Examples of Transient RC and RL Circuits.
The Series RLC Circuit

Impulse response of RC Circuit.

Let's examine the response of the circuit shown on Figure 1. The form of the source voltage Vs is shown on Figure 2.

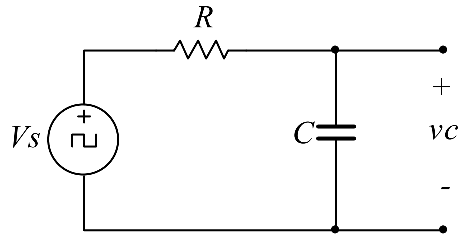


Figure 1. RC circuit

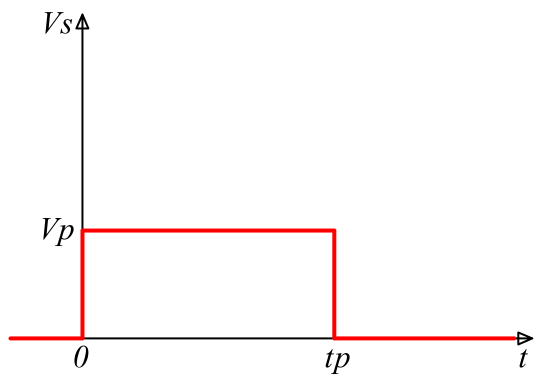


Figure 2.

We will investigate the response $vc(t)$ as a function of the τp and Vp.

The general response is given by:

$$vc(t) = Vp\left(1 - e^{\frac{-t}{RC}}\right) \quad 0 \le t \le tp \tag{1.1}$$

If $tp \gg RC$ the capacitor voltage at $t = tp$ is equal to Vp. Therefore for times $t > tp$ the response becomes

$$vc(t) = Vp\left(e^{\frac{-(t-tp)}{RC}}\right) \quad tp \le t \tag{1.2}$$

A general plot of the response is shown on Figure 3 for
$RC = 1\sec, \quad tp = 6\sec, \quad Vp = 10Volts$

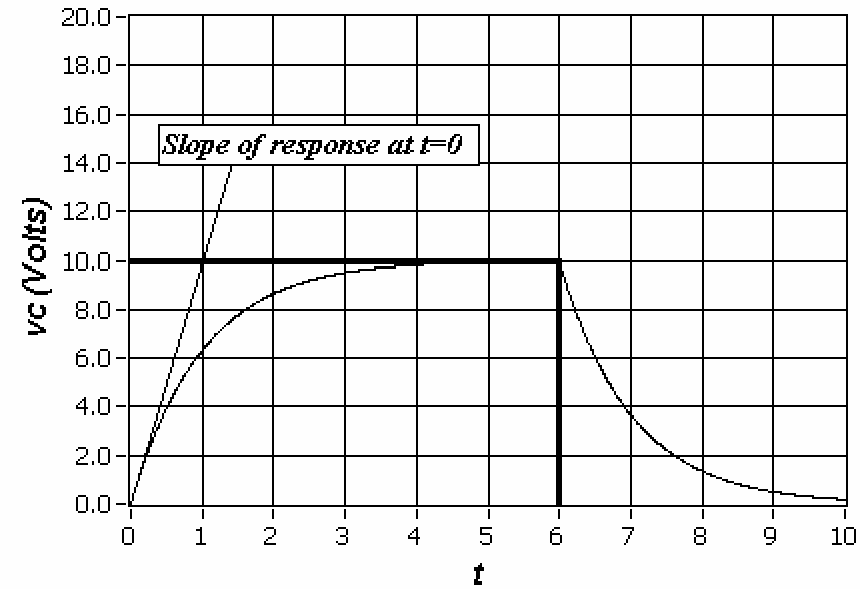


Figure 3

If the pulse becomes narrower, the value of *vc* will not reach the maximum value.

By expanding the exponential in Equation (1.1) we obtain,

$$vc(t) = Vp\left(1 - \left[1 - \frac{t}{RC} + \frac{1}{2}\left(\frac{t}{RC}\right)^2 - \frac{1}{6}\left(\frac{t}{RC}\right)^3 + \ldots\right]\right) \quad 0 \le t \le tp \tag{1.3}$$

