Chance constrained optimization

- chance constraints and percentile optimization
- chance constraints for log-concave distributions
- convex approximation of chance constraints

sources: Rockafellar & Uryasev, Nemirovsky & Shapiro

Chance constraints and percentile optimization

- 'chance constraints' (η is 'confidence level'):

$$\mathbf{Prob}(f_i(x,\omega) \le 0) \ge \eta$$

  - convex in some cases (later)
  - generally interested in $\eta = 0.9,\ 0.95,\ 0.99$
  - $\eta = 0.999$ meaningless (unless you're sure about the distribution tails)

- percentile optimization (γ is 'η-percentile'):

$$\begin{array}{ll} \mbox{minimize} & \gamma \\ \mbox{subject to} & \mathbf{Prob}(f_0(x,\omega) \le \gamma) \ge \eta \end{array}$$

  - convex or quasi-convex in some cases (later)

Value-at-risk and conditional value-at-risk

- value-at-risk of random variable z, at level η:

$$\mathbf{VaR}(z;\eta) = \inf\{\gamma \mid \mathbf{Prob}(z \le \gamma) \ge \eta\}$$

  – chance constraint $\mathbf{Prob}(f_i(x,\omega) \le 0) \ge \eta$ same as $\mathbf{VaR}(f_i(x,\omega);\eta) \le 0$

- conditional value-at-risk:

$$\mathbf{CVaR}(z;\eta) = \inf_{\beta} \left(\beta + 1/(1-\eta)\,\mathbf{E}(z-\beta)_+\right)$$

  – $\mathbf{CVaR}(z;\eta) \ge \mathbf{VaR}(z;\eta)$ (more on this later)

CVaR interpretation

(for continuous distributions)

- in $\mathbf{CVaR}$ definition, $\beta^\star = \mathbf{VaR}(z;\eta)$:

$$0 = \frac{d}{d\beta}\left(\beta + 1/(1-\eta)\,\mathbf{E}(z-\beta)_+\right) = 1 - 1/(1-\eta)\,\mathbf{Prob}(z \geq \beta)$$

  so $\mathbf{Prob}(z \geq \beta^\star) = 1-\eta$

- conditional tail expectation (or expected shortfall)

$$\begin{aligned}
\mathbf{E}(z|z \geq \beta^\star) &= \mathbf{E}(\beta^\star + (z-\beta^\star)|z \geq \beta^\star) \\
&= \beta^\star + \mathbf{E}((z-\beta^\star)_+)/\,\mathbf{Prob}(z \geq \beta^\star) \\
&= \mathbf{CVaR}(z;\eta)
\end{aligned}$$

Chance constraints for log-concave distributions

- suppose
  - ω has log-concave density $p(\omega)$
  - $C = \{(x, \omega) \mid f(x, \omega) \leq 0\}$ is convex in (x, ω)
- then

$$\mathbf{Prob}(f(x, \omega) \leq 0) = \int 1((x, \omega) \in C) p(\omega) \, d\omega$$

  is log-concave, since integrand is

- so chance constraint $\mathbf{Prob}(f(x, \omega) \leq 0) \geq \eta$ can be expressed as convex constraint

$$\log \mathbf{Prob}(f(x, \omega) \leq 0) \geq \log \eta$$

Linear inequality with normally distributed parameter

- consider $a^T x \leq b$, with $a \sim \mathcal{N}(\bar{a}, \Sigma)$
- then $a^T x - b \sim \mathcal{N}(\bar{a}^T x - b, x^T \Sigma x)$
- hence

$$\mathbf{Prob}(a^T x \leq b) = \Phi\left(\frac{b - \bar{a}^T x}{\sqrt{x^T \Sigma x}}\right)$$

- and so

$$\mathbf{Prob}(a^T x \leq b) \geq \eta \iff b - \bar{a}^T x \geq \Phi^{-1}(\eta)\|\Sigma^{1/2} x\|_2$$

a second-order cone constraint for $\eta \geq 0.5$ ($i.e.$, $\Phi^{-1}(\eta) \geq 0$)

Portfolio optimization example

- $x \in \mathbf{R}^n$ gives portfolio allocation; x_i is (fractional) position in asset i
- x must satisfy $\mathbf{1}^T x = 1$, $x \in \mathcal{C}$ (convex portfolio constraint set)
- portfolio return (say, in percent) is $p^T x$, where $p \sim \mathcal{N}(\bar{p}, \Sigma)$
  (a more realistic model is p log-normal)
- maximize expected return subject to limit on probability of loss

