Introduction to Simulation - Lecture 2

Equation Formulation Methods

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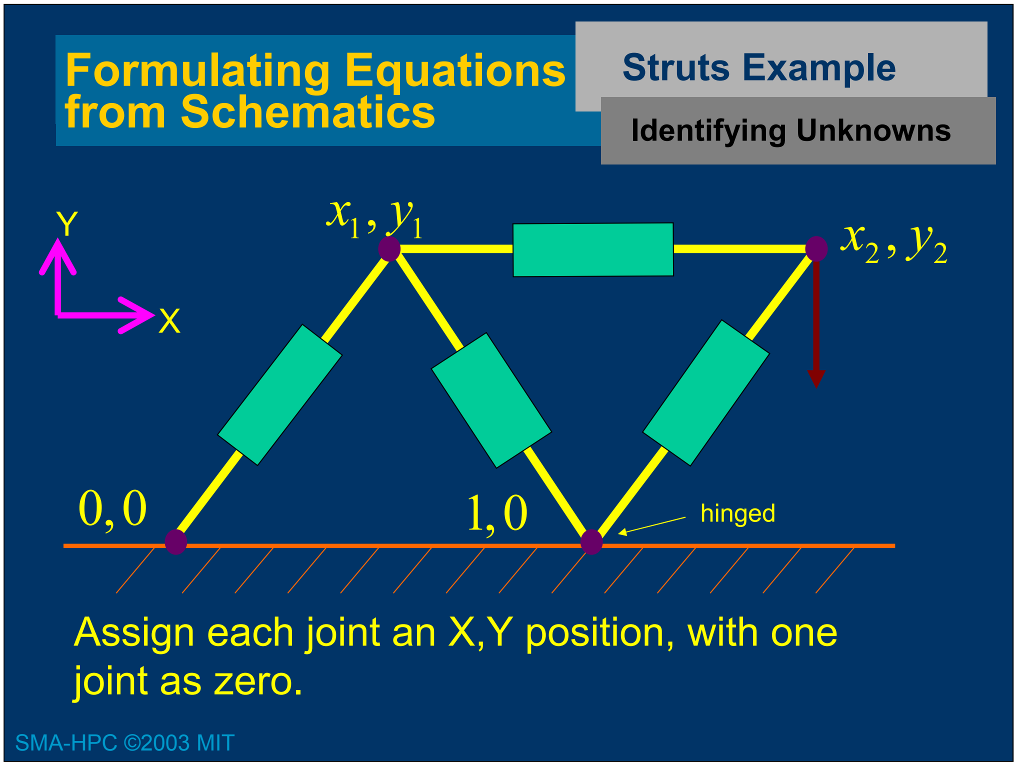
Thanks to Deepak Ramaswamy, Michal Rewienski, and Karen Veroy

Outline

- Formulating Equations from Schematics
  - Struts and Joints Example
- Matrix Construction From Schematics
  - “Stamping Procedure”
- Two Formulation Approaches
  - Node-Branch – More general but less efficient
  - Nodal – Derivable from Node-Branch



Given a schematic for the struts, the problem is to determine the joint positions and the strut forces.

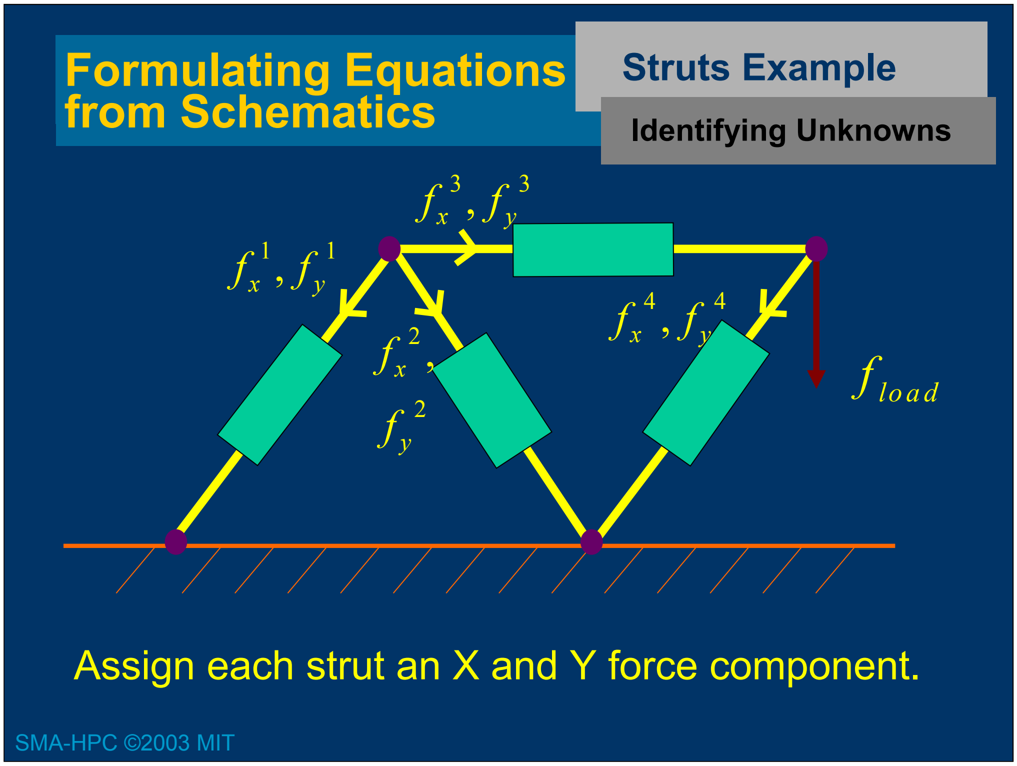
Recall the joints in the struts problem correspond physically to the location where steel beams are bolted together. The joints are also analogous to the nodes in the circuit, but there is an important difference. The joint position is a vector because one needs two (X,Y) (three (X,Y,Z)) coordinates to specify a joint position in two (three) dimensions.

The joint positions are labeled $x_1,y_1,x_2,y_2,\ldots.x_j,y_j$ where j is the number of joints whose positions are unknown. Like in circuits, in struts and joints there is also an issue about position reference. The position of a joint is usually specified with respect to a reference joint.

Note also the symbol

This symbol is used to denote a fixed structure (like a concrete wall, for example). Joints on such a wall have their positions fixed and usually one such joint is selected as the reference joint. The reference joint has the position 0,0

(0,0,0 in three dimensions).

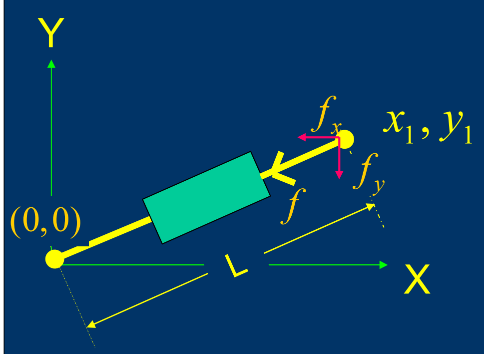


The second set of unknowns are the strut forces. Like the currents in the circuit examples, these forces can be considered "branch" quantities. There is again a complication due to the two dimensional nature of the problem, there is an x and a y component to the force. The strut forces are labeled $f_x^1, f_y^1, \ldots, f_x^s, f_y^s$ where s is the number of struts.

Formulating Equations from Schematics

Struts Example

Aside on Strut Forces



$$f = EA_c \frac{L_0 - L}{L_0} = \varepsilon\left(L_0 - L\right)$$

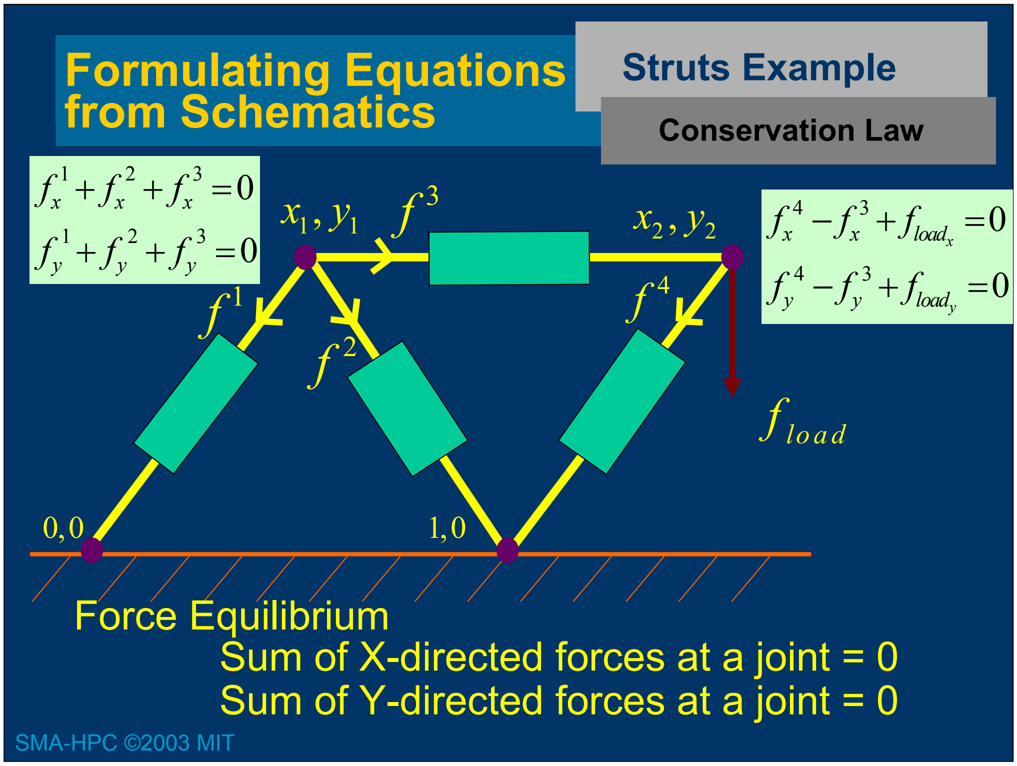
$$f_x = \frac{x_1}{L} f$$

$$f_y = \frac{y_1}{L} f$$

$$L = \sqrt{x_1^2 + y_1^2}$$

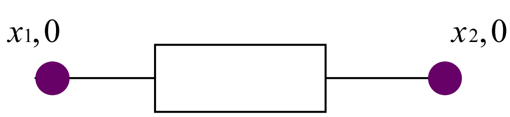
The force, f, in a stretched strut always acts along the direction of the strut, as shown in the figure. However, it will be necessary to sum the forces at a joint, individual struts connected to a joint will not all be in the same direction. So, to sum such forces, it is necessary to compute the components of the forces in the X and Y direction. Since one must have selected the directions for the X and Y axis once for a given problem, such axes are referred to as the "global" coordinate system. Then, one can think of the process of computing f_x, f_y shown in the figure as mapping from a local to a global coordinate system.