When $RC \gg t$ the higher order terms may be neglected resulting in

$$vc(t) \simeq Vp\frac{t}{RC} \quad 0 \le t \le tp \tag{1.4}$$

At the end of the pulse (at $t = tp$) the voltage becomes

$$vc(t = tp) \simeq \frac{Vptp}{RC} \tag{1.5}$$

For $t > tp$ the response becomes

$$vc = \frac{Vp\,tp}{RC}\left(e^{\frac{-(t-tp)}{RC}}\right) \tag{1.6}$$

The product $Vp\,tp$ is the area of the pulse and thus the response is proportional to that area. As the pulse becomes narrower (i.e. as $tp \to 0$) equation (1.6) simplifies to

$$vc \simeq \frac{Vp\,tp}{RC}\left(e^{\frac{-t}{RC}}\right) \tag{1.7}$$

If we constrain the area of the impulse to a constant $A = Vp\ tp$, then as the pulse becomes narrower, the amplitude Vp increases, resulting in an impulse of strength *A*. Therefore the response of an impulse of strength A is

$$vc = \frac{A}{RC}e^{\frac{-t}{RC}} \tag{1.8}$$

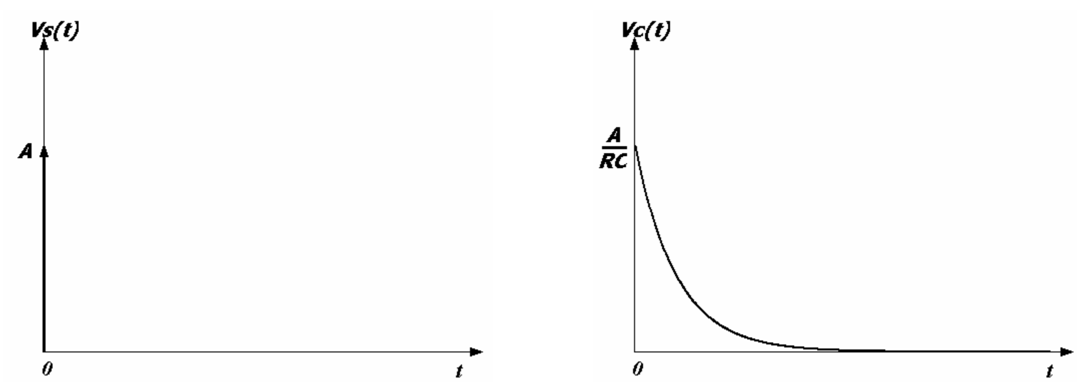


Figure 4. Impulse response of RC circuit

The spark plug in your car (a simplified model)

Consider the circuit shown on Figure 5. The battery *Vb* corresponds to the 12 Volt car battery. The spark plug is connected actors the inductor and current may flow though it only if the voltage across the gap of the plug exceeds a very large value (about 20 kV).

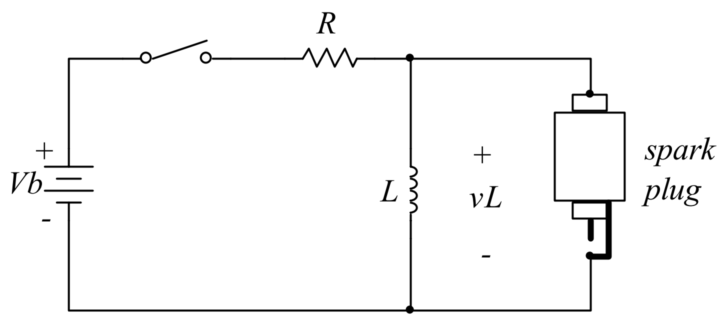


Figure 5

When the switch is closed, the current through the inductor reaches a maximum value of Vb/R. The equation that describes the evolution of the current with the switch closed is

$$i(t) = \frac{Vb}{R}\left(1 - e^{\frac{-t}{L/R}}\right) \qquad (1.9)$$

And the corresponding voltage across the inductor is given by

$$vL(t) = Vb\ e^{\frac{-t}{L/R}} \qquad (1.10)$$

When the switch is opened, the current path is effectively broken and thus the time rate of change of the current becomes arbitrarily large. Since the voltage is proportional to di/dt, the voltage developed across the inductor could become very large.

As an example, let's consider a system with a resistance of 5Ω, a solenoid with an inductance of 10mH connected to a 12 Volt battery. How long does it take for the solenoid to reach 99% of its maximum value? If the switch is opened in 1μs, what is the voltage developed across the solenoid?

The time constant of the system is

$$\frac{L}{R} = \frac{0.01}{5} = 0.002\,\text{sec}$$

The maximum current that can flow in the system is $\frac{12}{5}A = 2.4A$. The time to reach 99% of the maximum value is given by