- problem is

$$
\begin{array}{ll}
\text{maximize} & \mathbf{E}\, p^T x \\
\text{subject to} & \mathbf{Prob}(p^T x \leq 0) \leq \beta \\
& \mathbf{1}^T x = 1, \quad x \in \mathcal{C}
\end{array}
$$

- can be expressed as convex problem (provided $\beta \leq 1/2$)

$$
\begin{array}{ll}
\text{maximize} & \bar{p}^T x \\
\text{subject to} & \bar{p}^T x \geq \Phi^{-1}(1-\beta)\|\Sigma^{1/2} x\|_2 \\
& \mathbf{1}^T x = 1, \quad x \in \mathcal{C}
\end{array}
$$

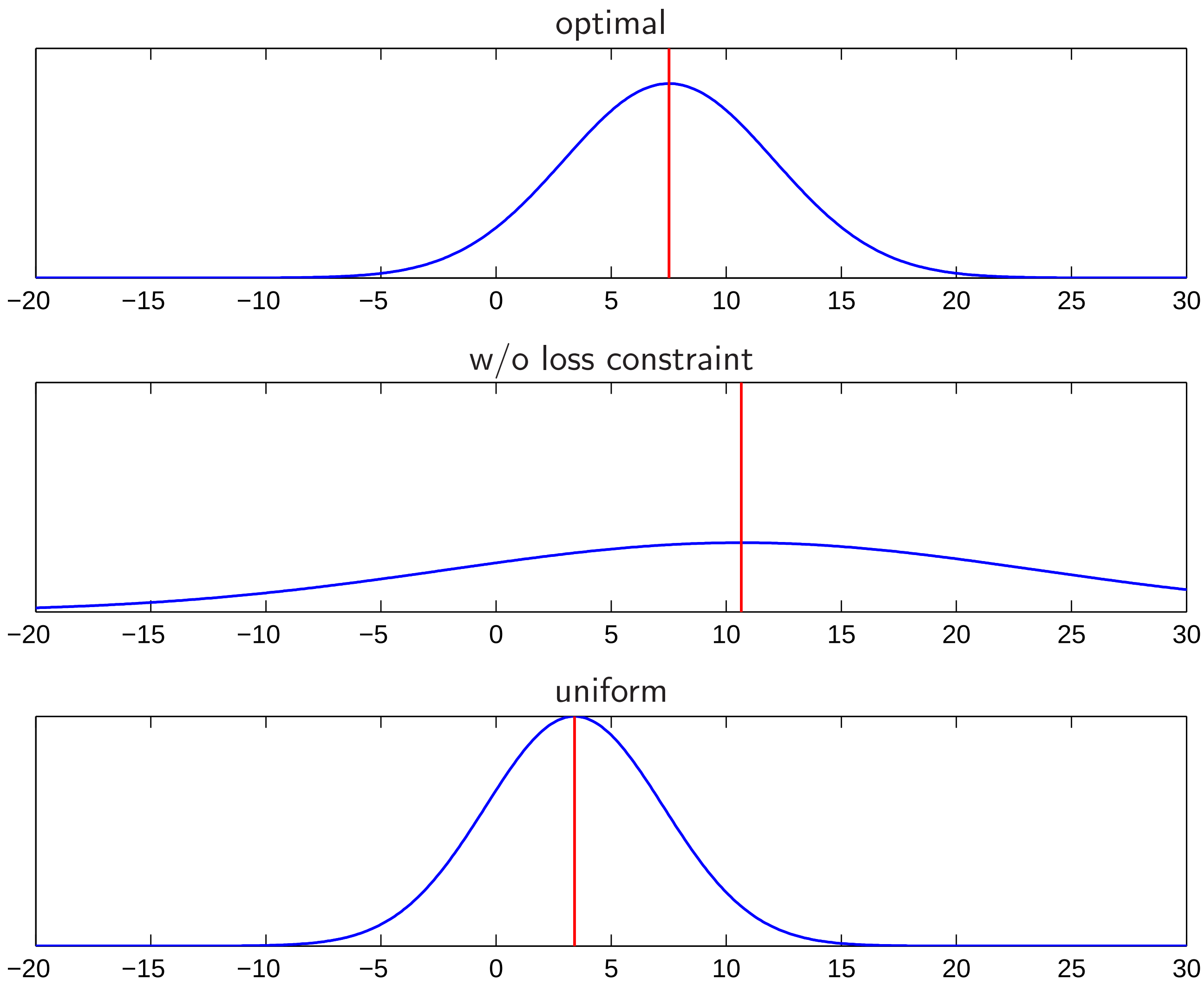
(an SOCP when $\mathcal{C}$ is polyhedron)