The formulas for determining f_x and f_y from f follow easily from the geometry depicted in the figure, one is imply projecting the vector force onto coordinate axes.

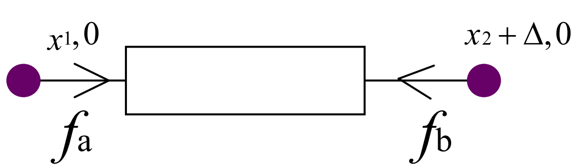


The conservation law for struts is usually referred to as requiring force equilibrium. There are some subtleties about signs, however. To begin, consider that the sum of X-directed forces at a joint must sum to zero otherwise the joint will accelerate in the X-direction. The Y-directed forces must also sum to zero to avoid joint acceleration in the Y direction.

To see the subtlety about signs, consider a single strut aligned with the X axis as shown below



If the strut is stretched by Δ, then the strut will exert force in attempt to contract, as shown below



The forces f_a and f_b , are equal in magnitude but opposite in sign. This is because f_a points in the positive X direction and f_b in the negative X direction.

If one examines the force equilibrium equation for the left-hand joint in the figure, then that equation will be of the form

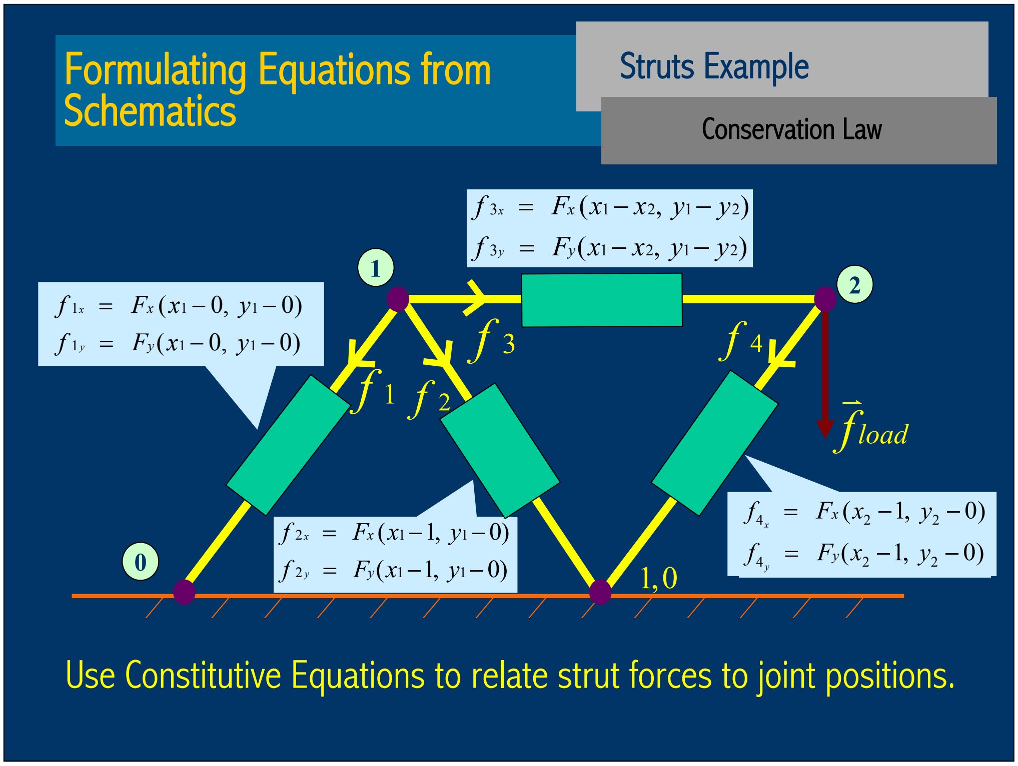
$$\text{Other forces} + f_a = 0$$

whereas the equilibrium equation for the right-hand joint will be

$$\text{Other forces} + f_b = \text{Other forces} - f_a = 0$$

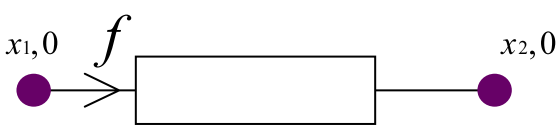
In setting up a system of equations for the strut, one need not include both f_a and f_b as separate variables in the system of equations. Instead, one can select either force and implicitly exploit the relationship between the forces on opposite sides of the strut.

As an example, consider that for strut 3 between joint 1 and joint 2 on the slide, we have selected to represent the force on the joint 1 side of the strut and labeled that force f_3. Therefore, for the conservation law associated with joint 1, force f_3 appears with a positive sign, but for the conservation law associated with joint 2, we need the opposite side force, $-f_3$. Although the physical mechanism seems quite different, this trick of representing the equations using only the force on one side of the strut as a variable makes an algebraic analogy with the circuit sum of currents law. That is, it appears as if a strut's force "leaves" one joint and "enters" another.



It is worth examining how the signs of the force are determined.

Again consider a single strut aligned with the X axis.



The X axis alignment can be used to simplify the relation between the force on the x_1 side and x_1 and x_2 to

$$f_x = \frac{x_1 - x_2}{|x_1 - x_2|} \in \frac{L_0 - |x_1 - x_2|}{L_0}$$

Note that there are two ways to make f_x negative and point in the negative x direction. Either $x_1 - x_2 > 0$, which corresponds to flipping the strut, or $|x_2 - x_1| < L_0$ which corresponds to compressing the strut.

Formulating Equations from Schematics

Struts Example

Summary

Unknowns for the Strut Example

Joint positions (except for a reference or fixed joints)

Strut forces

Equations for the Strut Example

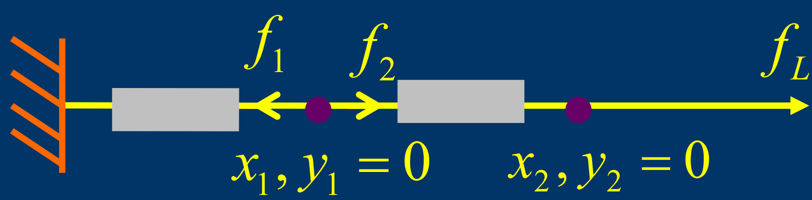
One set of conservation equations for each joint.

One set of constitutive equations for each strut.

Note that the **# equations = # unknowns**

Strut Example To Demonstrate Sign convention

Two Struts Aligned with the X axis



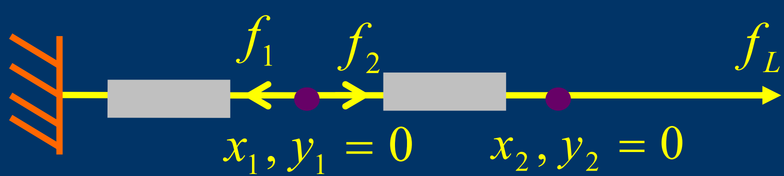
Conservation Law

At node 1: $f_{1x} + f_{2x} = 0$

At node 2: $-f_{2x} + f_L = 0$

Strut Example To Demonstrate Sign convention

Two Struts Aligned with the X axis



Constitutive Equations

$$f_{1x} = \frac{x_1 - 0}{|x_1 - 0|} \varepsilon\left(L_0 - |x_1 - 0|\right)$$

$$f_{2x} = \frac{x_1 - x_2}{|x_1 - x_2|} \varepsilon\left(L_0 - |x_1 - x_2|\right)$$

Strut Example To Demonstrate Sign convention

Two Struts Aligned with the X axis

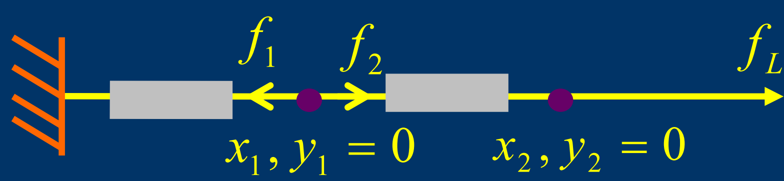
Reduced (Nodal) Equations

$$\frac{x_1}{|x_1|}\varepsilon\left(L_0-|x_1|\right)+\underbrace{\frac{x_1-x_2}{|x_1-x_2|}\varepsilon\left(L_0-|x_1-x_2|\right)}_{f_{2x}}=0$$

$$\underbrace{-\frac{x_1-x_2}{|x_1-x_2|}\varepsilon\left(L_0-|x_1-x_2|\right)}_{-f_{2x}}+f_L=0$$

Strut Example To Demonstrate Sign convention

Two Struts Aligned with the X axis



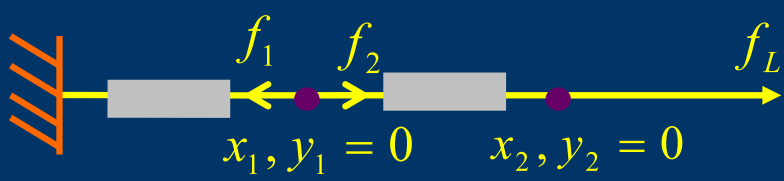
Solution of Nodal Equations

$f_L = 10$ (force in positive x direction)

$$x_1 = L_0 + \frac{10}{\varepsilon} \qquad x_2 = x_1 + L_0 + \frac{10}{\varepsilon}$$