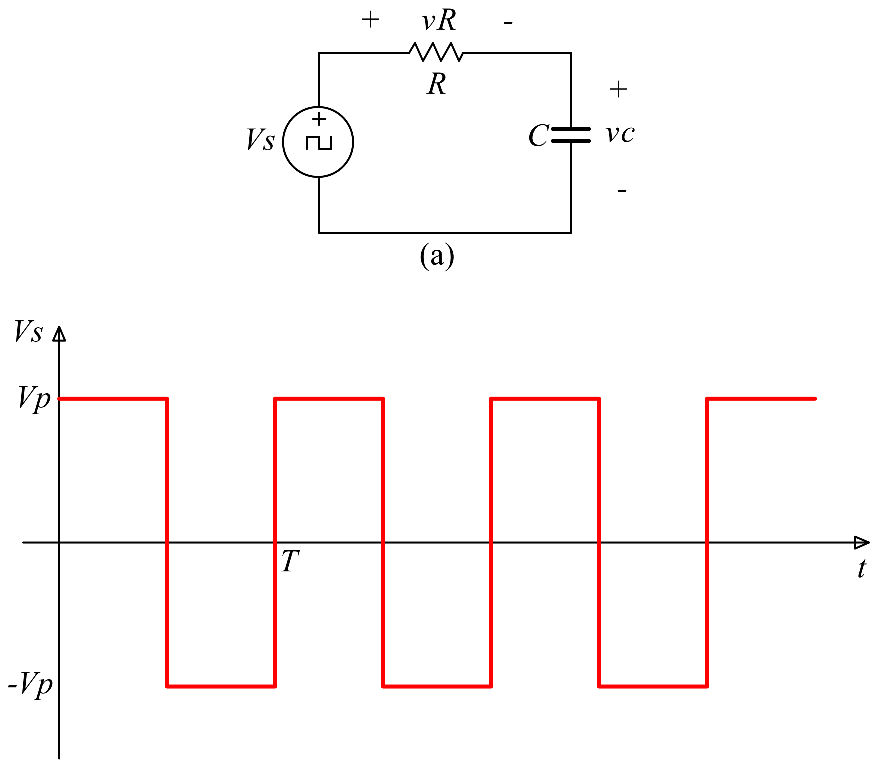
$$0.99 = 1 - e^{\frac{-t}{0.002}}$$

The voltage across the coil when the switch is opened is

$$v = L\frac{\Delta i}{\Delta t} = 0.01\frac{2.4}{1\times 10^{-6}} = 24kV$$

Response of *RC* circuit driven by a square wave.

Let's now consider the *RC* circuit shown on Figure 6(a) driven by a square wave signal of the form shown on Figure 6(b).



(b)
Figure 6

The response *vc(t)* is given by

$$\text{response} = \text{final value} + \left[\text{initial value - final value}\right]e^{\frac{-t}{\tau}} \qquad (1.11)$$

By assuming that the initial value of the voltage across the capacitor is $-Vp$ the response during the first half cycle of the square wave is

$$\begin{aligned} vc(t) &= Vp + \left[-Vp - Vp\right]e^{\frac{-t}{RC}} \\ &= Vp\left[1 - 2e^{\frac{-t}{RC}}\right] \end{aligned} \qquad (1.12)$$

During the second half cycle the initial condition is

$$vc(T/2) = Vp\left[1 \ - \ 2e^{\frac{-T/2}{RC}}\right] \tag{1.13}$$

And the complete response during the second half of the first cycle becomes

$$vc(t) \ = \ -Vp \ + \ \left[Vp\left[1 \ - \ 2e^{\frac{-T/2}{RC}}\right] + Vp\right]e^{\frac{-t}{RC}} \tag{1.14}$$

Similarly the response during the first part of the second cycle starts with the value of *vc* at *t=T* and evolves towards the value *Vp*.

If the time constant is small compared to the period of the square wave, the response will reach the maximum and minimum values of the square wave as shown on Figure 7, where $RC = 1\times 10^{-4}$ sec and thus *T/2=10RC*.

