L2: Combinational Logic Design

(Construction and Boolean Algebra)

Acknowledgements:

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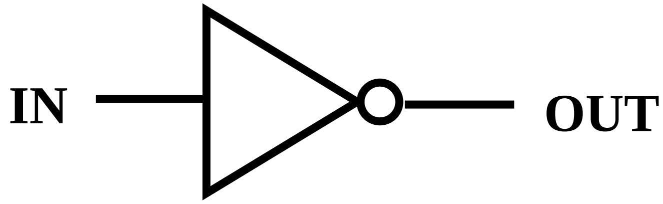
Prof. Randy Katz (Unified Microelectronics Corporation Distinguished Professor in Electrical Engineering and Computer Science at the University of California, Berkeley) and Prof. Gaetano Borriello (University of Washington Department of Computer Science & Engineering) from Chapter 2 of R. Katz, G. Borriello. *Contemporary Logic Design*. 2nd ed. Pentice-Hall/Pearson Education, 2005.

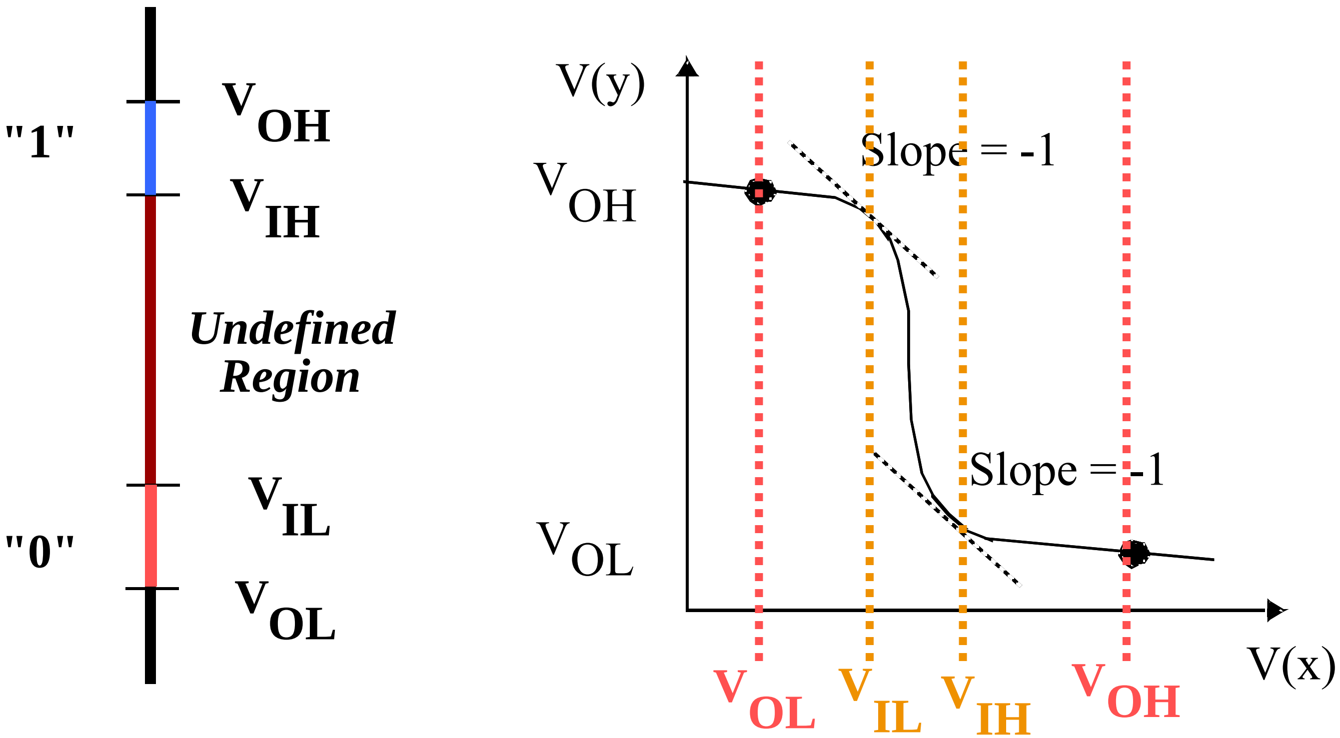
J. Rabaey, A. Chandrakasan, B. Nikolic. *Digital Integrated Circuits: A Design Perspective*. Prentice Hall/Pearson, 2003.