Example

- $n = 10$ assets, $\beta = 0.05$, $\mathcal{C} = \{x \mid x \succeq -0.1\}$
- compare
  - optimal portfolio
  - optimal portfolio w/o loss risk constraint
  - uniform portfolio $(1/n)\mathbf{1}$

portfolio	$\mathbf{E}\, p^T x$	$\mathbf{Prob}(p^T x \leq 0)$
optimal	7.51	5.0%
w/o loss constraint	10.66	20.3%
uniform	3.41	18.9%

return distributions:



Convex approximation of chance constraint bound

- assume $f_i(x, \omega)$ is convex in x
- suppose $\phi : \mathbf{R} \to \mathbf{R}$ is nonnegative convex nondecreasing, with $\phi(0) = 1$
- for any $\alpha_i > 0$, $\phi(z/\alpha_i) \geq 1(z > 0)$ for all z, so

$$\mathbf{E}\,\phi(f_i(x, \omega)/\alpha_i) \geq \mathbf{Prob}(f_i(x, \omega) > 0)$$

- hence (convex) constraint

$$\mathbf{E}\,\phi(f_i(x, \omega)/\alpha_i) \leq 1 - \eta$$

  ensures chance constraint $\mathbf{Prob}(f_i(x, \omega) \leq 0) \geq \eta$ holds

- this holds for any $\alpha_i > 0$; we now show how to optimize over α_i

- write constraint as

$$\mathbf{E}\,\alpha_i\phi(f_i(x,\omega)/\alpha_i) \leq \alpha_i(1-\eta)$$

  - (perspective function) $v\phi(u/v)$ is convex in (u,v) for $v>0$, nondecreasing in u
  - so composition $\alpha_i\phi(f_i(x,\omega)/\alpha_i)$ is convex in (x,α_i) for $\alpha_i>0$
  - hence constraint above is convex in x *and* α_i
  - so we can optimize over x *and* $\alpha_i>0$ via convex optimization

- yields a convex stochastic optimization problem that is a conservative approximation of the chance-constrained problem

- we'll look at some special cases

Markov chance constraint bound

- taking $\phi(u) = (u+1)_+$ gives Markov bound: for any $\alpha_i > 0$,

$$\mathbf{Prob}(f_i(x,\omega) > 0) \le \mathbf{E}(f_i(x,\omega)/\alpha_i + 1)_+$$

- convex approximation constraint

$$\mathbf{E}\,\alpha_i(f_i(x,\omega)/\alpha_i + 1)_+ \le \alpha_i(1-\eta)$$

can be written as

$$\mathbf{E}(f_i(x,\omega) + \alpha_i)_+ \le \alpha_i(1-\eta)$$

- we can optimize over x and $\alpha_i \ge 0$

Interpretation via conditional value-at-risk

- write conservative approximation as

$$\frac{\mathbf{E}(f_i(x,\omega)+\alpha_i)_+}{1-\eta} - \alpha_i \leq 0$$

- LHS is convex in (x, α_i), so minimum over α_i,

$$\inf_{\alpha_i > 0} \left(\frac{\mathbf{E}(f_i(x,\omega)+\alpha_i)_+}{1-\eta} - \alpha_i \right)$$

  is convex in x

- this is $\mathbf{CVaR}(f_i(x,\omega);\eta)$ (can show $\alpha_i > 0$ can be dropped)

- so convex approximation replaces $\mathbf{VaR}(f_i(x,\omega);\eta) \leq 0$ with $\mathbf{CVaR}(f_i(x,\omega);\eta) \leq 0$ which is convex in x