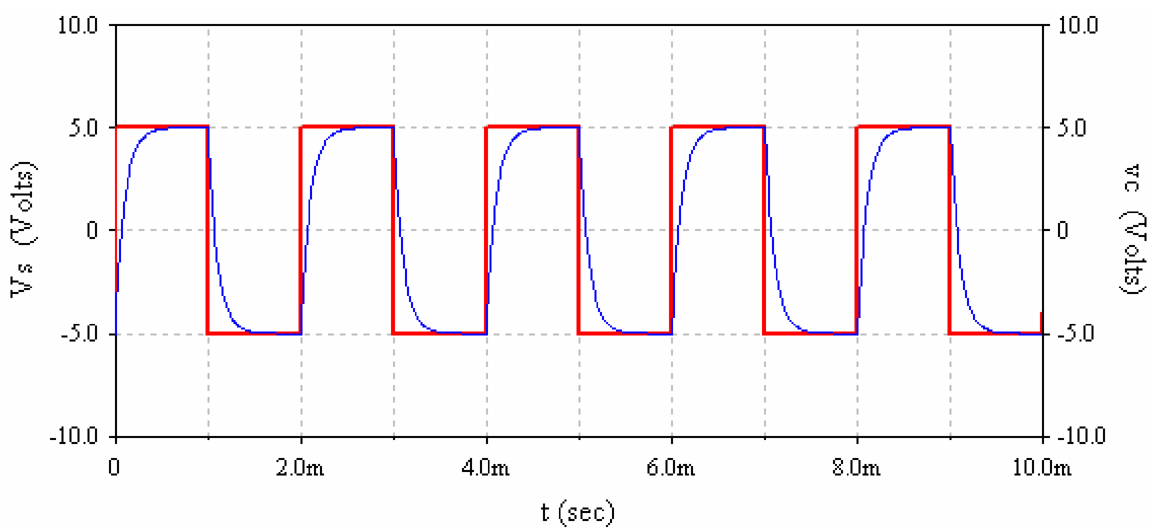


Figure 7

As the time constant *RC* increases, it takes longer for the response to reach the maximum value. Figure 8 shows a plot of the response for *T/2=RC*. Note that the response does not reach the maximum values of the input signal and the average value of the response is equal to the average value of the input signal.

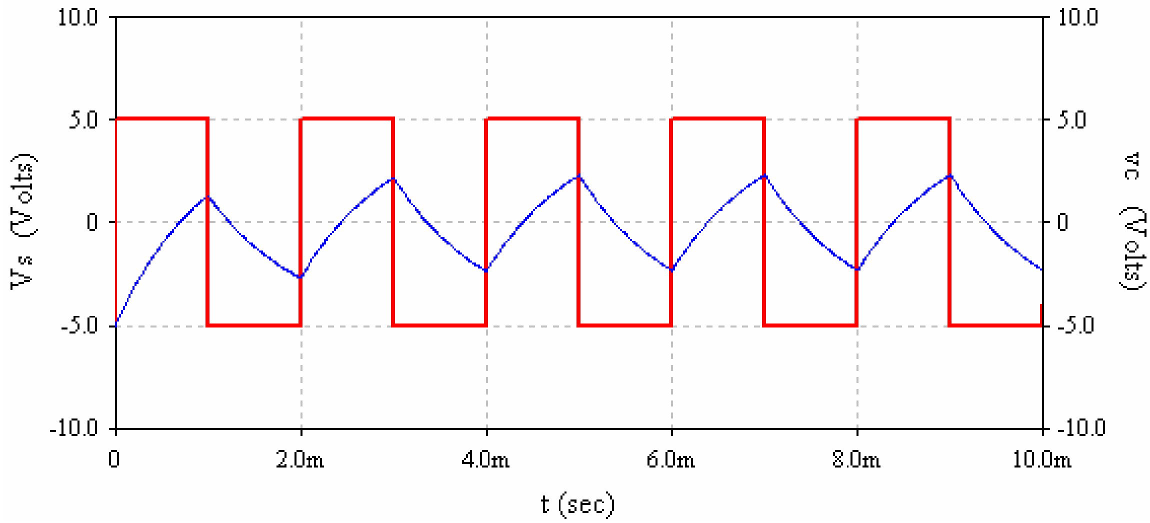
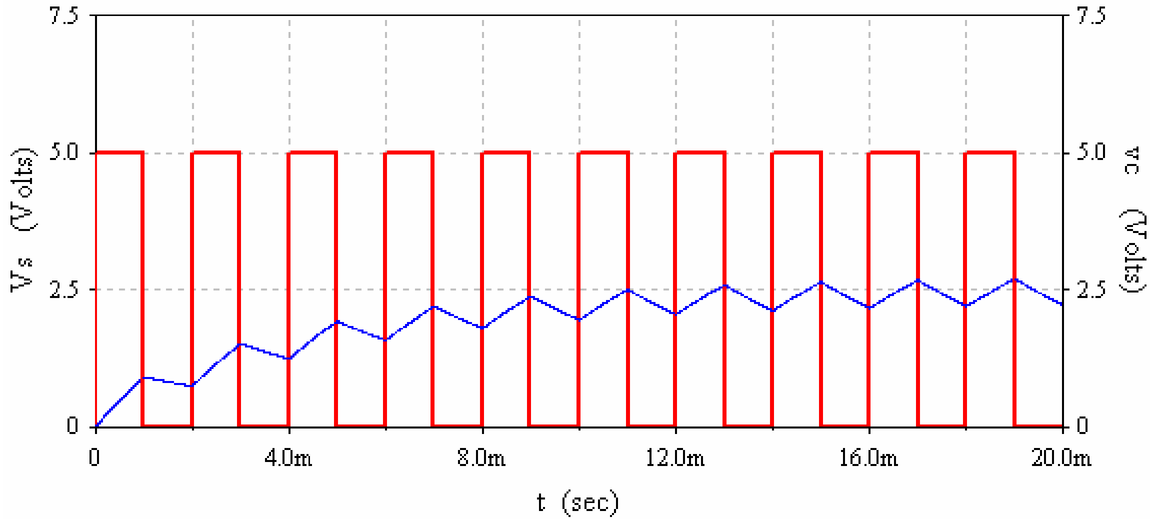


Figure 8

Figure 9(a) and Figure 9(b) show the system response for *RC=5T/2* for a square wave with a duty factor of 50% that varies between 0 and 5 Volts. Notice that the average value is reached within a certain number of oscillations and that there is a variation of the response "ripple" about the average value. The magnitude of this ripple is inversely proportional to the time constant *RC*.

