{bmatrix} \frac{1}{32} \\ \frac{1}{4} \end{bmatrix} \tag{6.41}$$

and set

$$x[n] = g_1 q_1[n] + g_2 q_2[n] \tag{6.42}$$

to get the closed-loop matrix

$$\mathbf{A} + \mathbf{b}\mathbf{g}^T = \begin{bmatrix} 1 + \frac{g_1}{32} & \frac{1}{4} + \frac{g_2}{32} \\ \frac{g_1}{4} & 1 + \frac{g_2}{4} \end{bmatrix} . \tag{6.43}$$

The fastest possible closed-loop response in this DT model is the deadbeat behavior described earlier in Example 6.1, obtained by placing both closed-loop natural frequencies at 0, i.e., choosing the closed-loop characteristic polynomial to be $\nu(\lambda) = \lambda^2$. A little bit of algebra shows that g_1 and g_2 need to satisfy the following equations for this to be achieved:

$$\begin{aligned} \frac{g_1}{32} + \frac{g_2}{4} &= -2 \\ -\frac{g_1}{32} + \frac{g_2}{4} &= -1 \ . \end{aligned} \tag{6.44}$$

Solving these simultaneously, we get $g_1 = -16$ and $g_2 = -6$. We have not shown how to analyze system behavior when there are repeated eigenvalues, but in the particular instance of repeated eigenvalues at 0, it is easy to show that the state will die to 0 in a finite number of steps — at most two steps, for this second-order system. To establish this, note that with the above choice of $\mathbf{g}$ we get

$$\mathbf{A} + \mathbf{b}\mathbf{g}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{16} \\ -4 & -\frac{1}{2} \end{bmatrix} , \tag{6.45}$$

so

$$\left(\mathbf{A} + \mathbf{b}\mathbf{g}^T\right)^2 = 0 , \tag{6.46}$$

which shows that any nonzero initial condition will vanish in two steps. In practice, such deadbeat behavior may not be attainable, as unduly large control effort — rudder angles, in the case of the ship — would be needed. One is likely therefore to aim for slower decay of the error.

Typically, we do not have direct measurements of the state variables, only knowledge of the control input, along with noisy measurements of the system output. The state may then be reconstructed using an observer that produces asymptotically convergent estimates of the state variables, under the assumption that the system (6.20), (6.21) is observable. We shall see in more detail shortly that one can do quite well using the state estimates produced by the observer, in place of direct state measurements, in a feedback control scheme.

6.4.1 Proof of Eigenvalue Placement Results

This subsection presents the proof of the main result claimed earlier for state feedback, namely that it can yield any (monic, real-coefficient) closed-loop characteristic polynomial $\nu(\lambda)$ that includes among its roots all the unreachable eigenvalues of the original system. We shall also demonstrate that the closed-loop transfer function is given by the expression in (6.31).

First transform the open-loop system (6.20), (6.21) to modal coordinates; this changes nothing essential in the system, but simplifies the derivation. Using the same notation for modal coordinates as in the previous chapter, the closed-loop system is now defined by the equations

$$r_i[n+1] = \lambda_i r_i[n] + \beta_i x[n] \ , \quad i = 1, 2, \ldots, L \tag{6.47}$$

$$x[n] = \gamma_1 r_1[n] + \cdots + \gamma_L r_L[n] + p[n] \ , \tag{6.48}$$

where

$$\begin{pmatrix} \gamma_1 & \cdots & \gamma_L \end{pmatrix} = \mathbf{g}^T \mathbf{V} \ , \tag{6.49}$$

and $\mathbf{V}$ is the modal matrix, whose columns are the eigenvectors of the open-loop system. The γ_i are therefore just the state-feedback gains in modal coordinates.

Now using (6.47) and (6.48) to evaluate the transfer function from $p[n]$ to $x[n]$, we get

$$\frac{X(z)}{P(z)} = \Big(1 - \sum_{1}^{L} \frac{\gamma_i \beta_i}{z - \lambda_i}\Big)^{-1} = \frac{a(z)}{\nu(z)} \ . \tag{6.50}$$

To obtain the second equality in the above equation, we have used the following facts: (ii) the open-loop characteristic polynomial $a(z)$ is given by (6.22), and this is what appears in the numerator of (6.50; (ii) the poles of this transfer function must be the closed-loop poles of the system, and its denominator degree must equal its numerator degree, so the denominator of this expression must be the closed-loop characteristic polynomial $\nu(z)$. Then using (6.24), we find that the overall transfer function from the input $p[n]$ of the closed-loop system to the output $y[n]$ is

$$\frac{Y(z)}{P(z)} = \frac{Y(z)}{X(z)} \frac{X(z)}{P(z)} \tag{6.51}$$

$$= \frac{\eta(z)}{a(z)} \frac{a(z)}{\nu(z)} \tag{6.52}$$

$$= \frac{\eta(z)}{\nu(z)} \ . \tag{6.53}$$

The conclusion from all this is that state feedback has changed the denominator of the input-output transfer function expression from $a(z)$ in the open-loop case to $\nu(z)$ in the closed-loop case, and has accordingly modified the characteristic polynomial and poles. State feedback has left unchanged the numerator polynomial $\eta(z)$ from which the zeros are selected; all roots of $\eta(z)$ that are not canceled by roots of $\nu(z)$ will appear as zeros of the closed-loop transfer function.