Chebyshev chance constraint bound

- taking $\phi(u) = (u+1)_+^2$ yields Chebyshev bound: for any $\alpha_i > 0$,

$$\mathbf{Prob}(f_i(x,\omega) > 0) \leq \mathbf{E}(f_i(x,\omega)/\alpha_i + 1)_+^2$$

- convex approximation constraint

$$\mathbf{E}\,\alpha_i(f_i(x,\omega)/\alpha_i + 1)_+^2 \leq \alpha_i(1-\eta)$$

can be written as

$$\mathbf{E}(f_i(x,\omega) + \alpha_i)_+^2/\alpha_i \leq \alpha_i(1-\eta)$$

Traditional Chebyshev bound

- dropping subscript $+$ we get more conservative constraint

$$\mathbf{E}\,\alpha_i(f_i(x,\omega)/\alpha_i + 1)^2 \leq \alpha_i(1-\eta)$$

which we can write as

$$2\,\mathbf{E}\,f_i(x,\omega) + (1/\alpha_i)\,\mathbf{E}\,f_i(x,\omega)^2 + \alpha_i\eta \leq 0$$

- minimizing over α_i gives $\alpha_i = \left(\mathbf{E}\,f_i(x,\omega)^2/\eta\right)^{1/2}$; yields constraint

$$\mathbf{E}\,f_i(x,\omega) + \left(\eta\,\mathbf{E}\,f_i(x,\omega)^2\right)^{1/2} \leq 0$$

which depends only on first and second moments of f_i

Example

- $f_i(x) = a^T x - b$, where a is random with $\mathbf{E}\, a = \bar{a}$, $\mathbf{E}\, aa^T = \Sigma$
- traditional Chebyshev approximation of chance constraint is

$$\bar{a}^T x - b + \eta^{1/2} \left(x^T \Sigma x - 2b\bar{a}^T x + b^2 \right)^{1/2} \leq 0$$

- can write as second-order cone constraint

$$\bar{a}^T x - b + \eta^{1/2} \|(z, y)\|_2 \leq 0$$

with $z = \Sigma^{1/2} x - b\Sigma^{-1/2}\bar{a}$, $\quad y = b\left(1 - \bar{a}^T \Sigma^{-1} \bar{a}\right)^{1/2}$

- can interpret as certainty-equivalent constraint, with norm term as 'extra margin'

Chernoff chance constraint bound

- taking $\phi(u) = \exp u$ yields Chernoff bound: for any $\alpha_i > 0$,

$$\mathbf{Prob}(f_i(x,\omega) > 0) \leq \mathbf{E}\exp(f_i(x,\omega)/\alpha_i)$$

- convex approximation constraint

$$\mathbf{E}\,\alpha_i \exp(f_i(x,\omega)/\alpha_i) \leq \alpha_i(1-\eta)$$

can be written as

$$\log \mathbf{E}\exp(f_i(x,\omega)/\alpha_i) \leq \log(1-\eta)$$

(LHS is cumulant generating function of $f_i(x,\omega)$, evaluated at $1/\alpha_i$)

Example

- maximize a linear revenue function (say) subject to random linear constraints holding with probability η:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \mathbf{Prob}(\max(Ax - b) \leq 0) \geq \eta \end{array}$$

with variable $x \in \mathbf{R}^n$; $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$ random (Gaussian)

- Markov/CVaR approximation:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \mathbf{E}(\max(Ax - b) + \alpha)_+ \leq \alpha(1 - \eta) \end{array}$$

with variables $x \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$

- Chebyshev approximation:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \mathbf{E}(\max(Ax-b)+\alpha)_+^2/\alpha \leq \alpha(1-\eta) \end{array}$$

with variables $x \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$

- optimal values of these approximate problems are lower bounds for original problem

- instance with $n = 5$, $m = 10$, $\eta = 0.9$
- solve approximations with sampling method with $N = 1000$ training samples, validate with $M = 10000$ samples
- compare to solution of deterministic problem

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & \mathbf{E}\, Ax \leq \mathbf{E}\, b \end{array}$$

- estimates of $\mathbf{Prob}(\max(Ax - b) \leq 0)$ on training/validation data

	$c^T x$	train	validate
Markov	3.60	0.97	0.96
Chebyshev	3.43	0.97	0.96
deterministic	7.98	0.04	0.03

- PDF of $\max(Ax - b)$ for Markov approximation solution