Review: Noise Margin



Truth Table

IN	OUT
0	1
1	0

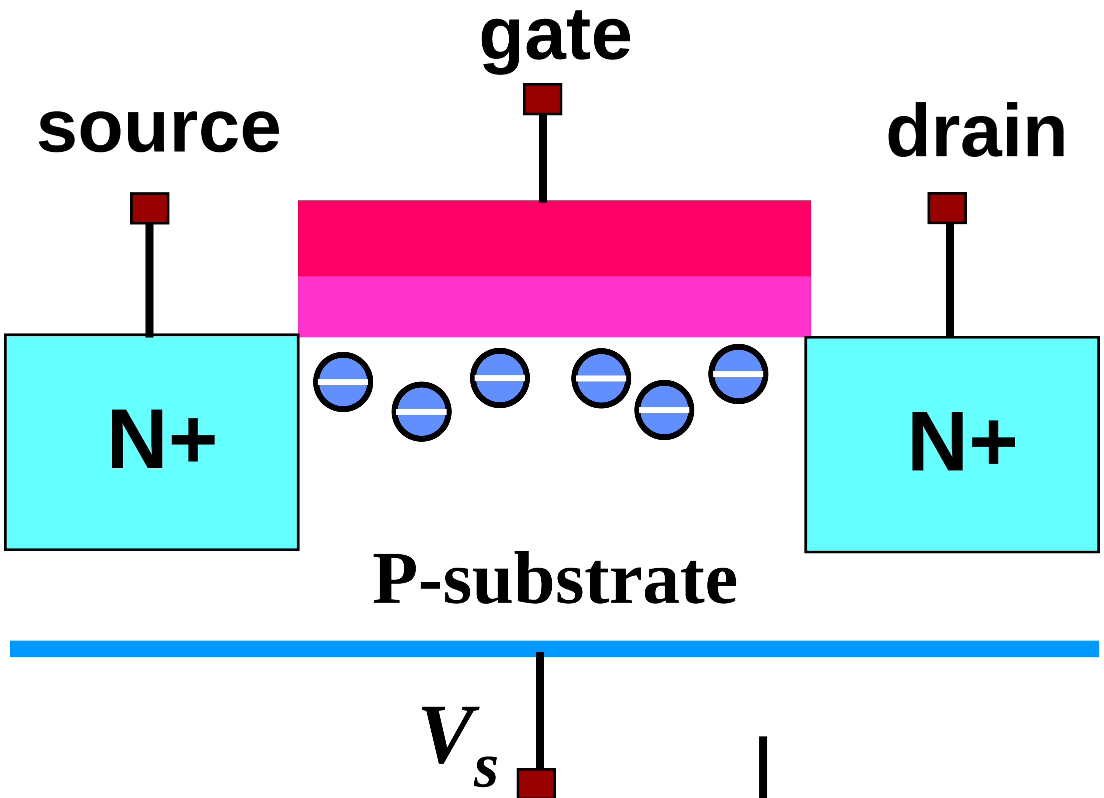


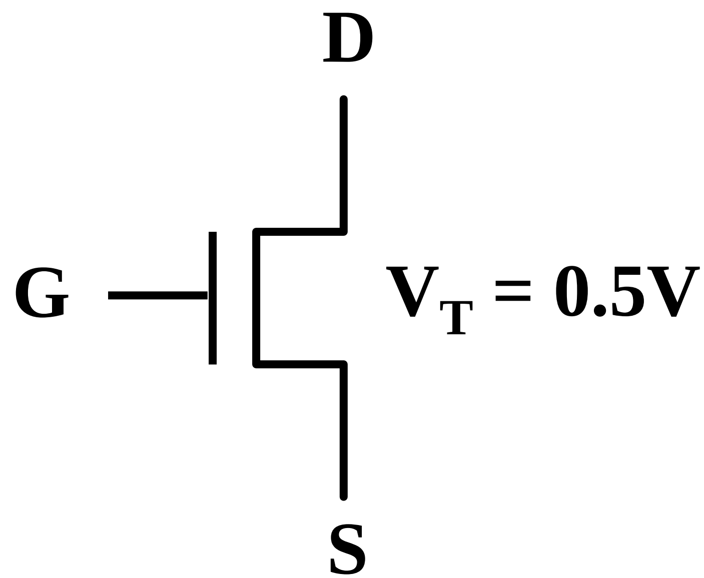
$NM_L = V_{IL} - V_{OL}$

$NM_H = V_{OH} - V_{IH}$

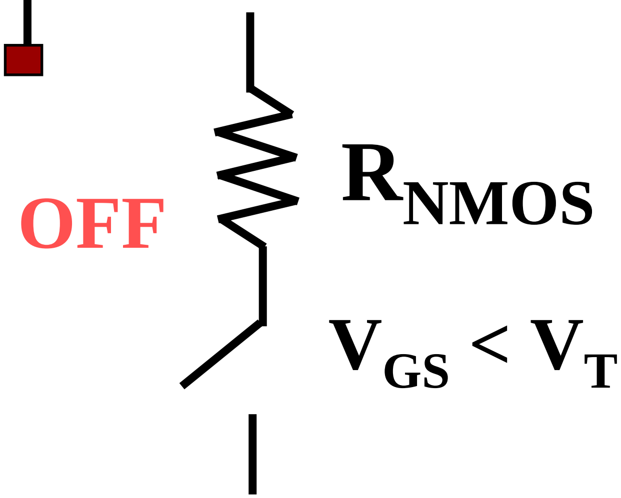
- **Large noise margins** protect against various noise sources