This is the first step that one must take when an AC signal is converted to DC. Next week, when we learn about the diode, we will explore this circuit further.


7.5
5.0
2.5
0
Vs (Volts)
vc (Volts)
0
4.0m
8.0m
12.0m
16.0m
20.0m
t (sec)
(a)

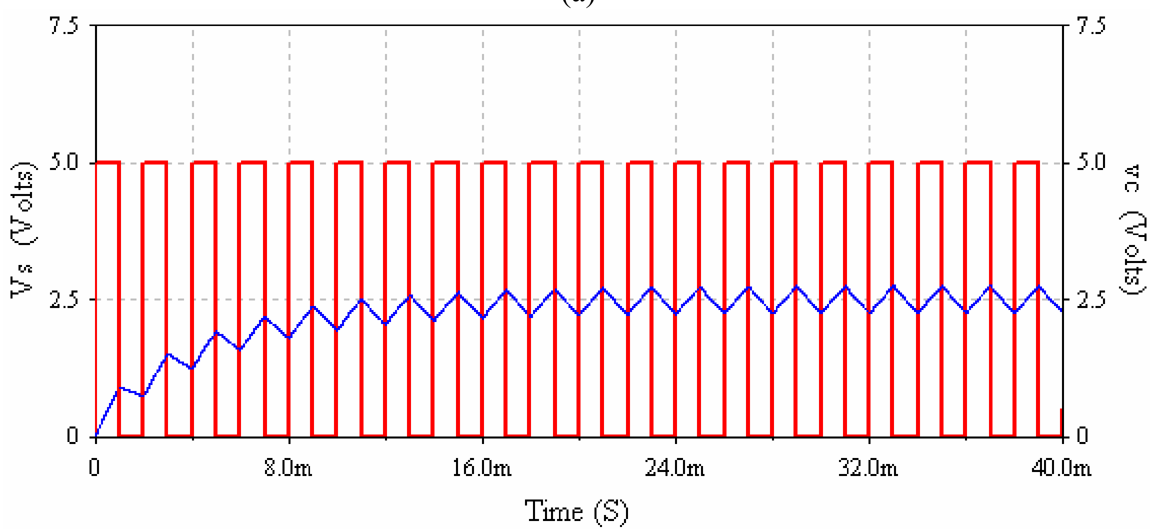

7.5
5.0
2.5
0
Vs (Volts)
vc (Volts)
0
8.0m
16.0m
24.0m
32.0m
40.0m
Time (S)
(b)

Figure 9

Second Order Circuits

Series *RLC* circuit

The circuit shown on Figure 10 is called the series *RLC* circuit. We will analyze this circuit in order to determine its transient characteristics once the switch *S* is closed.

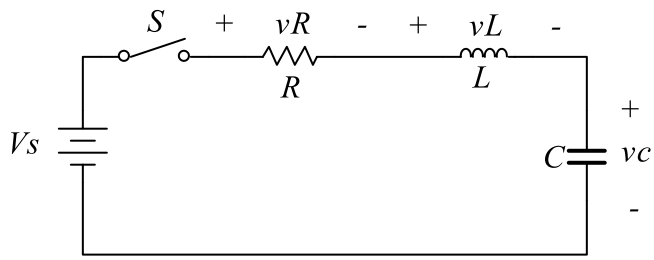


Figure 10

The equation that describes the response of the system is obtained by applying KVL around the mesh

$$vR + vL + vc = Vs \tag{1.15}$$

The current flowing in the circuit is

$$i = C\frac{dvc}{dt} \tag{1.16}$$

And thus the voltages *vR* and *vL* are given by

$$vR = iR = RC\frac{dvc}{dt} \tag{1.17}$$

$$vL = L\frac{di}{dt} = LC\frac{d^2vc}{dt^2} \tag{1.18}$$

Substituting Equations (1.17) and (1.18) into Equation (1.15) we obtain

$$\frac{d^2vc}{dt^2} + \frac{R}{L}\frac{dvc}{dt} + \frac{1}{LC}vc = \frac{1}{LC}Vs \tag{1.19}$$

The solution to equation (1.19) is the linear combination of the homogeneous and the particular solution $vc = vc_p + vc_h$

The particular solution is

$$vc_p = Vs \tag{1.20}$$

And the homogeneous solution satisfies the equation

$$\frac{d^2 vc_h}{dt^2} + \frac{R}{L}\frac{dvc_h}{dt} + \frac{1}{LC} vc_h = 0 \tag{1.21}$$

Assuming a homogeneous solution is of the form Ae^{st} and by substituting into Equation (1.21) we obtain the characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{1.22}$$