Strut Example To Demonstrate Sign convention

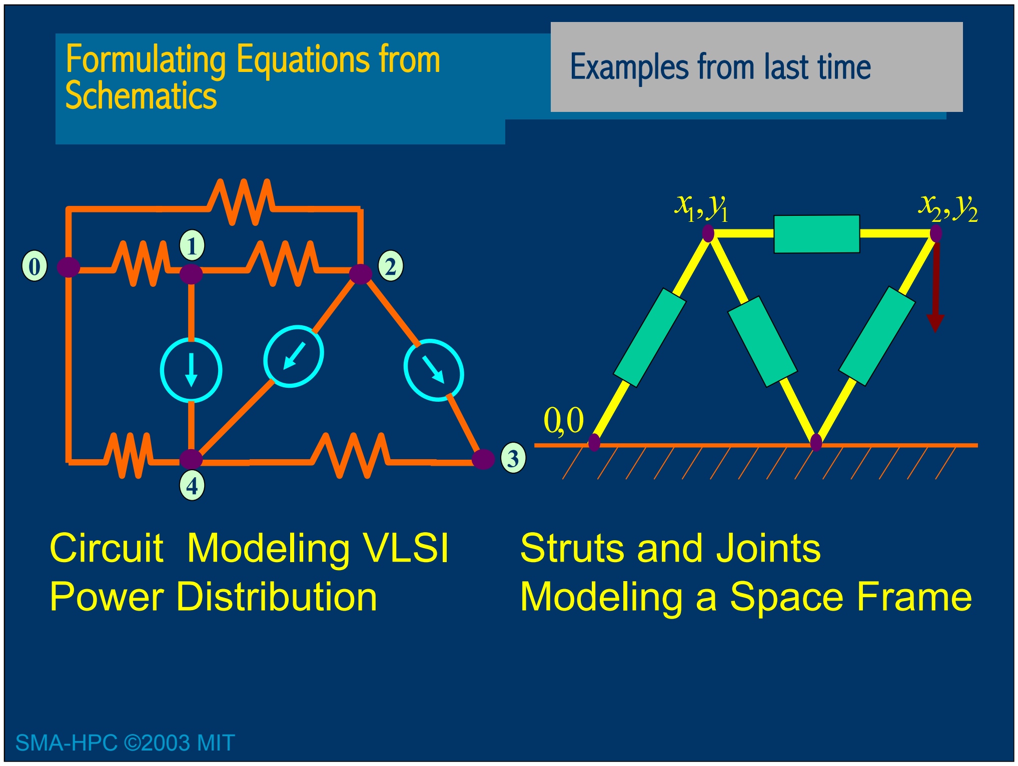
Two Struts Aligned with the X axis



Notice the signs of the forces

$f_{2x} = 10$ (force in positive x direction)

$f_{1x} = -10$ (force in negative x direction)


Formulating Equations from Schematics
Examples from last time
0
1
2
3
4
x1, y1
x2, y2
0,0
Circuit Modeling VLSI Power Distribution
Struts and Joints Modeling a Space Frame
SMA-HPC ©2003 MIT

Formulating Equations from Schematics

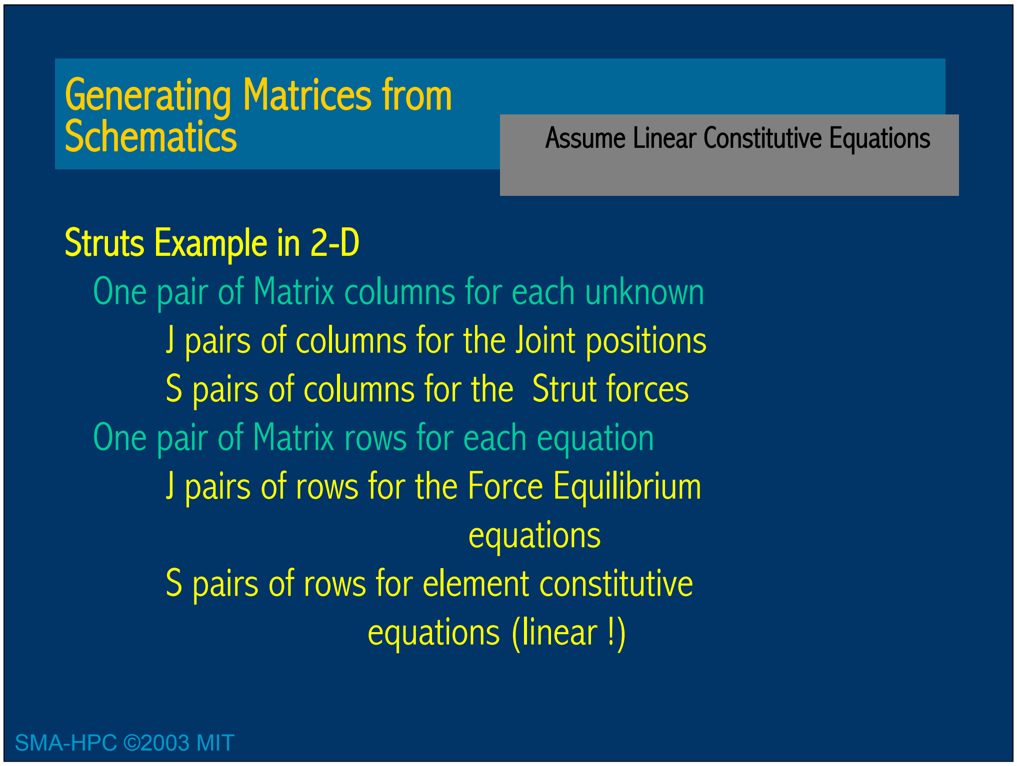
- Two Types of Unknowns
  - Circuit - Node voltages, element currents
  - Struts - Joint positions, strut forces
- Two Types of Equations
  - Conservation Law
    - Circuit - Sum of Currents at each node = 0
    - Struts - Sum of Forces at each joint = 0
  - Constitutive Relations
    - Circuit – branch (element) current proportional to branch (element) voltage
    - Struts - branch (strut) force proportional to branch (strut) displacement

Generating Matrices from Schematics

Assume Linear Constitutive Equations…

Circuit Example

- One Matrix column for each unknown
  - N columns for the Node voltage
  - B columns for the Branch currents
- One Matrix row for each equation
  - N rows for KCL
  - B rows for element constitutive equations (linear !)



Generating Matrices from Schematics
Assume Linear Constitutive Equations
Struts Example in 2-D
One pair of Matrix columns for each unknown
J pairs of columns for the Joint positions
S pairs of columns for the Strut forces
One pair of Matrix rows for each equation
J pairs of rows for the Force Equilibrium equations
S pairs of rows for element constitutive equations (linear !)
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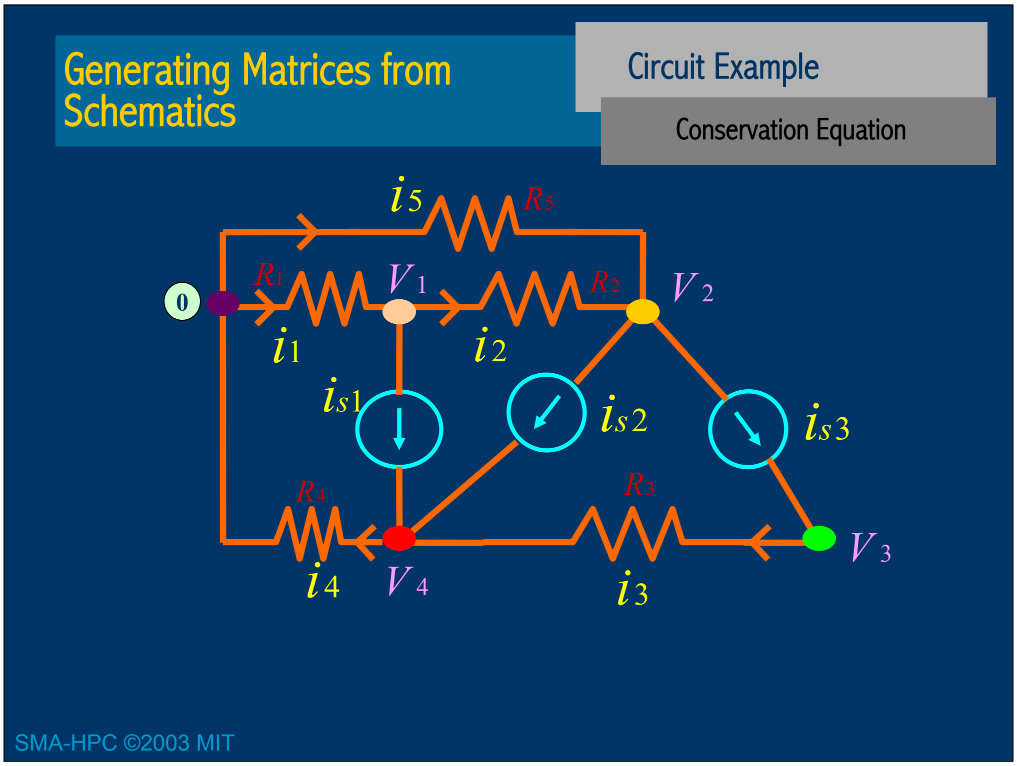


To generate a matrix equation for the circuit, we begin by writing the KCL equation at each node in terms of the branch currents and the source currents. In particular, we write

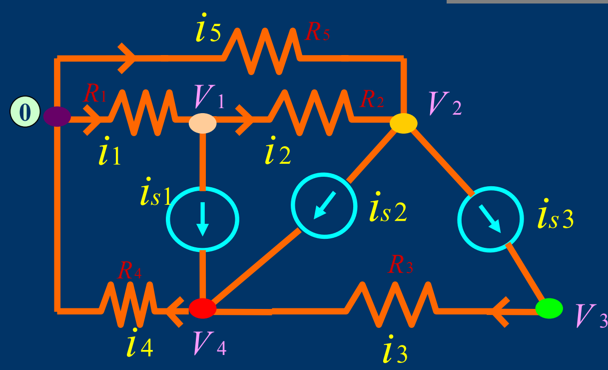
$$\sum \text{signed branch currents} = \sum \text{signed source currents}$$

where the sign of a branch current in the equation is positive if the current is leaving the node and negative otherwise. The sign of the source current in the equation is positive if the current is entering the node and negative otherwise.