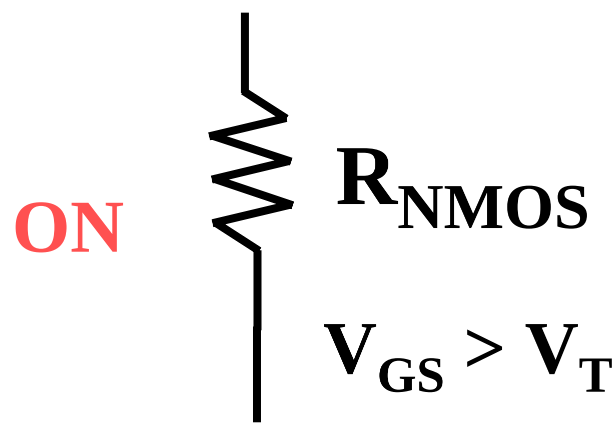
MOS Technology: The NMOS Switch





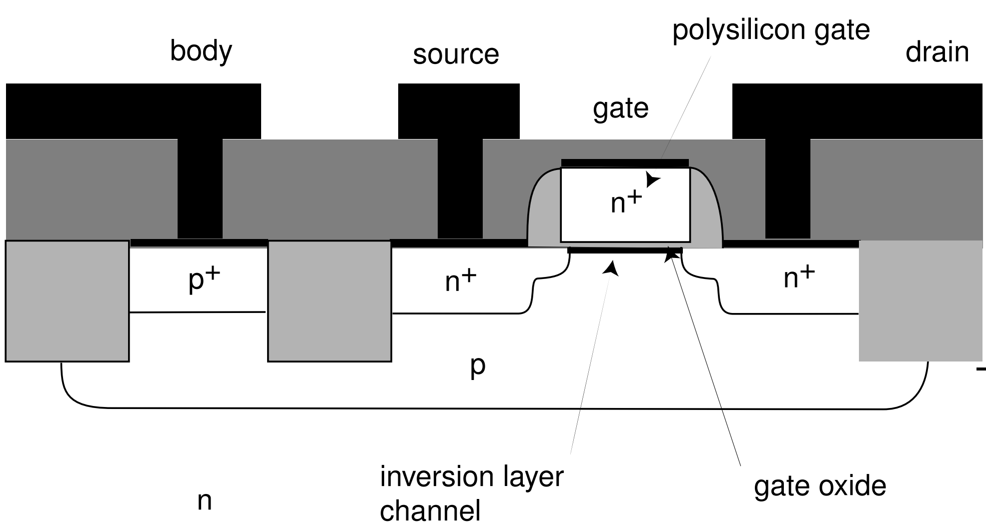
Switch Model

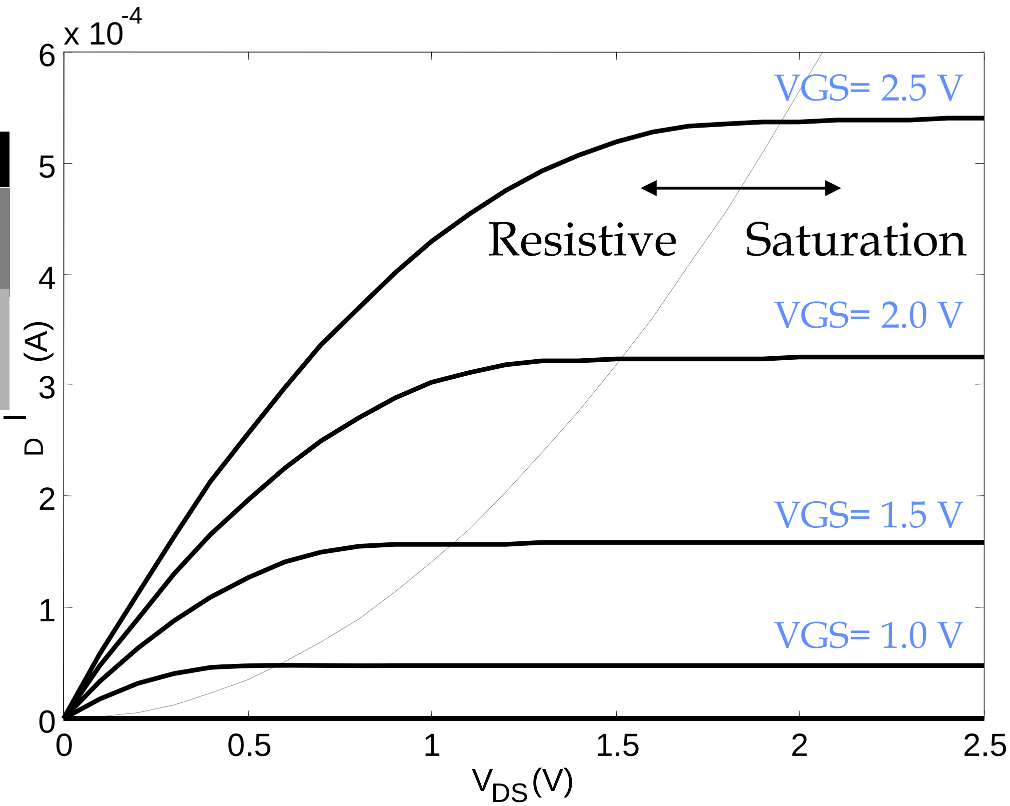


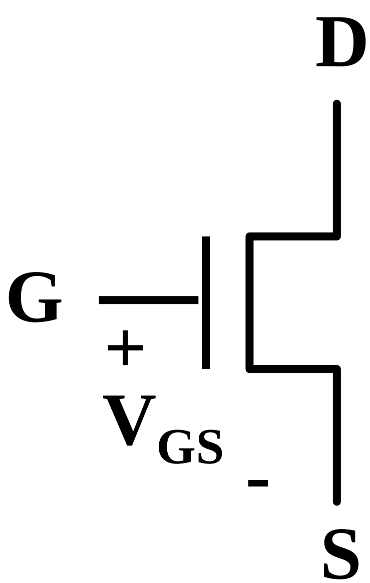


NMOS ON when Switch Input is High

NMOS Device Characteristics

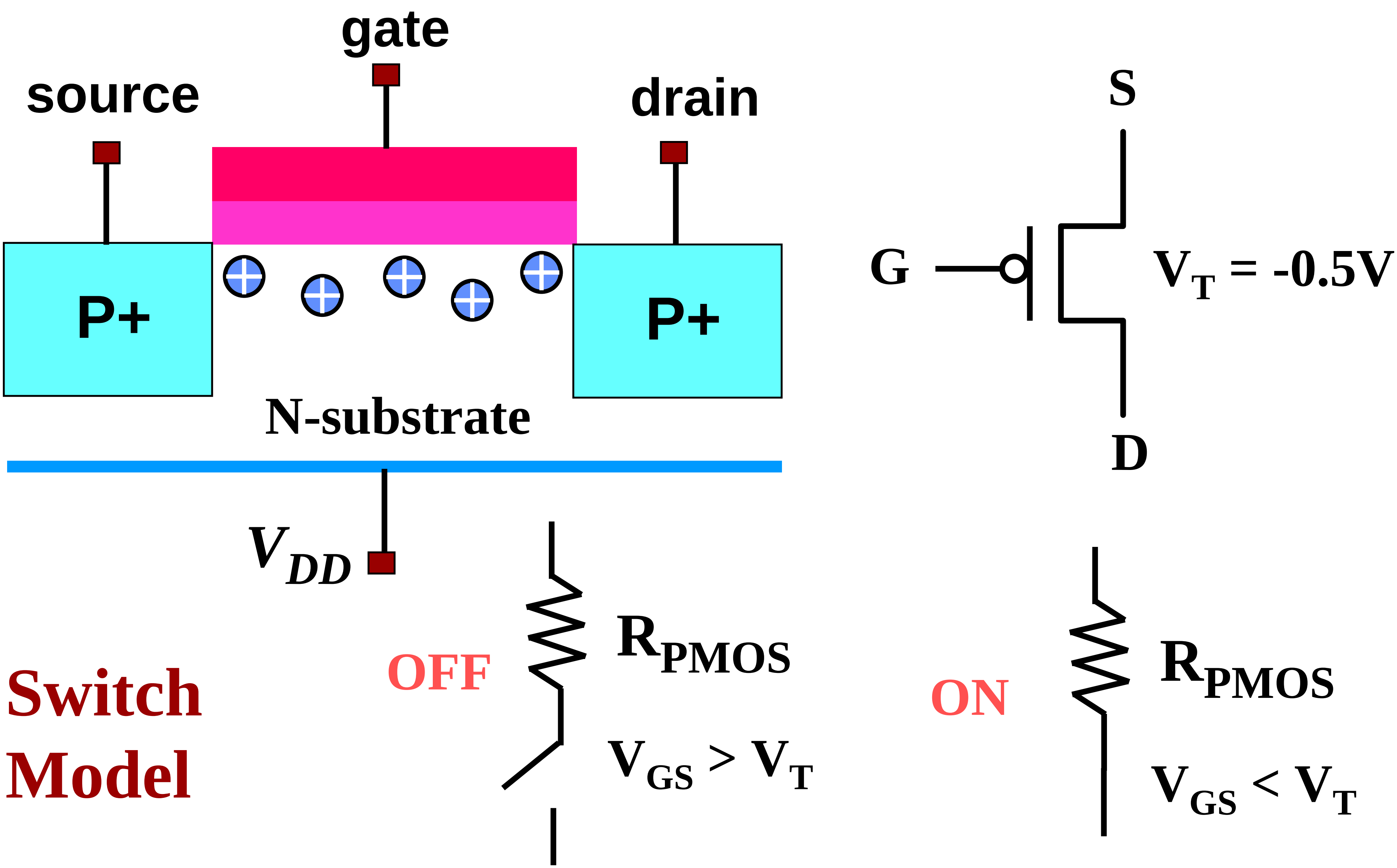






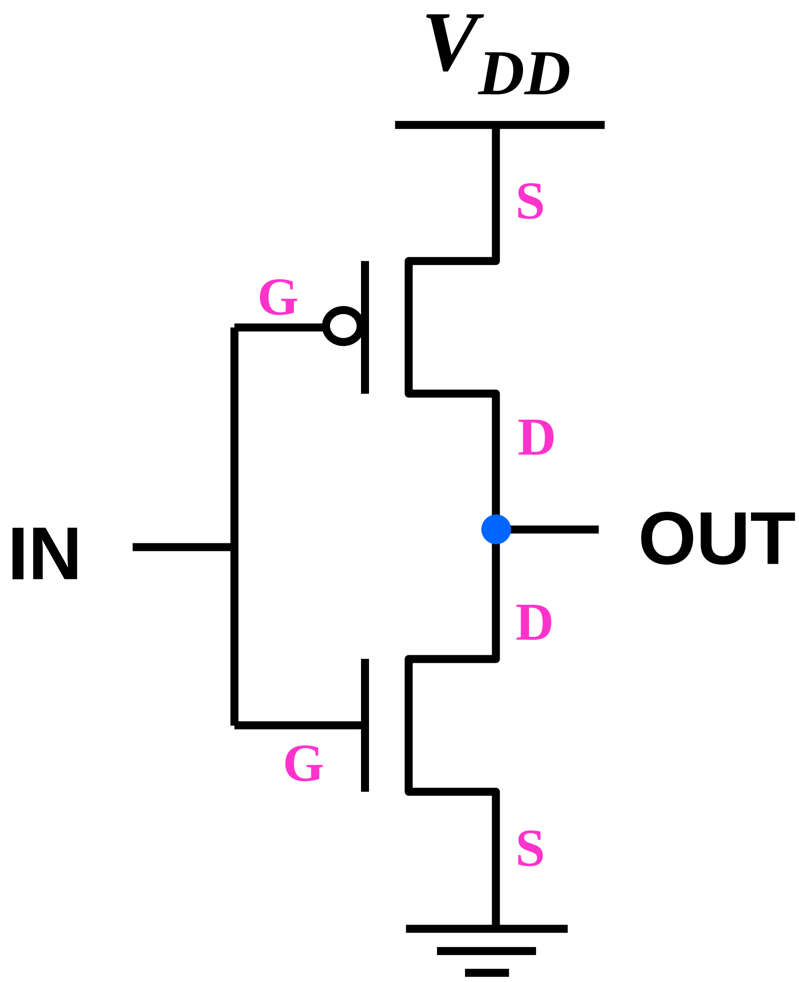
$V_T = 0.5V$

- MOS is a very non-linear.
- Switch-resistor model sufficient for first order analysis.