By defining

$$\alpha = \frac{R}{2L} \tag{1.23}$$

And

$$\omega_o = \frac{1}{\sqrt{LC}} \tag{1.24}$$

The characteristic equation becomes

$$s^2 + 2\alpha\, s + \omega_o^2 = 0 \tag{1.25}$$

The roots of the characteristic equation are

$$s1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \tag{1.26}$$

$$s2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \tag{1.27}$$

And the homogeneous solution becomes

$$vc_h = A_1 e^{s1t} + A_2 e^{s2t} \tag{1.28}$$

The total solution now becomes

$$vc = Vs + A_1 e^{s1t} + A_2 e^{s2t} \tag{1.29}$$

The parameters A1 and A2 are constants and can be determined by the application of the initial conditions of the system $vc(t=0)$ and $\frac{dvc(t=0)}{dt}$.

The value of the term $\sqrt{\alpha^2 - \omega_o^2}$ determines the behavior of the response. Three types of responses are possible:

1. $\alpha = \omega_o$ then $s1$ and $s2$ are equal and real numbers: no oscillatory behavior
   Critically Damped System

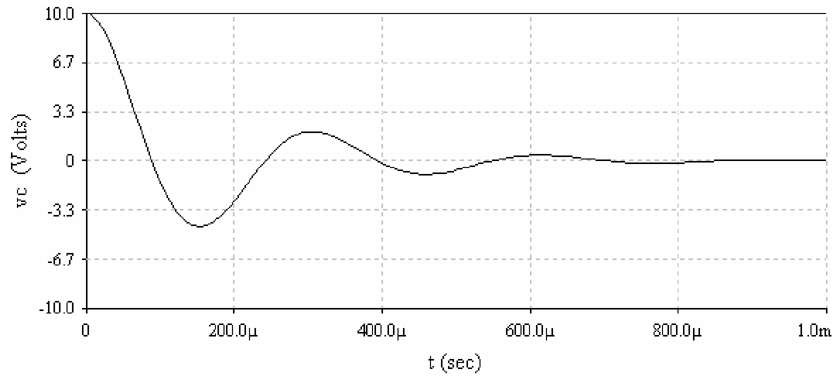
2. $\alpha > \omega_o$. Here $s1$ and $s2$ are real numbers but are unequal: no oscillatory behavior
   Over Damped System
   $vc = Vs + A_1 e^{s1t} + A_2 e^{s2t}$

3. $\alpha < \omega_o$. $\sqrt{\alpha^2 - \omega_o^2} = j\sqrt{\omega_o^2 - \alpha^2}$ In this case the roots s1 and s2 are complex numbers: $s1 = -\alpha + j\sqrt{\omega_o^2 - \alpha^2}$, $s2 = -\alpha - j\sqrt{\omega_o^2 - \alpha^2}$. System exhibits oscillatory behavior
   Under Damped System

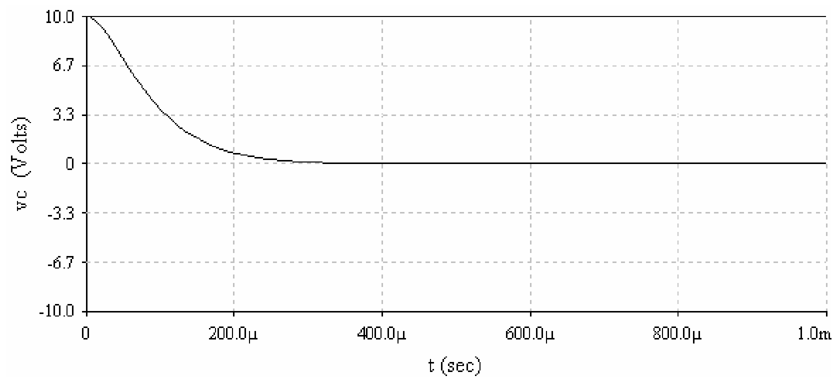
Important observations for the series RLC circuit.

- As the resistance increases the value of α increases and the system is driven towards an over damped response.
- The frequency $\omega_o = \frac{1}{\sqrt{LC}}$ (rad/sec) is called the natural frequency of the system or the resonant frequency.
- The quantity $\sqrt{\frac{L}{C}}$ has units of resistance

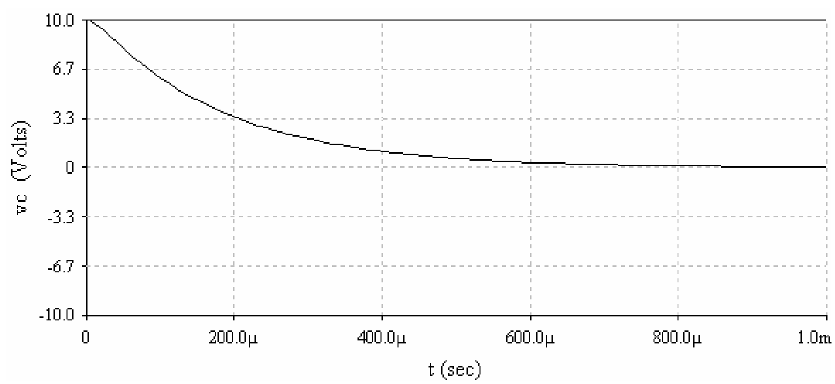
Figure 11 shows the response of the series RLC circuit with L=47mH, C=47nF and for three different values of R corresponding to the underdamped, critically damped and overdamped case. We will construct this circuit in the laboratory and examine its behavior in more detail.