Generating Matrices from Schematics

Circuit Example

Conservation Equation



- $-i_1 + i_2 = -i_{s1}$
- $-i_2 - i_5 = -i_{s2} - i_{s3}$
- $i_3 = i_{s3}$
- $-i_3 + i_4 = i_{s_1} + i_{s_2}$

Generating Matrices from Schematics

Circuit Example

Conservation Equation

Matrix Form for the Equations

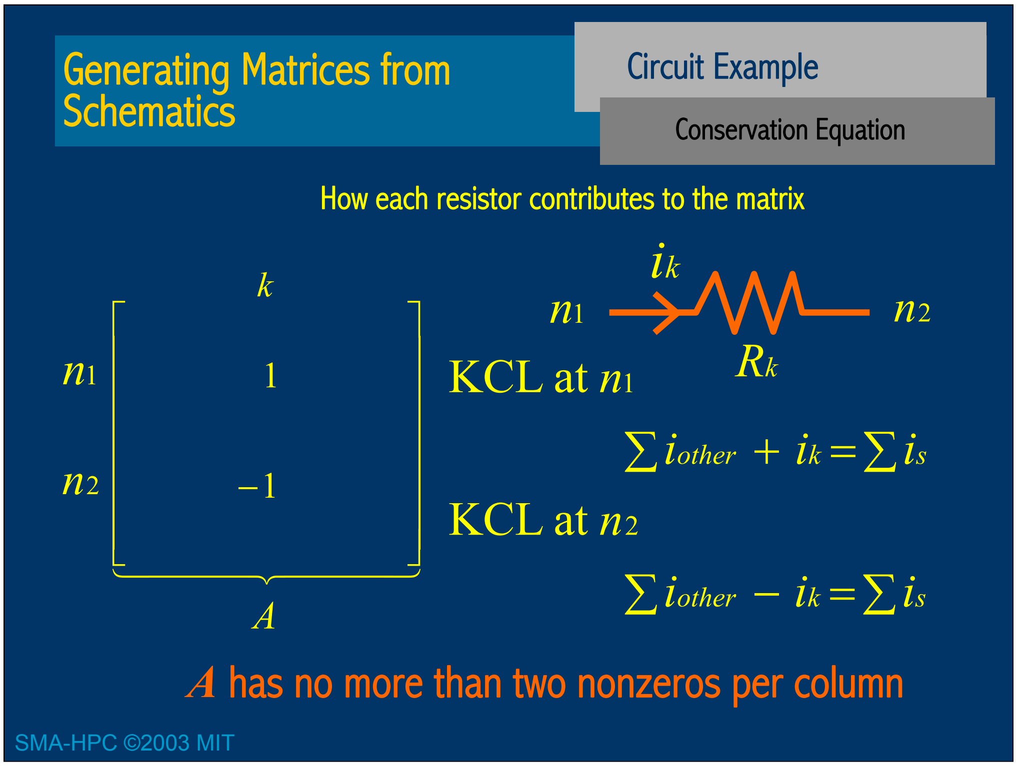
One Row for each KCL Equation (N rows, nodes 1–4)

$$\overbrace{\begin{bmatrix} -1 & 1 & & & \\ & -1 & & & -1 \\ & & 1 & & \\ & & -1 & 1 & \end{bmatrix}}^{A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -i_{s_1} \\ -i_{s_2} - i_{s_3} \\ i_{s_3} \\ i_{s_1} + i_{s_2} \end{bmatrix}}_{\text{Right Hand Side for Source Currents}}$$

B: One column for each branch current

Right Hand Side for Source Currents

The matrix A is usually not square



What happens to the matrix when one end of a resistor is connected to the reference (or the zero node).

In that case, there is only one contribution to the kth column of the matrix, as shown below

$$\begin{array}{c} \\ n_1 \\ \\ \end{array} \begin{bmatrix} & \overset{k}{} & \\ & 1 & \\ & & \\ \end{bmatrix}$$

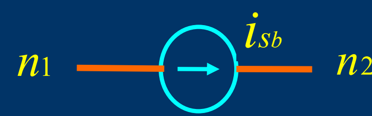
Generating Matrices from Schematics

Circuit Example

Conservation Equation

How each current source contributes to the Right Hand Side

$$
\begin{matrix} n_1 \\ n_2 \end{matrix}
\underbrace{\begin{bmatrix} \sum i_{Sother} + i_{Sb} \\ \sum i_{Sother} - i_{Sb} \end{bmatrix}}_{\text{RHS}}
$$



KCL at n_1

$$\sum i_b\text{'s} = \sum i_{Sother} + i_{Sb}$$

KCL at n_2

$$\sum i_b\text{'s} = \sum i_{Sother} - i_{Sb}$$

Generating Matrices from Schematics

Conservation Matrix Equation Generation Algorithm

For each resistor

$$\text{if } (n_1 > 0)\ A(n_1, b) = 1$$
$$\text{if } (n_2 > 0)\ A(n_2, b) = -1$$

Set I_s = zero vector

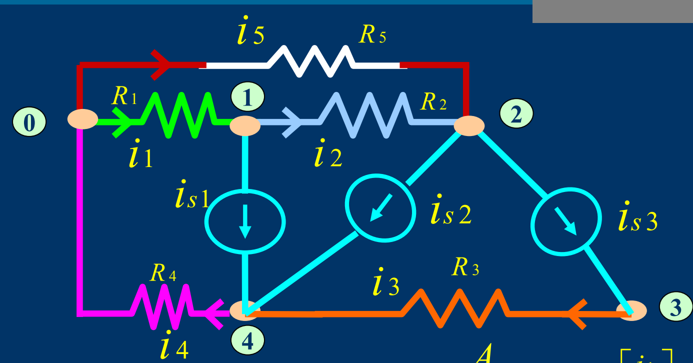
For each current source

$$\text{if } (n_1 > 0)\ I_s(n_1) = I_s(n_1) - i_{sb}$$
$$\text{if } (n_2 > 0)\ I_s(n_2) = I_s(n_1) + i_{sb}$$

Generating Matrices from Schematics

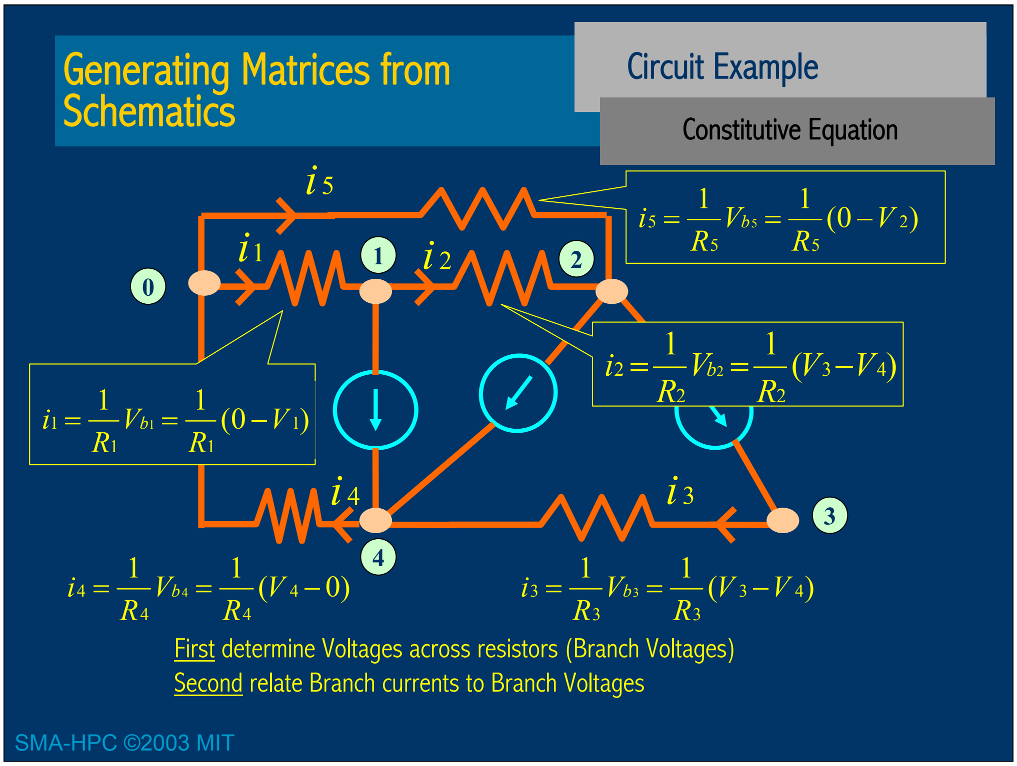
Circuit Example

Conservation Equation

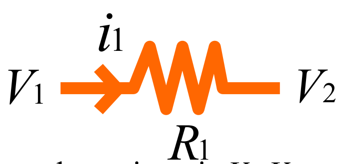


$$\underbrace{\begin{bmatrix} -1 & 1 & & & \\ & -1 & & & -1 \\ & & 1 & & \\ & & -1 & 1 & \end{bmatrix}}_{A} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} -i_{s1} \\ -i_{s2} - i_{s3} \\ i_{s3} \\ i_{s_1} + i_{s_2} \end{bmatrix}$$

The rows of A are labelled with nodes 1, 2, 3, 4 and its columns with i_1, i_2, i_3, i_4, i_5.



The current through a resistor is related to the voltage across the resistor, which in turn is related to the node voltages. Consider the resistor below.