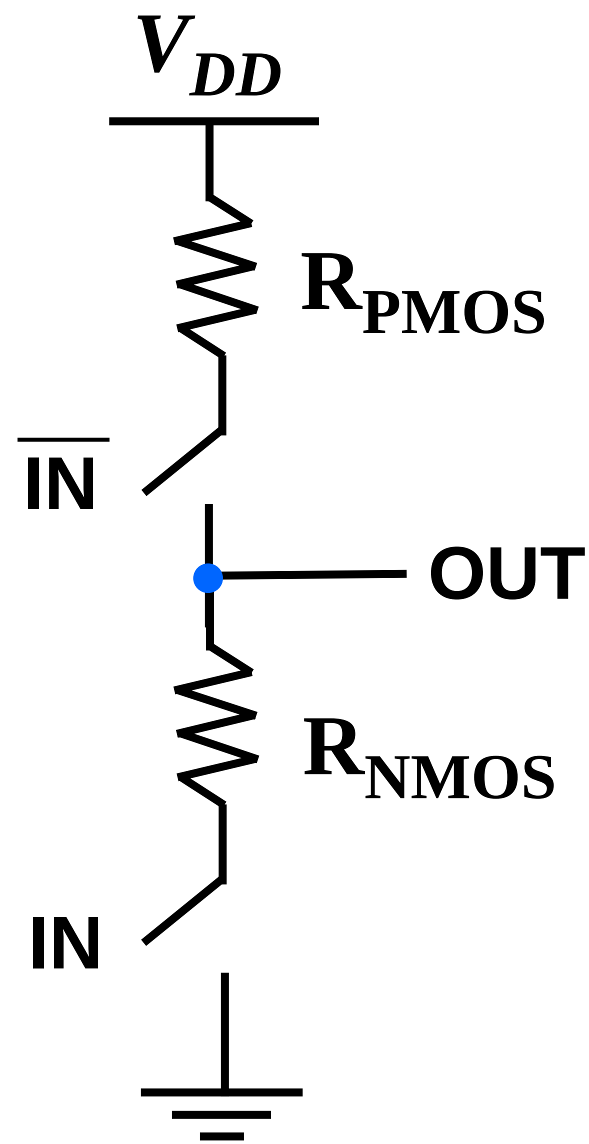


PMOS ON when Switch Input is Low

The CMOS Inverter

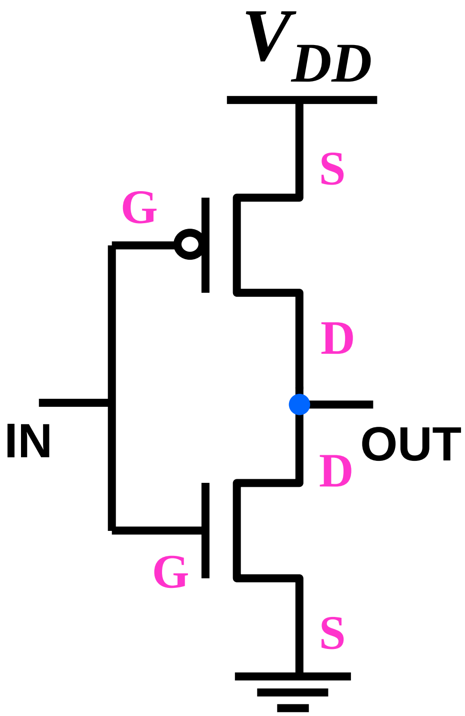


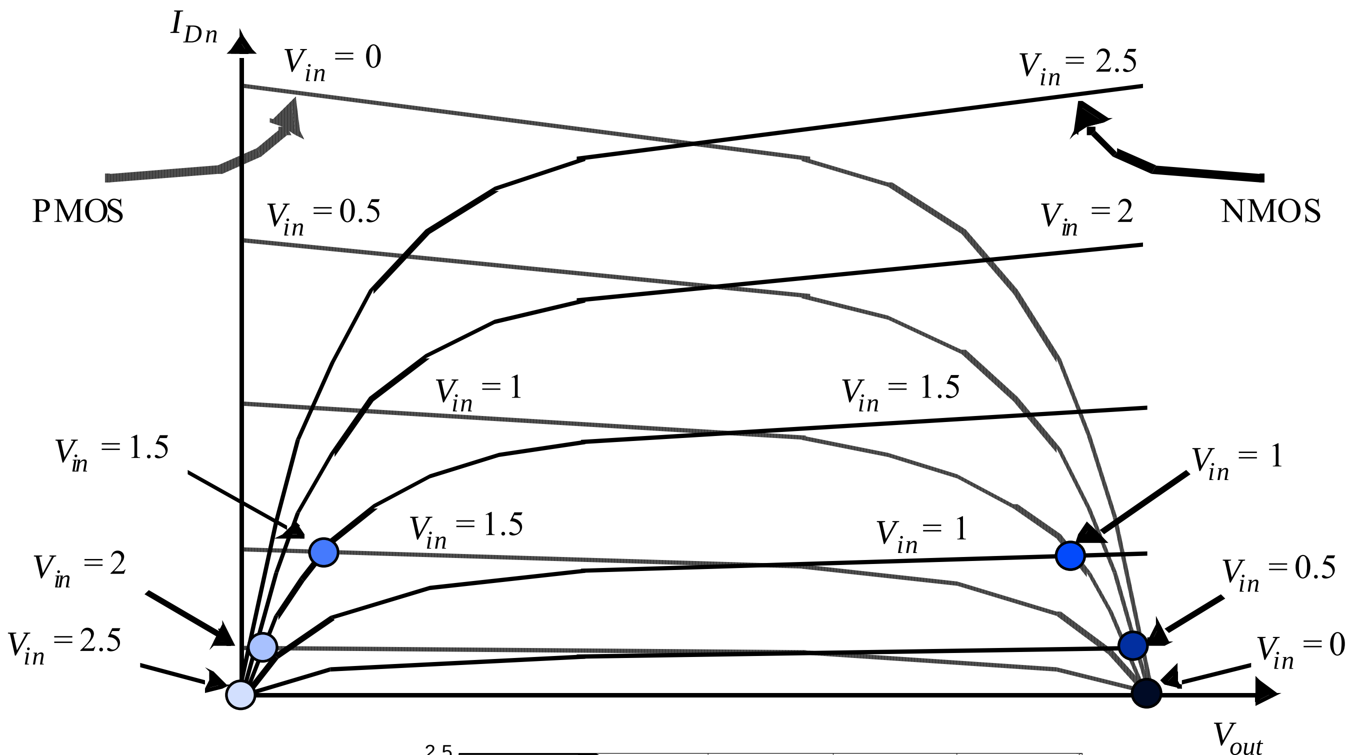
Switch Model

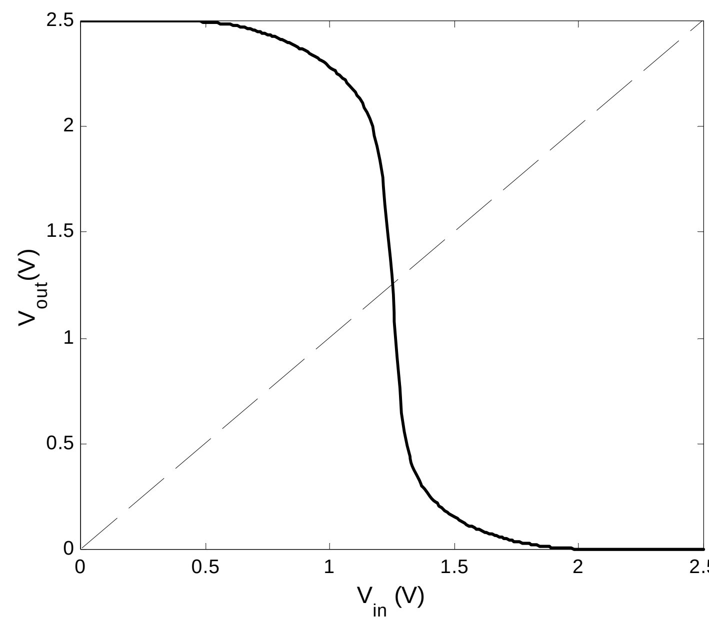


Rail-to-rail Swing in CMOS

Inverter VTC: Load Line Analysis





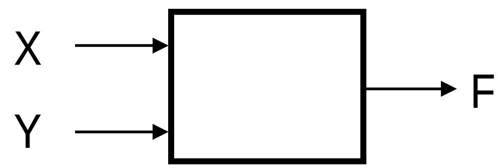


CMOS gates have:

- Rail-to-rail swing (0V to V_{DD})
- Large noise margins
- "zero" static power dissipation

Possible Function of Two Inputs

There are 16 possible functions of 2 input variables:



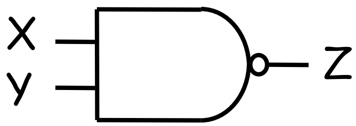
X	Y	16 possible functions (F_0–F_{15})															
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
		0	X AND Y		X		Y	X XOR Y	X OR Y	X NOR Y NOT (X OR Y)	X = Y	NOT Y		NOT X		X NAND Y NOT (X AND Y)	1

In general, there are 2 $^{(2^n)}$ functions of n inputs

Common Logic Gates

Gate	Symbol	Truth-Table	Expression

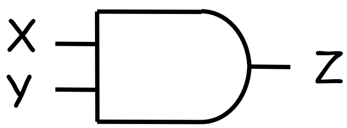
NAND



X	Y	Z
0	0	1
0	1	1
1	0	1
1	1	0

$$Z = \overline{X \bullet Y}$$

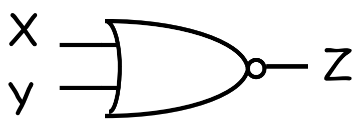