(a) Under Damped. R=500Ω



(b) Critically Damped. R=2000 Ω



(c) Over Damped. R=4000 Ω

Figure 11

The *LC* circuit.

In the limit $R \to 0$ the *RLC* circuit reduces to the lossless *LC* circuit shown on Figure 12.

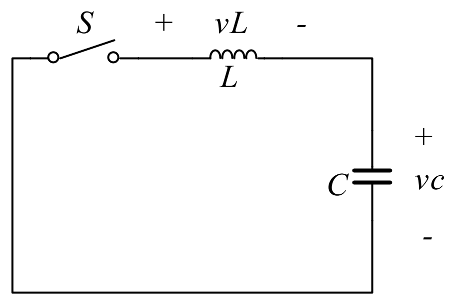


Figure 12

The equation that describes the response of this circuit is

$$\frac{d^2vc}{dt^2} + \frac{1}{LC}vc = 0 \tag{1.30}$$

Assuming a solution of the form Ae^{st} the characteristic equation is

$$s^2 + \omega_o^2 = 0 \tag{1.31}$$

Where $\omega_o = \dfrac{1}{\sqrt{LC}}$

The two roots are

$$s1 = +j\omega_o \tag{1.32}$$

$$s2 = -j\omega_o \tag{1.33}$$

And the solution is a linear combination of $A1e^{s1t}$ and $A2e^{s2t}$

$$vc(t) = A1e^{j\omega_o t} + A2e^{-j\omega_o t} \tag{1.34}$$

By using Euler's relation Equation (1.34) may also be written as

$$vc(t) = B1\cos(\omega_o t) + B2\sin(\omega_o t) \tag{1.35}$$

The constants *A1, A2* or *B1*, *B2* are determined from the initial conditions of the system.

For $vc(t=0)=Vo$ and for $\frac{dvc(t=0)}{dt}=0$ (no current flowing in the circuit initially) we have from Equation (1.34)

$$A1 + A2 = Vo \tag{1.36}$$

And

$$j\omega_o A1 - j\omega_o A2 = 0 \tag{1.37}$$

Which give

$$A1 = A2 = \frac{Vo}{2} \tag{1.38}$$

And the solution becomes

$$\begin{aligned} vc(t) &= \frac{Vo}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right) \\ &= Vo\cos(\omega_o t) \end{aligned} \tag{1.39}$$

The current flowing in the circuit is

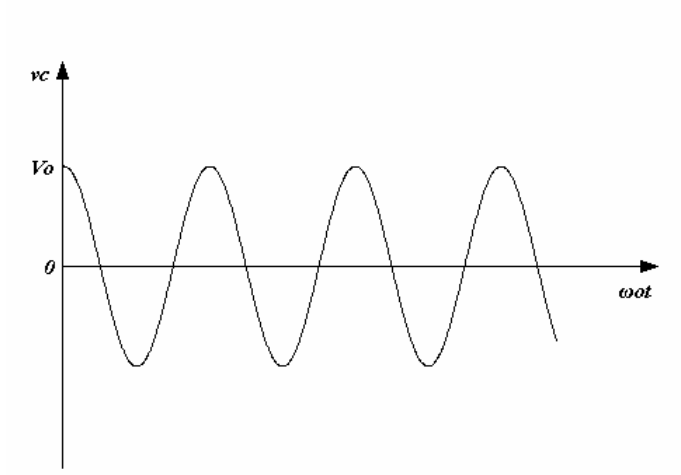
$$\begin{aligned} i &= C\frac{dvc}{dt} \\ &= -CVo\omega_o \sin(\omega_o t) \end{aligned} \tag{1.40}$$

And the voltage across the inductor is easily determined from KVL or from the element relation of the inductor $vL = L\frac{di}{dt}$