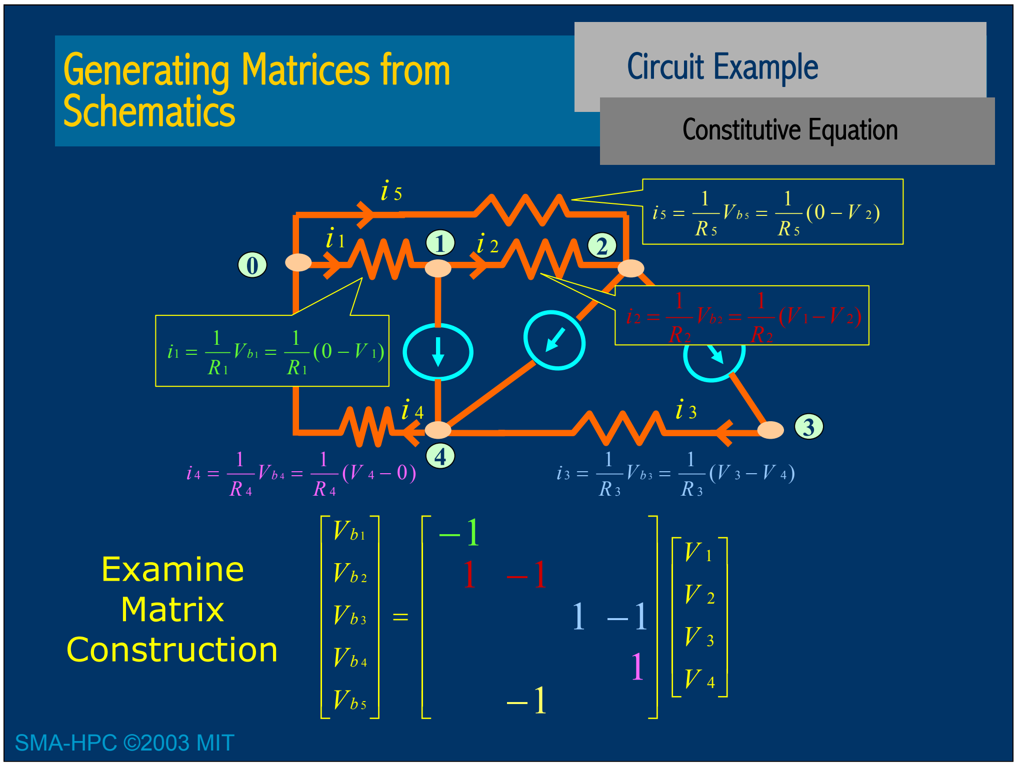


The voltage across the resistor is V_1-V_2 and the current through the resistor is

$$i_1 = \frac{1}{R_1}(V_1 - V_2)$$

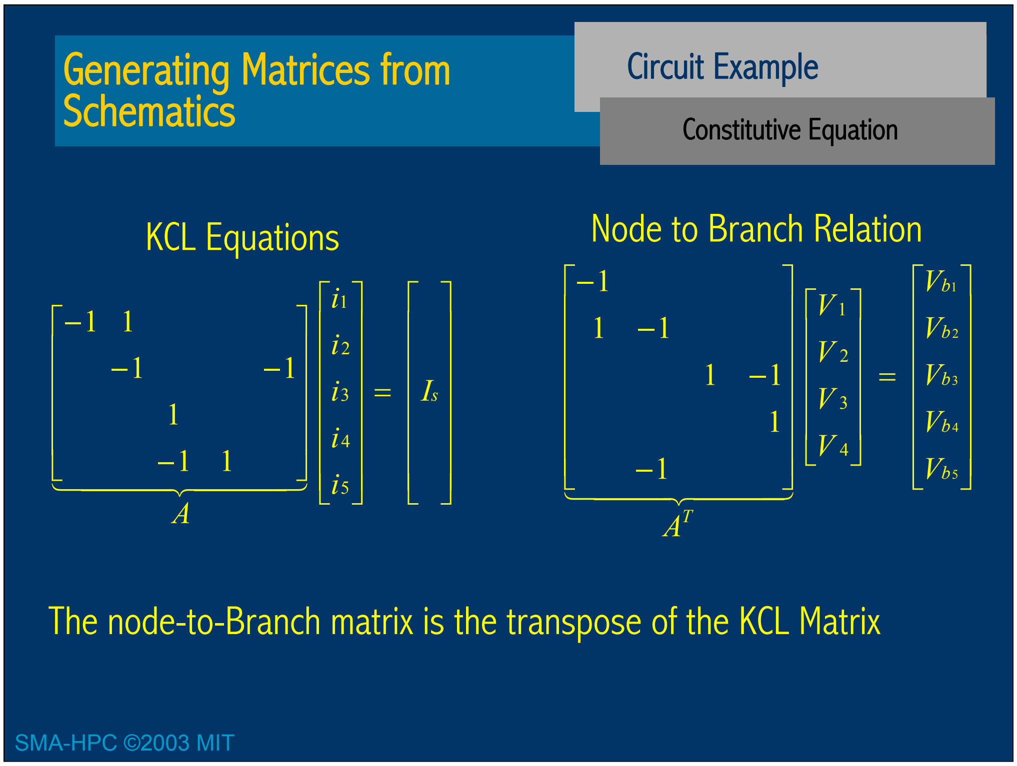
Notice the sign, i , is positive if $V_1 > V_2$.

In order to construct a matrix representation of the constitutive equations, the first step is to relate the node voltages to the voltages across resistors, the branch voltages.

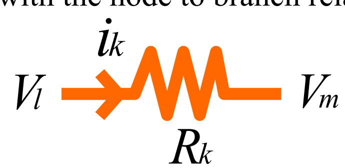


To generate a matrix equation that relates the node voltages to the branch voltages, one notes that the voltage across a branch is just the difference between the node voltages at the ends of the branch. The sign is determined by the direction of the current, which points from the positive node to the negative node.

Since there are B branch voltages and N node voltages, the matrix relating the two has B rows and N columns.



A relation exists between the matrix associated with the conservation law (KCL) and the matrix associated with the node to branch relation. To see this, examine a single resistor.



For the conservation law, branch **k** contributes two non-zeros to the **k**th column of ***A*** as in

$$\begin{array}{c} l \\ \\ m \end{array} \underbrace{\begin{bmatrix} & \overset{k}{1} & \\ & & \\ & -1 & \end{bmatrix}}_{A} \begin{bmatrix} I_1 \\ \vdots \\ \vdots \\ I_B \end{bmatrix} = \begin{bmatrix} \\ I_s \\ \\ \end{bmatrix}$$

Note that the voltage across branch **k** is $V_l - V_m$, so the **k**th branch contributes two non-zeros to the **k**th row of the node branch relation as in

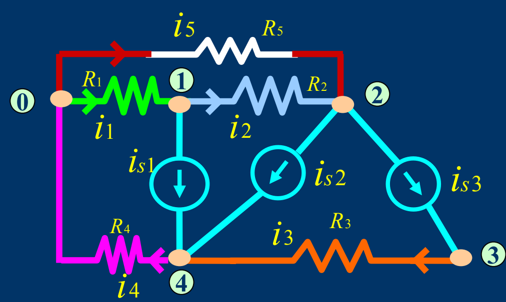
$$
\begin{array}{c} \\ k \\ \\ \end{array}
\begin{bmatrix} & l & & m & \\ & & & & \\ & 1 & & -1 & \\ & & & & \end{bmatrix}
\begin{bmatrix} V_1 \\ : \\ : \\ : \\ V_N \end{bmatrix} = \begin{bmatrix} V_{b1} \\ \\ \\ \\ V_{bB} \end{bmatrix}
$$

It is easy to see that each branch element will contribute a column to the incidence matrix A, and will contribute the transpose of that column, a row, to the node-to-branch relation.

Generating Matrices from Schematics

Circuit Example

Constitutive Equation



$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \overbrace{\begin{bmatrix} \frac{1}{R_1} & & & & \\ & \frac{1}{R_2} & & & \\ & & \frac{1}{R_3} & & \\ & & & \frac{1}{R_4} & \\ & & & & \frac{1}{R_5} \end{bmatrix}}^{\alpha} \begin{bmatrix} V_{b1} \\ V_{b2} \\ V_{b3} \\ V_{b4} \\ V_{b5} \end{bmatrix}$$

The k[th] resistor contributes $\frac{1}{R_k}$ to $\alpha(k,k)$

The matrix α relates branch voltages to branch currents.

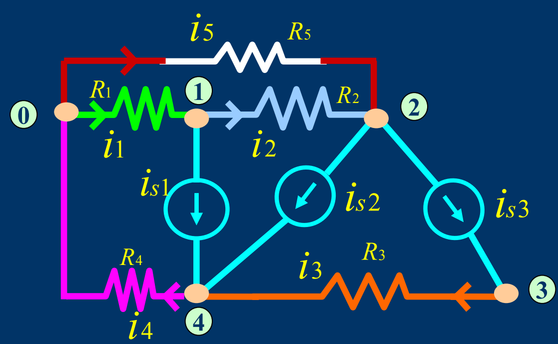
- One row for each unknown current.
- One column for each associated branch voltage.

The matrix α is square and diagonal.

Generating Matrices from Schematics

Circuit Example

Constitutive Equation



$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} - \alpha A^T \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{V_S}$$

The node voltages can be related to branch currents.

- A^T relates node voltages to branch voltages.
- α relates branch voltages to branch currents.

Generating Matrices from Schematics

Circuit Example

Node-Branch Form

$$\begin{matrix} B \updownarrow \\ \\ N \updownarrow \end{matrix} \begin{bmatrix} I & -\alpha A^T \\ & \\ A & 0 \end{bmatrix} \begin{bmatrix} I_b \\ \\ V_N \end{bmatrix} = \begin{bmatrix} 0 \\ \\ I_s \end{bmatrix}$$

$$\quad \underset{B}{\leftrightarrow} \qquad \underset{N}{\leftrightarrow}$$

N = Number of Nodes with unknown voltages

B = Number of Branches with unknown currents

$$I_b - \alpha A^T V_N = 0 \qquad \text{Constitutive Relation}$$

$$A I_b = I_s \qquad \text{Conservation Law}$$

Generating Matrices from Schematics

Struts Example

In 2-D

One pair of columns for each unknown

- J pairs of columns for the Joint positions
- S pairs of columns for the Strut positions