AND



X	Y	Z
0	0	0
0	1	0
1	0	0
1	1	1

$$Z = X \bullet Y$$

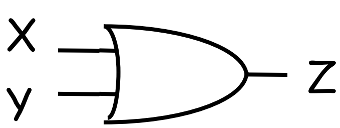
NOR



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	0

$$Z = \overline{X + Y}$$

OR



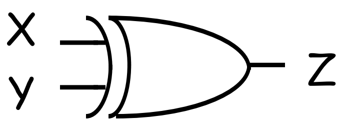
X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	1

$$Z = X + Y$$

Exclusive (N)OR Gate

XOR

$(X \oplus Y)$



X	Y	Z
0	0	0
0	1	1
1	0	1
1	1	0

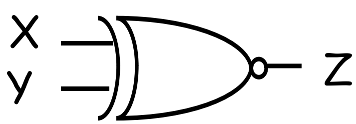
$Z = X\,\overline{Y} + \overline{X}\,Y$

X or Y but not both

("inequality", "difference")

XNOR

$\overline{(X \oplus Y)}$



X	Y	Z
0	0	1
0	1	0
1	0	0
1	1	1

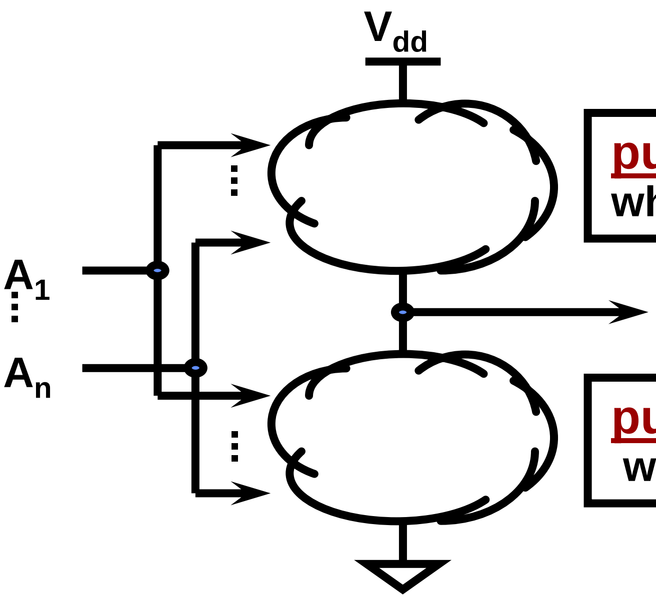
$Z = \overline{X}\,\overline{Y} + X\,Y$

X and Y the same

("equality")

Widely used in arithmetic structures such as adders and multipliers

Generic CMOS Recipe

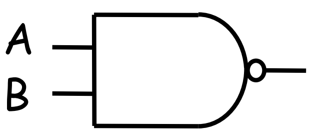


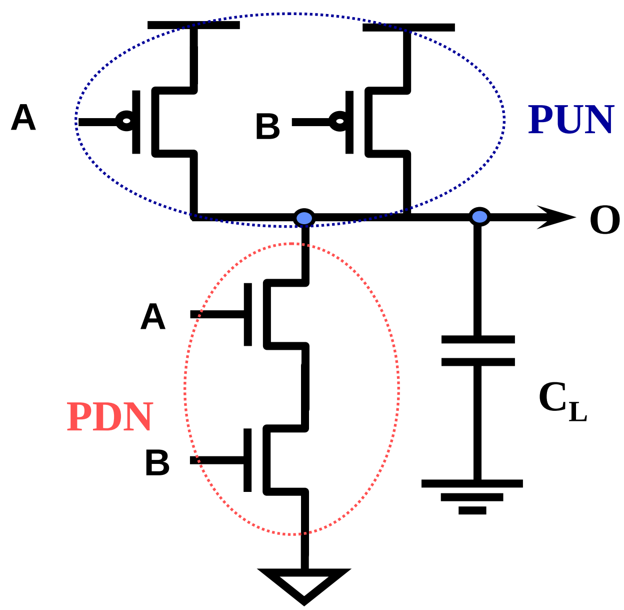
pullup: make this connection when we want $F(A_1,\ldots,A_n) = 1$

pulldown: make this connection when we want $F(A_1,\ldots,A_n) = 0$

Note: CMOS gates result in inverting functions!