Inverting (6.50), we find

$$\frac{\nu(z)}{a(z)} = 1 - \sum_{1}^{L} \frac{\gamma_i \beta_i}{z - \lambda_i} \ . \tag{6.54}$$

Hence, given the desired closed-loop characteristic polynomial $\nu(\lambda)$, we can expand $\nu(z)/a(z)$ in a partial fraction expansion, and determine the state feedback gain γ_i (in modal coordinates) for each i by dividing the coefficient of $1/(z - \lambda_i)$ by $-\beta_i$, assuming this is nonzero, i.e., assuming the ith mode is reachable. If the jth mode is unreachable, so $\beta_j = 0$, then λ_j does not appear as a pole on the right side of (6.54), which must mean that $\nu(z)$ has to contain $z - \lambda_j$ as a factor (in order for this factor to cancel out on the left side of the equation), i.e., every unreachable natural frequency of the open-loop system has to remain as a natural frequency of the closed-loop system.

6.5 OBSERVER-BASED FEEDBACK CONTROL

The obstacle to state feedback is the general unavailability of direct measurements of the state. All we typically have are knowledge of what control signal $x[n]$ we are applying, along with (possibly noise-corrupted) measurements of the output $y[n]$, and a nominal model of the system. We have already seen how to use this information to estimate the state variables, using an observer or state estimator. Let us therefore consider what happens when we use the state estimate provided by the observer, rather than the (unavailable) actual state, in the feedback control law (6.25). With this substitution, (6.25) is modified to

$$\begin{aligned} x[n] &= p[n] + \mathbf{g}^T\widehat{\mathbf{q}}[n] \\ &= p[n] + \mathbf{g}^T(\mathbf{q}[n] - \widetilde{\mathbf{q}}[n]) \; . \end{aligned} \tag{6.55}$$

The overall closed-loop system is then as shown in Figure 6.5, and is governed by the following state-space model, obtained by combining the representations of the subsystems that make up the overall system, namely the plant (6.1), observer error dynamics (6.9), and feedback control law (6.55):

$$\begin{bmatrix} \mathbf{q}[n+1] \\ \widetilde{\mathbf{q}}[n+1] \end{bmatrix} = \begin{bmatrix} \mathbf{A}+\mathbf{b}\mathbf{g}^T & -\mathbf{b}\mathbf{g}^T \\ 0 & \mathbf{A}+\ell\mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{q}[n] \\ \widetilde{\mathbf{q}}[n] \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} p[n] + \begin{bmatrix} \mathbf{I} \\ \mathbf{I} \end{bmatrix} \mathbf{w}[n] + \begin{bmatrix} 0 \\ \ell \end{bmatrix} \zeta[n] \; . \tag{6.56}$$

Note that we have reverted here to the more elaborate plant representation in (6.1), (6.2) rather than the streamlined one in (6.20), (6.21), in order to display the effect of plant disturbance and measurement error on the overall closed-loop system. (Instead of choosing the state vector of the overall system to comprise the state vector $\mathbf{q}[n]$ of the plant and the state vector $\widetilde{\mathbf{q}}[n]$ of the error equation, we could equivalently have picked $\mathbf{q}[n]$ and $\widehat{\mathbf{q}}[n]$. The former choice leads to more transparent expressions.)

The (block) triangular structure of the state matrix in (6.56) allows us to conclude that the natural frequencies of the overall system are simply the eigenvalues of $\mathbf{A}+\mathbf{b}\mathbf{g}^T$ along with those of $\mathbf{A}+\ell\mathbf{c}^T$. (This is not hard to demonstrate, either based on the definition of eigenvalues and eigenvectors, or using properties of determinants, but we omit the details.) In other words, our observer-based feedback control law results in a nicely behaved closed-loop system, with natural frequencies that are the union of those obtained with perfect state feedback and those obtained for the observer error equation. Both sets of natural frequencies can be arbitrarily selected, provided the open-loop system is reachable and observable. One would normally pick the modes that govern observer error decay to be faster than those associated with state feedback, in order to have reasonably accurate estimates available to the feedback control law before the plant state can wander too far away from what is desired.

The other interesting fact is that the transfer function from $p[n]$ to $y[n]$ in the new closed-loop system is exactly what would be obtained with perfect state feedback, namely the transfer function in (6.46). The reason is that the condition under which the transfer function is computed — as the input-output response when starting from the zero state — ensures that the observer starts up from the same initial condition as the plant. This in turn ensures that there is no estimation error, so the estimated state is as good as the true state. Another way to see this is to note that the observer error modes are unobservable from the available measurements.

The preceding observer-based compensator is the starting point for a very general and powerful approach to control design, one that carries over to the multi-input,

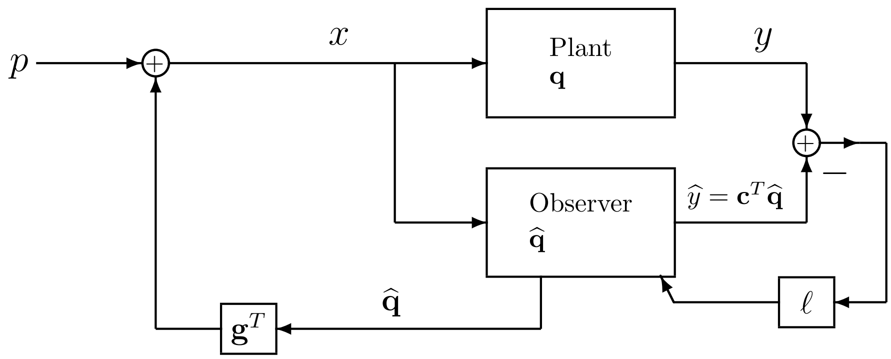

FIGURE 6.5 Observer-based compensator, feeding back an LTI combination of the estimated state variables.

multi-output case. With the appropriate embellishments around this basic structure, one can obtain every possible stabilizing LTI feedback controller for the system (6.20), (6.21). Within this class of controllers, we can search for those that have good robustness properties, in the sense that they are relatively immune to the uncertainties in our models. Further exploration of all this has to be left to more advanced courses.