One pair of Matrix Rows for each Equation

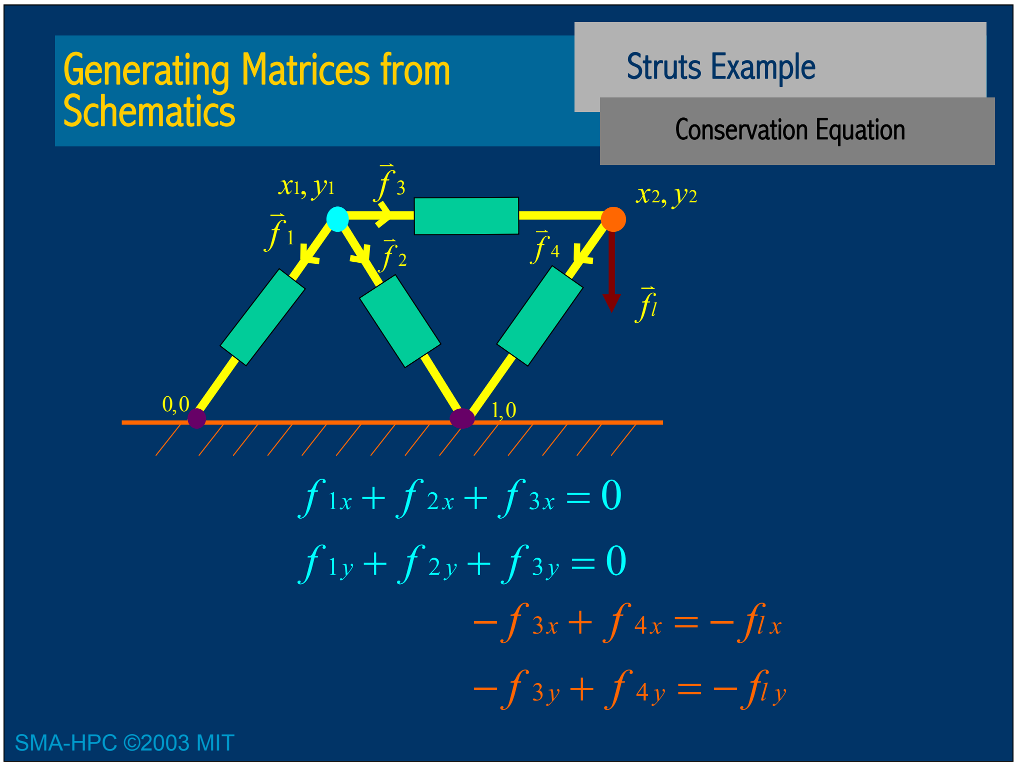
- J pairs of rows for the force equilibrium equations
- S pairs of rows for the Linearized constitutive relations.

Generating Matrices from Schematics

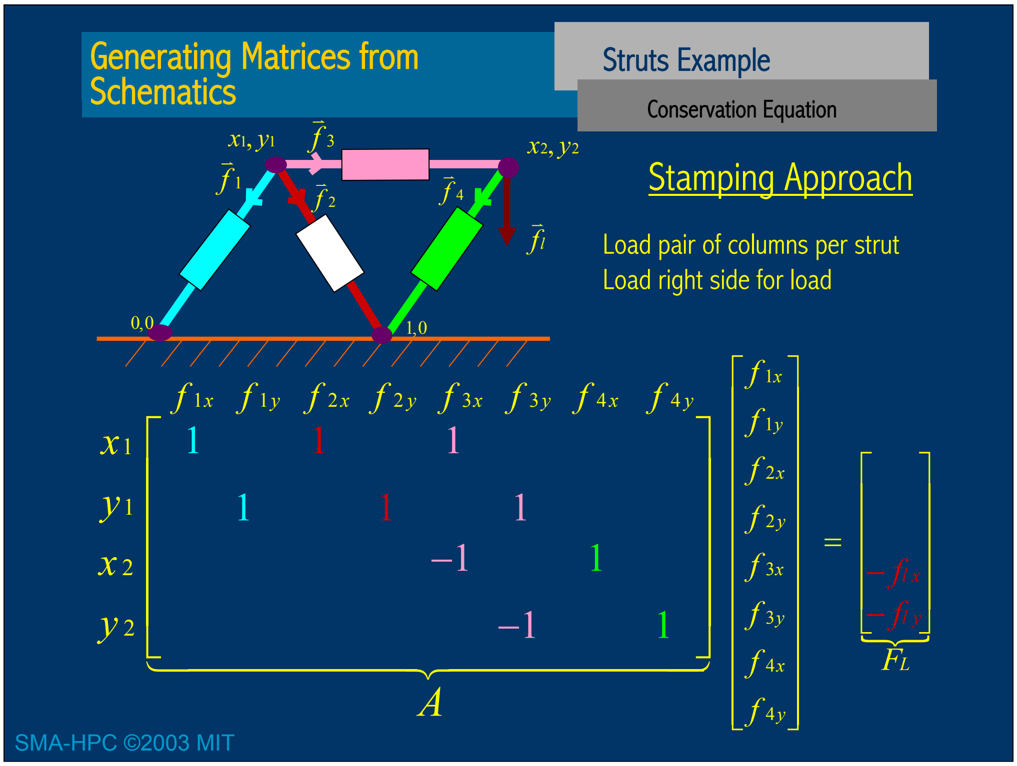
Struts Example

Follow Approach Parallel to Circuits

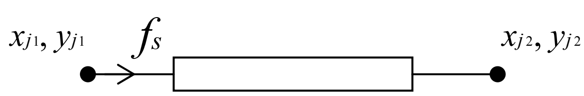
1) Form an "Incidence Matrix", A, from Conservation Law.
2) Determine strut deformation using A^T.
3) Use linearized constitutive equations to relate strut deformation
4) Combine (1),(2), and (3) to generate a node-branch form.



As a reminder, the conservation equation for struts is naturally divided in pairs. At each joint the sum of X-directed forces = 0 and the sum of Y-directed forces = 0. Note that the load force is known, so it appears on the right hand side of the equation.



Note that the incidence matrix, A, for the strut problem is very similar to the incidence matrix for the circuit problem, except the two dimensional forces and positions generate 2x2 blocks in the incidence matrix. Consider a single strut



The force equilibrium equations for the two joints at the ends of the strut are

<u>At joint j_1</u>

$$\sum_{j1} f_{Xother} + f_{Sx} = -\sum_{j1} f_{lx}$$

$$\sum_{j1} f_{yother} + f_{Sy} = -\sum_{j1} f_{ly}$$

<u>At joint j_2</u>

$$\sum_{j2} f_{Xother} - f_{Sx} = -\sum_{j2} f_{lx}$$

$$\sum_{j2} f_{yother} - f_{Sy} = -\sum_{j2} f_{ly}$$

Examining what goes in the matrix leads to a picture

$$
\begin{array}{c} \\ x_{j1} \\ y_{j1} \\ x_{j2} \\ y_{j2} \\ \\ \end{array}
\begin{array}{c} \quad f_{Sx} \quad f_{Sy} \\
\left[\begin{array}{ccc}
& & \\
\cdots \; \cdots & \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} & \\
\cdots \; \cdots & \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} & \\
& &
\end{array} \right]
\end{array}
$$

Note that the matrix entries are 2x2 blocks. Therefore, the individual entries in the matrix block for strut S's contribution to $j1$'s conservation equation need specific indices and we use $j1_x$, $j1_y$ to indicate the two rows and S_x, S_y to indicate the two columns.

Conservation Matrix Generation Algorithm

For each strut

If (j_1 is not fixed) $A(j_{1x}, b_x) = 1 \quad A(j_{1y}, b_y) = 1$

If (j_2 is not fixed) $A(j_{2x}, b_x) = -1 \quad A(j_{2y}, b_y) = -1$

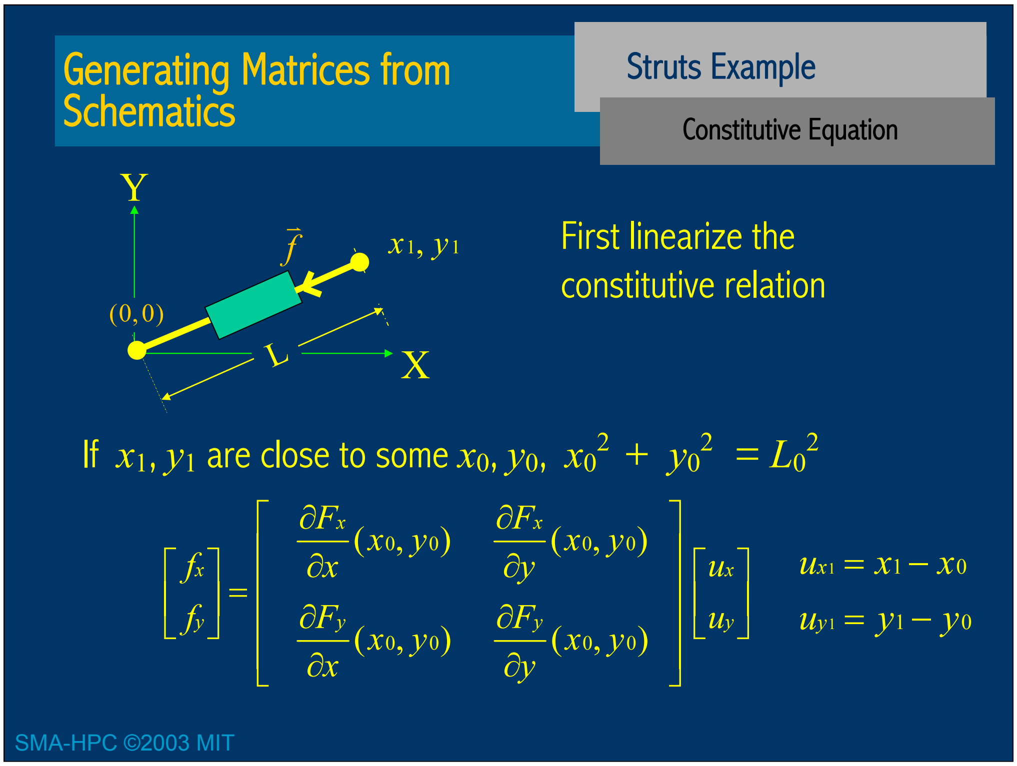
For each load

If (j_1 is not fixed)

$$F_L(j1_x) = F_L(j1_x) - f_{load_x}$$

$$F_L(j1_y) = F_L(j1_x) - f_{load_y}$$

A has at most 2 non-zeros / column

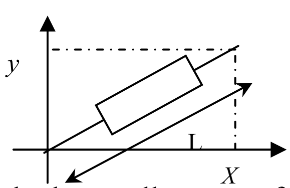


As shown before, the force through a strut is

$$f_x = F_x(x, y) = \frac{x}{L}\varepsilon\left(L_0 - L\right)$$

$$f_y = F_y(x, y) = \frac{y}{L}\varepsilon\left(L_0 - L\right)$$

where $L = \sqrt{x^2 + y^2}$ and x, y are as in



If x and y are perturbed a small amount from some x_0, y_0 such that $x_0^2 + y_0^2 = L_0^2$, then since $F_x(x_0, y_0) = 0$

$$f_x \simeq \frac{\partial F_x}{\partial x}(x_0, y_0)\,(x_1 - x_0) + \frac{\partial F_x}{\partial y}(x_0, y_0)\,(y_1 - y_0)$$

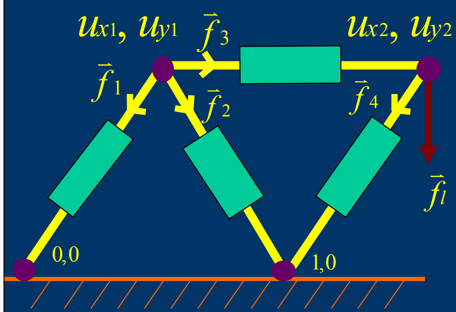
and a similar expression holds for y.