(easier to build NAND vs. AND)





A	B	PDN	PUN	O
0	0	Off	On	1
0	1	Off	On	1
1	0	Off	On	1
1	1	On	Off	0

How do you build a 2-input NOR Gate?

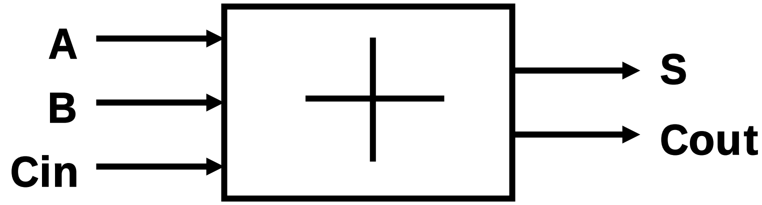
Theorems of Boolean Algebra (I)

- **Elementary**
  1. $X + 0 = X$ — 1D. $X \bullet 1 = X$
  2. $X + 1 = 1$ — 2D. $X \bullet 0 = 0$
  3. $X + X = X$ — 3D. $X \bullet X = X$
  4. $(\overline{\overline{X}}) = X$
  5. $X + \overline{X} = 1$ — 5D. $X \bullet \overline{X} = 0$
- **Commutativity:**
  6. $X + Y = Y + X$ — 6D. $X \bullet Y = Y \bullet X$
- **Associativity:**
  7. $(X + Y) + Z = X + (Y + Z)$ — 7D. $(X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)$
- **Distributivity:**
  8. $X \bullet (Y + Z) = (X \bullet Y) + (X \bullet Z)$ — 8D. $X + (Y \bullet Z) = (X + Y) \bullet (X + Z)$
- **Uniting:**
  9. $X \bullet Y + X \bullet \overline{Y} = X$ — 9D. $(X + Y) \bullet (X + \overline{Y}) = X$
- **Absorption:**
  10. $X + X \bullet Y = X$ — 10D. $X \bullet (X + Y) = X$
  11. $(X + \overline{Y}) \bullet Y = X \bullet Y$ — 11D. $(X \bullet \overline{Y}) + Y = X + Y$

- **Factoring:**
  - **12.** $(X \cdot Y) + (X \cdot Z) = X \cdot (Y + Z)$
  - **12D.** $(X + Y) \cdot (X + Z) = X + (Y \cdot Z)$
- **Consensus:**
  - **13.** $(X \cdot Y) + (Y \cdot Z) + (\overline{X} \cdot Z) = X \cdot Y + \overline{X} \cdot Z$
  - **13D.** $(X + Y) \cdot (Y + Z) \cdot (\overline{X} + Z) = (X + Y) \cdot (\overline{X} + Z)$
- **De Morgan's:**
  - **14.** $\overline{(X + Y + \ldots)} = \overline{X} \cdot \overline{Y} \cdot \ldots$
  - **14D.** $\overline{(X \cdot Y \cdot \ldots)} = \overline{X} + \overline{Y} + \ldots$
- **Generalized De Morgan's:**
  - **15.** $\overline{f}(X1,X2,\ldots,Xn,0,1,+,\cdot) = f(\overline{X1},\overline{X2},\ldots,\overline{Xn},1,0,\cdot,+)$
- **Duality**
  - Dual of a Boolean expression is derived by replacing • by +, + by •, 0 by 1, and 1 by 0, and leaving variables unchanged
  - $f(X1,X2,\ldots,Xn,0,1,+,\cdot) \Leftrightarrow f(X1,X2,\ldots,Xn,1,0,\cdot,+)$

Simple Example: One Bit Adder

- **1-bit binary adder**
  - inputs: A, B, Carry-in
  - outputs: Sum, Carry-out



A	B	Cin	S	Cout
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Sum-of-Products Canonical Form

$$S = \overline{A}\,\overline{B}\,Cin + \overline{A}\,B\,\overline{Cin} + A\,\overline{B}\,\overline{Cin} + A\,B\,Cin$$

$$Cout = \overline{A}\,B\,Cin + A\,\overline{B}\,Cin + A\,B\,\overline{Cin} + A\,B\,Cin$$

- **Product term (or minterm)**
  - ANDed product of literals – input combination for which output is true
  - Each variable appears exactly once, in true or inverted form (but not both)

Simplify Boolean Expressions

$$Cout = \overline{A}\,B\,Cin + A\,\overline{B}\,Cin + A\,B\,\overline{Cin} + A\,B\,Cin$$

$$= \overline{A}\,B\,Cin + A\,B\,Cin + A\,\overline{B}\,Cin + A\,B\,Cin + A\,B\,\overline{Cin} + A\,B\,Cin$$

$$= (\overline{A} + A)\,B\,Cin + A\,(\overline{B} + B)\,Cin + A\,B\,(\overline{Cin} + Cin)$$

$$= B\,Cin + A\,Cin + A\,B$$

$$= (B + A)\,Cin + A\,B$$

$$S = \overline{A}\,\overline{B}\,Cin + \overline{A}\,B\,\overline{Cin} + A\,\overline{B}\,\overline{Cin} + A\,B\,Cin$$

$$= (\overline{A}\,\overline{B} + A\,B)\,Cin + (A\,\overline{B} + \overline{A}\,B)\,\overline{Cin}$$

$$= \overline{(A \oplus B)}\,Cin + (A \oplus B)\,\overline{Cin}$$

$$= A \oplus B \oplus Cin$$

Sum-of-Products & Product-of-Sum

- **Product term** (or minterm): ANDed product of literals – input combination for which output is true

A	B	C	minterms	
0	0	0	$\overline{A}\,\overline{B}\,\overline{C}$	m0
0	0	1	$\overline{A}\,\overline{B}\,C$	m1
0	1	0	$\overline{A}\,B\,\overline{C}$	m2
0	1	1	$\overline{A}\,B\,C$	m3
1	0	0	$A\,\overline{B}\,\overline{C}$	m4
1	0	1	$A\,\overline{B}\,C$	m5
1	1	0	$A\,B\,\overline{C}$	m6
1	1	1	$A\,B\,C$	m7

short-hand notation form in terms of 3 variables

F in canonical form:

$$F(A, B, C) = \Sigma m(1,3,5,6,7)$$
$$= m1 + m3 + m5 + m6 + m7$$
$$F = \overline{A}\,\overline{B}\,C + \overline{A}\,B\,C + A\,\overline{B}\,C + A\,B\,\overline{C} + ABC$$

canonical form ≠ minimal form

$$F(A, B, C) = \overline{A}\,\overline{B}\,C + \overline{A}\,B\,C + A\overline{B}\,C + ABC + AB\overline{C}$$
$$= (\overline{A}\,\overline{B} + \overline{A}\,B + A\overline{B} + AB)C + AB\overline{C}$$
$$= ((\overline{A} + A)(\overline{B} + B))C + AB\overline{C}$$
$$= C + AB\overline{C} = AB\overline{C} + C = AB + C$$