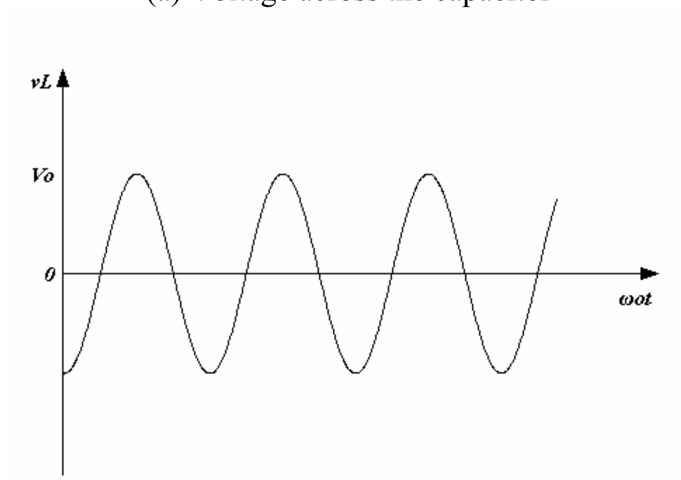
$$\begin{aligned} vL &= -vc \\ &= -Vo\cos(\omega_o t) \end{aligned} \tag{1.41}$$

Figure 13 shows the plots of $vc(t)$, $vL(t)$, *and* $i(t)$. Note the 180 degree phase difference between *vc(t)* and *vL(t)* and the 90 degree phase difference between *vL(t)* and *i(t)*.

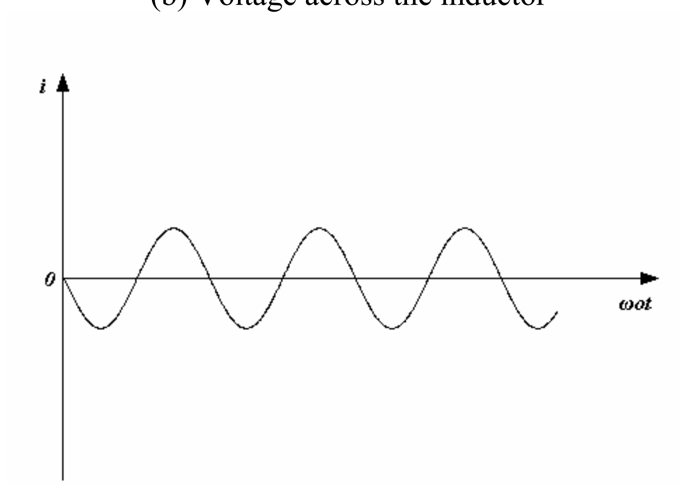
Figure 14 shows a plot of the energy in the capacitor and the inductor as a function of time. Note that the energy is exchanged between the capacitor and the inductor in this lossless system



(a) Voltage across the capacitor

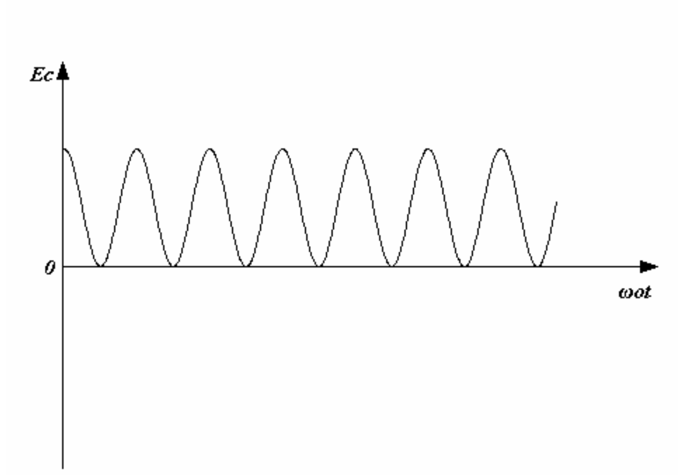


(b) Voltage across the inductor

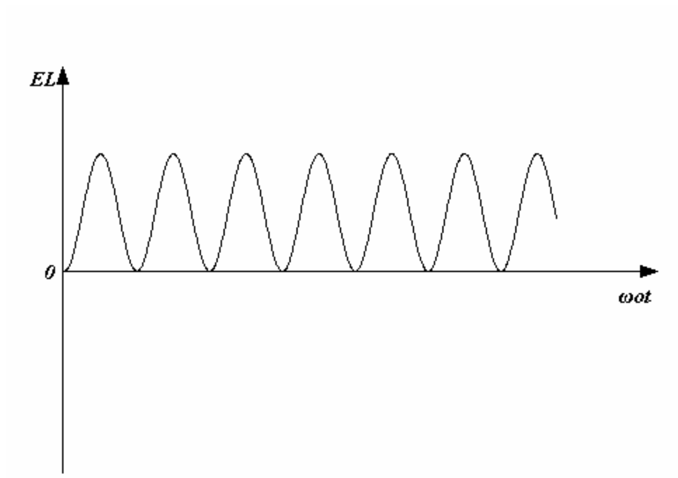


(c)Current flowing in the ciruit

Figure 13



(a) Energy stored in the capacitor



(b) Energy stored in the inductor

Figure 14