One should note that rotating the strut, even without stretching it, will violate the small perturbation conditions. The Taylor series expression will not give good approximate forces, because they will point in an incorrect direction.

Generating Matrices from Schematics

Struts Example

Constitutive Equation



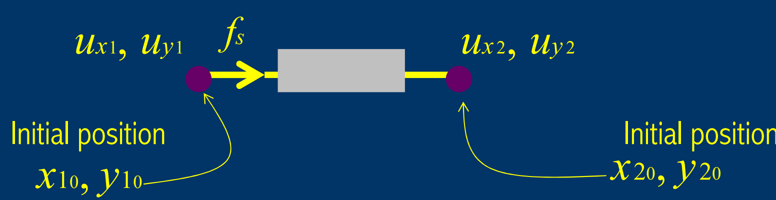
$$\begin{bmatrix} f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\\ f_{3x}\\ f_{3y}\\ f_{4x}\\ f_{4y} \end{bmatrix} - \underbrace{\begin{bmatrix} \alpha_{11} & & & \\ & \alpha_{22} & & \\ & & \alpha_{33} & \\ & & & \alpha_{44} \end{bmatrix}}_{\alpha} A^T \begin{bmatrix} u_{x1}\\ u_{y1}\\ u_{x2}\\ u_{y2} \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}$$

Generating Matrices from Schematics

Struts Example

Constitutive Equation

The $\alpha(s,s)$ block



$$\alpha(s,s) = \begin{bmatrix} \frac{\partial F_x}{\partial x}(x_{20}-x_{10}, y_{20}-y_{10}) & \frac{\partial F_x}{\partial y}(x_{20}-x_{10}, y_{20}-y_{10}) \\ \frac{\partial F_y}{\partial x}(x_{20}-x_{10}, y_{20}-y_{10}) & \frac{\partial F_y}{\partial y}(x_{20}-x_{10}, y_{20}-y_{10}) \end{bmatrix}$$

Generating Matrices from Schematics

Struts Example

Node-Branch From

$$\begin{matrix} 2\cdot S \updownarrow \\ \\ 2\cdot J \updownarrow \end{matrix}\begin{bmatrix} I & -\alpha A^T \\ & \\ A & 0 \end{bmatrix}\begin{bmatrix} f_s \\ \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ \\ f_L \end{bmatrix}$$

$$\quad\quad \overset{\leftrightarrow}{2\cdot S} \quad\quad \overset{\leftrightarrow}{2\cdot J}$$

S =Number of Struts

J = Number of unfixed Joints

$f_s = \alpha A^T u = 0$ Constitutive Equation

$A f_s = 0$ Conservation Law

Generating Matrices from Schematics

Struts Example

Comparison

$$\begin{bmatrix} I & -\alpha A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} f_s \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ f_L \end{bmatrix}$$

Rows: $2 \cdot S$, $2 \cdot J$; Columns: $2 \cdot S$, $2 \cdot J$

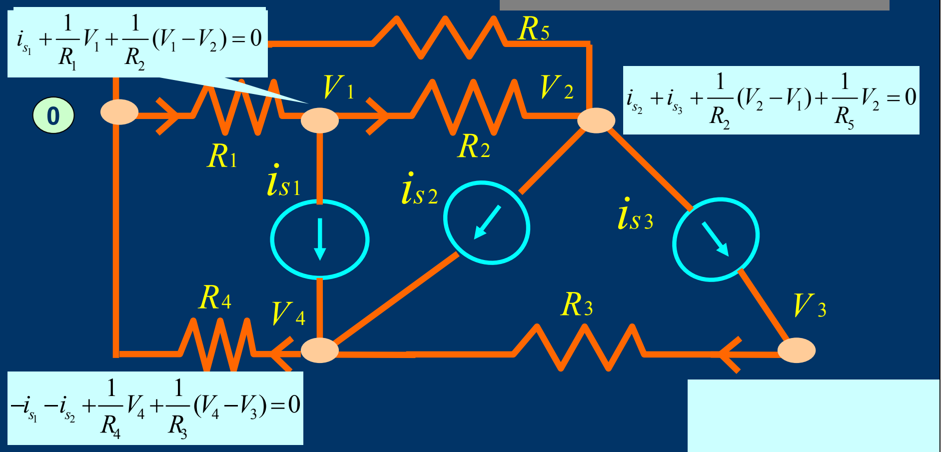
$$\begin{bmatrix} I & -\alpha A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} I_b \\ V_N \end{bmatrix} = \begin{bmatrix} V_s \\ I_s \end{bmatrix}$$

Rows: B, N; Columns: B, N

Nodal Formulation

Generating Matrices

Circuit Example

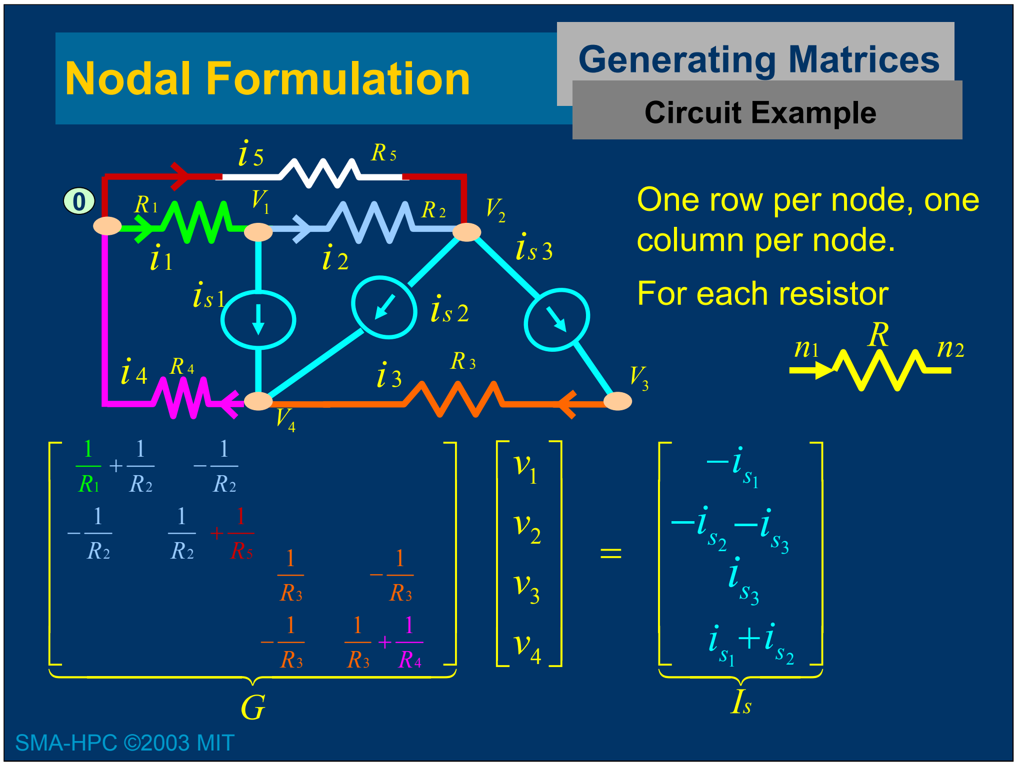


$$i_{s_1} + \frac{1}{R_1}V_1 + \frac{1}{R_2}(V_1 - V_2) = 0$$

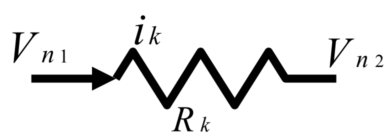
$$i_{s_2} + i_{s_3} + \frac{1}{R_2}(V_2 - V_1) + \frac{1}{R_5}V_2 = 0$$

$$-i_{s_1} - i_{s_2} + \frac{1}{R_4}V_4 + \frac{1}{R_3}(V_4 - V_3) = 0$$

1) Number the nodes with one node as 0.
2) Write a conservation law at each node. except (0) in terms of the node voltages !



Examining the nodal equations one sees that a resistor contributes a current to two equations, and its current is dependent on two voltages.