- **Sum term** (or maxterm) - ORed sum of literals – input combination for which output is false

A	B	C	maxterms	
0	0	0	$A + B + C$	M0
0	0	1	$A + B + \overline{C}$	M1
0	1	0	$A + \overline{B} + C$	M2
0	1	1	$A + \overline{B} + \overline{C}$	M3
1	0	0	$\overline{A} + B + C$	M4
1	0	1	$\overline{A} + B + \overline{C}$	M5
1	1	0	$\overline{A} + \overline{B} + C$	M6
1	1	1	$\overline{A} + \overline{B} + \overline{C}$	M7

short-hand notation for maxterms of 3 variables

F in canonical form:

$$F(A, B, C) = \Pi M(0,2,4)$$
$$= M0 \cdot M2 \cdot M4$$
$$= (A + B + C)\,(A + \overline{B} + C)\,(\overline{A} + B + C)$$

canonical form ≠ minimal form

$$F(A, B, C) = (A + B + C)\,(A + \overline{B} + C)\,(\overline{A} + B + C)$$
$$= (A + B + C)\,(A + \overline{B} + C)$$
$$(A + B + C)\,(\overline{A} + B + C)$$
$$= (A + C)\,(B + C)$$

Mapping Between Forms

1. **Minterm to Maxterm conversion:**
   rewrite minterm shorthand using maxterm shorthand
   replace minterm indices with the indices not already used

   E.g., $F(A,B,C) = \Sigma m(3,4,5,6,7) = \Pi M(0,1,2)$

2. **Maxterm to Minterm conversion:**
   rewrite maxterm shorthand using minterm shorthand
   replace maxterm indices with the indices not already used

   E.g., $F(A,B,C) = \Pi M(0,1,2) = \Sigma m(3,4,5,6,7)$

3. **Minterm expansion of F to Minterm expansion of F':**
   in minterm shorthand form, list the indices not already used in F

   E.g., $F(A,B,C) = \Sigma m(3,4,5,6,7) \longrightarrow F'(A,B,C) = \Sigma m(0,1,2)$
   $= \Pi M(0,1,2) \longrightarrow = \Pi M(3,4,5,6,7)$

4. **Minterm expansion of F to Maxterm expansion of F':**
   rewrite in Maxterm form, using the same indices as F

   E.g., $F(A,B,C) = \Sigma m(3,4,5,6,7) \longrightarrow F'(A,B,C) = \Pi M(3,4,5,6,7)$
   $= \Pi M(0,1,2) \longrightarrow = \Sigma m(0,1,2)$

The Uniting Theorem

- Key tool to simplification: $A(\bar{B} + B) = A$
- Essence of simplification of two-level logic
  - Find two element subsets of the ON-set where only one variable changes its value – this single varying variable can be eliminated and a single product term used to represent both elements

$$F = \bar{A}\,\bar{B} + A\bar{B} = (\bar{A} + A)\bar{B} = \bar{B}$$

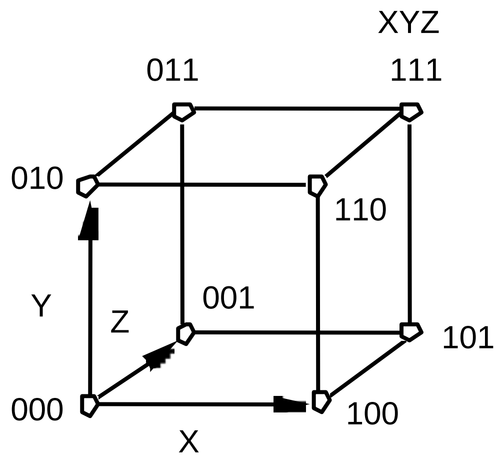
A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

B has the same value in both on-set rows
– B remains

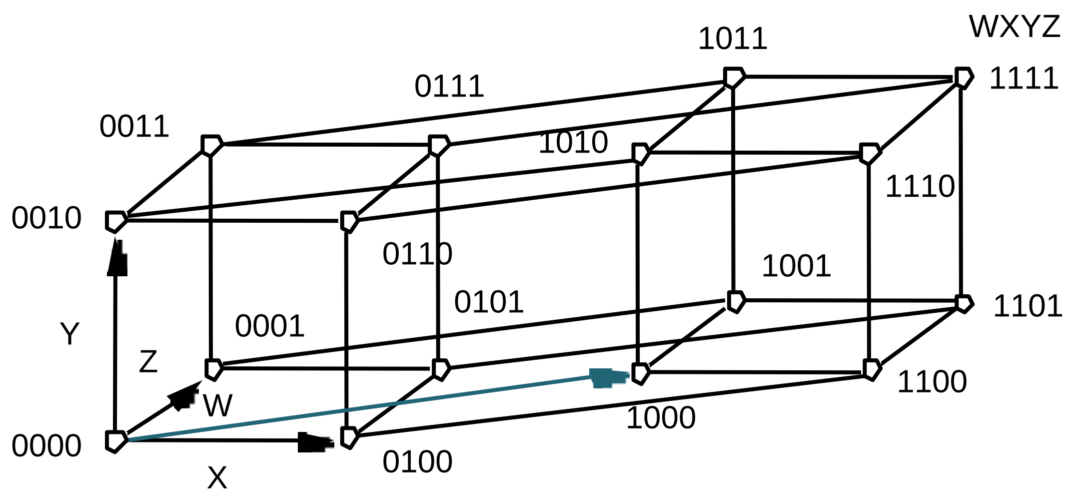
A has a different value in the two rows
– A is eliminated

Boolean Cubes

- Just another way to represent truth table
- Visual technique for identifying when the uniting theorem can be applied
- n input variables = n-dimensional "cube"



3-cube

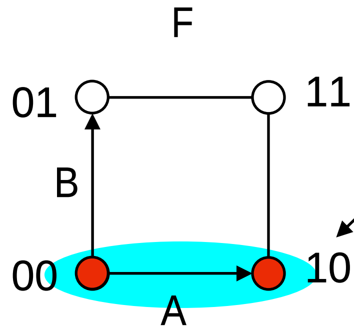


4-cube

Mapping Truth Tables onto Boolean Cubes

- **Uniting theorem**

A	B	F
0	0	1
0	1	0
1	0	1
1	1	0



Circled group of the on-set is called the *adjacency* plane. Each adjacency plane corresponds to a product term.