KCL at node n_1 $\quad \sum i_{others} + \frac{1}{R_k}(V_{n1} - V_{n2}) = -i_s$

KCL at node n_2 $\quad \sum i_{others} - \frac{1}{R_k}(V_{n1} - V_{n2}) = i_s$

So, the matrix entries associated with R_k are

$$
\begin{array}{c} \\ n_1 \\ n_2 \\ \\ \end{array}
\begin{array}{c} \qquad n_1 \qquad\quad n_2 \\
\left[\begin{array}{cccc} & & & \\ \cdots\cdots & \frac{1}{R_k} & -\frac{1}{R_k} & \\ \cdots\cdots & -\frac{1}{R_k} & \frac{1}{R_k} & \\ & & & \end{array}\right] \end{array}
$$

Nodal Formulation

Generating Matrices

Circuit Example

Nodal Matrix Generation Algorithm

if $(n_1 > 0)$ & $(n_2 > 0)$

$$G(n_1,n_2) = G(n_1,n_2) - \frac{1}{R} \quad , \quad G(n_2,n_1) = G(n_2,n_1) - \frac{1}{R}$$

$$G(n_1,n_1) = G(n_1,n_1) + \frac{1}{R} \quad , \quad G(n_2,n_2) = G(n_2,n_2) + \frac{1}{R}$$

else if $(n_1 > 0)$

$$G(n_1,n_1) = G(n_1,n_1) + \frac{1}{R}$$

else

$$G(n_2,n_2) = G(n_2,n_2) + \frac{1}{R}$$

Generating Matrices

Nodal Formulation

$$N \updownarrow \; G \; V_n = I_s$$
$$\leftrightarrow$$
$$N$$

(Resistor Networks)

$$2 \cdot J \updownarrow \; G \; u_j = F_L$$
$$\leftrightarrow$$
$$2 \cdot J$$

(Struts and Joints)

Nodal Formulation

Comparing to Node-Branch form

	Node-Branch Matrix	Nodal Matrix
Constitutive / Conservation Law	$\begin{bmatrix} I & -\alpha A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} I_b \\ V_N \end{bmatrix} = \begin{bmatrix} 0 \\ I_s \end{bmatrix}$	$[G][V_N] = [I_s]$

Nodal Formulation

G matrix properties

Diagonally Dominant ….

$$|G_{ii}| \geq \sum_{j \neq i} |G_{ij}|$$

Symmetric ……………..

$$G_{ij} = G_{ji}$$

Smaller …..

$$N \times N \ll (N + B) \times (N + B)$$
$$2J \times 2J \ll (2J + 2S) \times (2J + 2S)$$

Node-Branch formulation

$$\underbrace{\begin{bmatrix} I & -\alpha A^T \\ A & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} I_b \\ V_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ I_s \end{bmatrix}}_{b}$$

- Not Symmetric or Diagonally Dominant
- Matrix is $(n+b)$ x $(n+b)$

Nodal Formulation

Deriving Formulation From Node-Branch

$$I_b - \alpha A^T V_N = 0$$

$$A \cdot (I_b - \alpha A^T V_N) = A \cdot 0$$

$$A I_b = I_s$$

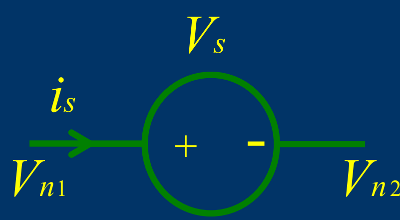
$$\Rightarrow \underbrace{A \alpha A^T}_{G} V_N = I_s$$

Nodal Formulation

Problem element

Voltage Source

Voltage source



Constitutive Equation

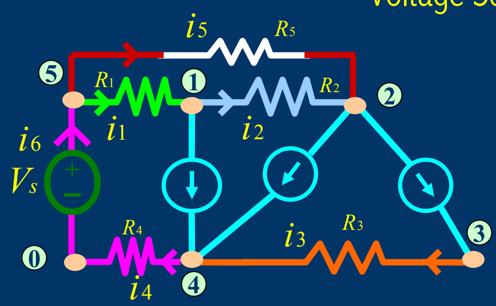
$$0 \cdot i_s + V_{n1} - V_{n2} = V_s$$

Nodal Formulation

Problem Element

Voltage Source

Can form Node-Branch Constitutive Equation with Voltage Sources



$$\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&&\\&&&&1&\\&&&&&0\end{bmatrix}\begin{bmatrix}i_1\\i_2\\i_3\\i_4\\i_5\\i_6\end{bmatrix} - \alpha A^T \begin{bmatrix}V_1\\V_2\\V_3\\V_4\\V_5\end{bmatrix} = \underbrace{\begin{bmatrix}0\\0\\0\\0\\0\\-V_s\end{bmatrix}}_{V_s}$$

$$\alpha = \begin{bmatrix}\frac{1}{R_1}&&&&&\\&\frac{1}{R_2}&&&&\\&&\frac{1}{R_3}&&&\\&&&\frac{1}{R_4}&&\\&&&&\frac{1}{R_5}&\\&&&&&1\end{bmatrix}$$

Nodal Formulation

Problem Element

Voltage Source

Cannot Derive Nodal Formulation

Resistor currents

Voltage source currents missing

$$\begin{bmatrix} I_{bR} \\ 0 \end{bmatrix} - \alpha A^T V_N = 0 \quad \text{(Constitutive Equation)}$$

$$A \cdot \begin{bmatrix} I_{bR} \\ 0 \end{bmatrix} - A \alpha A^T V_N = 0 \quad \text{(Multiply by } A\text{)}$$

$$A \cdot I_b = I_s \quad \text{(Conservation Law)}$$

Cannot Eliminate $A \cdot \begin{bmatrix} I_{bR} \\ 0 \end{bmatrix}$!

Nodal Formulation requires Constitutive relations in the form

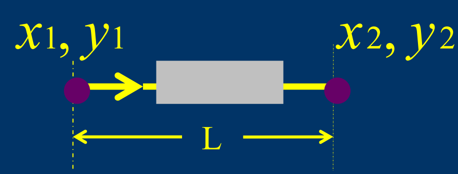
Conserved Quantity = F (Node Voltages) !

Nodal Formulation

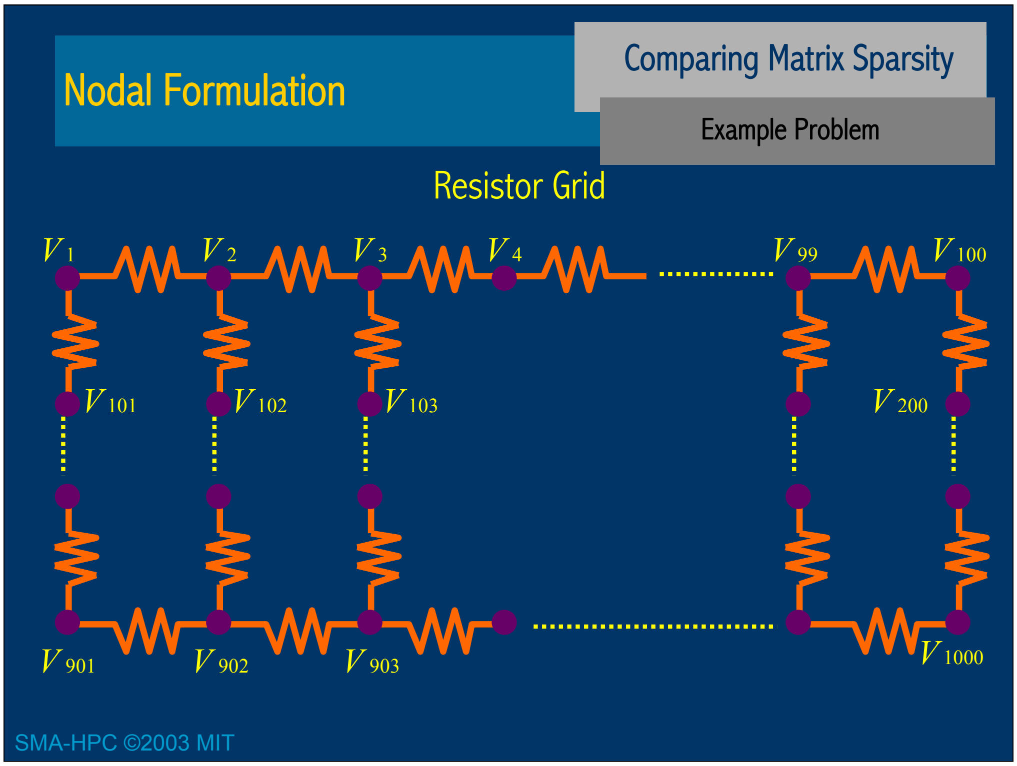
Problem Element

Rigid Rod

Rigid rod



$$\begin{bmatrix} 0 & 0 \\ (y_1-y_2) & (x_1-x_2)\end{bmatrix}\begin{bmatrix} f_x \\ f_y\end{bmatrix} + \begin{bmatrix}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \\ 0\end{bmatrix} = \begin{bmatrix} L \\ 0 \end{bmatrix}$$


Nodal Formulation
Comparing Matrix Sparsity
Example Problem
Resistor Grid
V 1
V 2
V 3
V 4
V 99
V 100
V 101
V 102
V 103
V 200
V 901
V 902
V 903
V 1000
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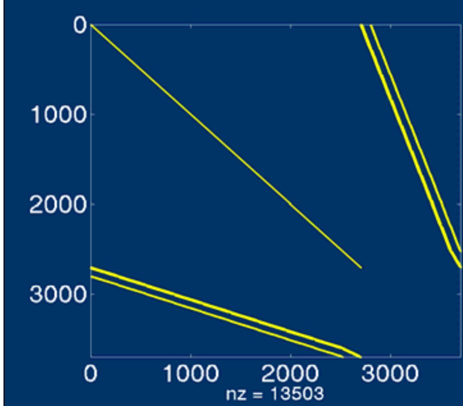
Nodal Formulation

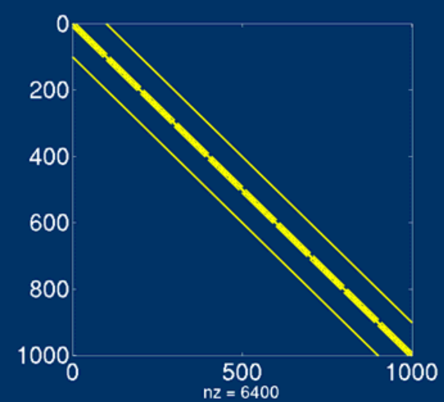
Comparing Matrix Sparsity

Example Problem

Matrix non-zero locations for 100 x 10 Resistor Grid



Node-Branch



Nodal

Summary of key points…...

- Developed algorithms for automatically constructing matrix equations from schematics using conservation law + constitutive equations.
- Looked at two node-branch and nodal forms.

Summary of key points

- Node-branch
  - General constitutive equations
  - Large sparser system
  - No diagonal dominance
- Nodal
  - Conserved quantity must be a function of node variables
  - Smaller denser system.
  - Diagonally dominant & symmetric.