ON-set = solid nodes
OFF-set = empty nodes

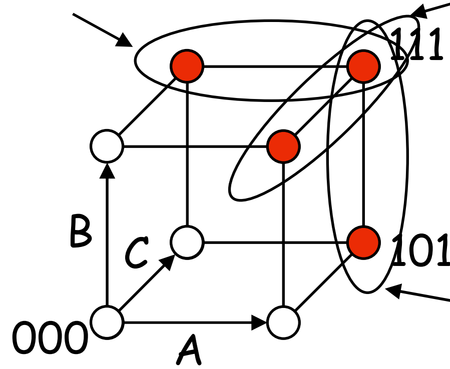
A varies within face, B does not this face represents the literal $\overline{B}$

- **Three variable example: Binary full-adder carry-out logic**

A	B	Cin	Cout
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$(\overline{A}+A)BCin$

$AB(\overline{Cin}+Cin)$

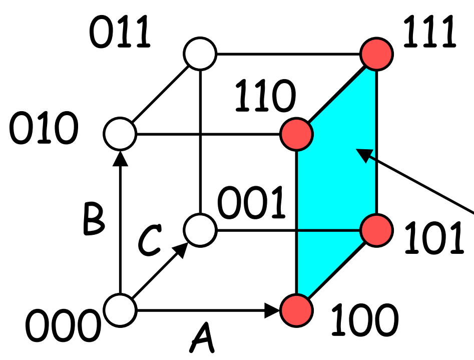


$A(B+\overline{B})Cin$

Cout = BCin+AB+ACin

The on-set is completely covered by the combination (OR) of the subcubes of lower dimensionality - note that “111” is covered three times

Higher Dimension Cubes



F(A,B,C) = Σm(4,5,6,7)

on-set forms a square
i.e., a cube of dimension 2 (2-D adjacency plane)

represents an expression in one variable
i.e., 3 dimensions – 2 dimensions

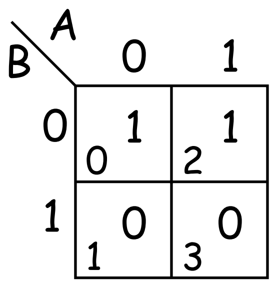
A is asserted (true) and unchanged
B and C vary

This subcube represents the literal A

- **In a 3-cube (three variables):**
  - **0-cube, i.e., a single node, yields a term in 3 literals**
  - **1-cube, i.e., a line of two nodes, yields a term in 2 literals**
  - **2-cube, i.e., a plane of four nodes, yields a term in 1 literal**
  - **3-cube, i.e., a cube of eight nodes, yields a constant term "1"**
- **In general,**
  - **m-subcube within an n-cube (m < n) yields a term with n – m literals**

Karnaugh Maps

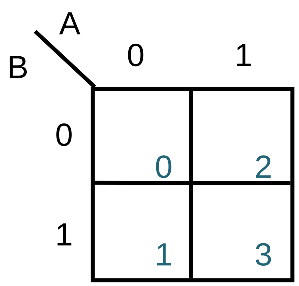
- **Alternative to truth-tables to help visualize adjacencies**
  - **Guide to applying the uniting theorem - On-set elements with only one variable changing value are adjacent unlike in a linear truth-table**



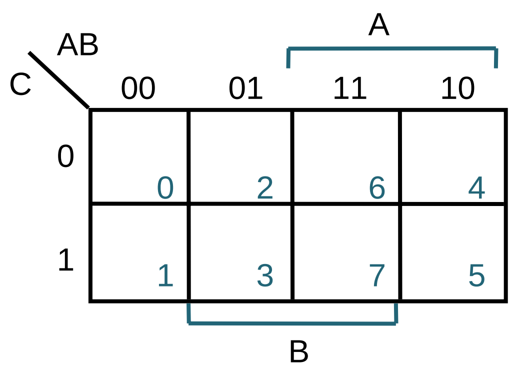
A	B	F
0	0	1
0	1	0
1	0	1
1	1	0

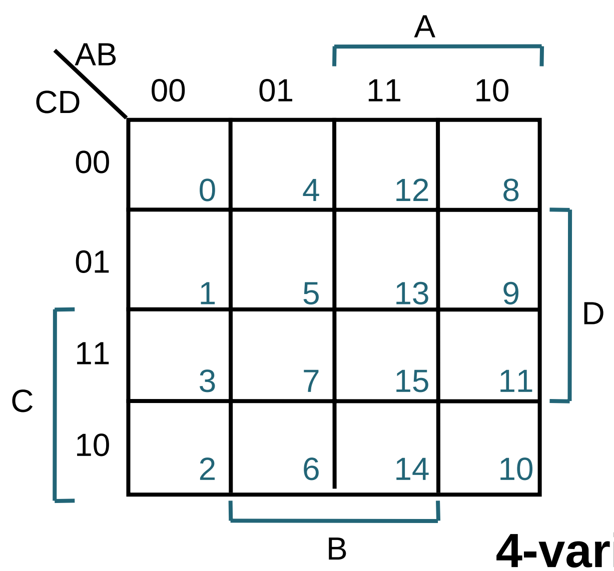
- **Numbering scheme based on Gray–code**
  - **e.g., 00, 01, 11, 10 (only a single bit changes in code for adjacent map cells)**

2-variable K-map



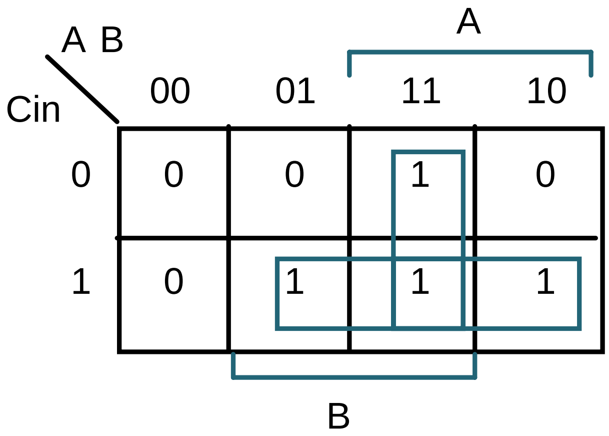
3-variable K-map



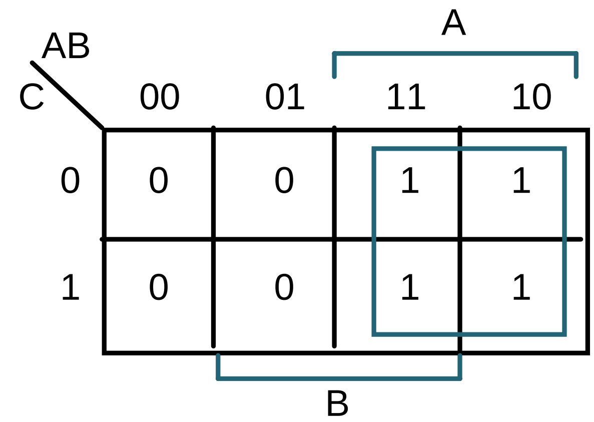


4-variable K-map

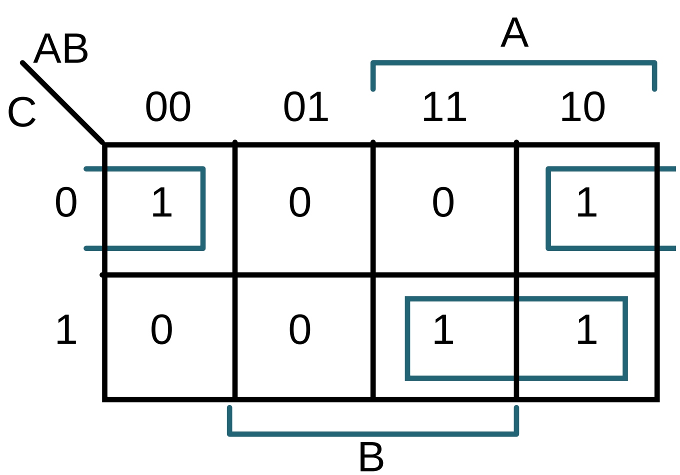
K-Map Examples



Cout =

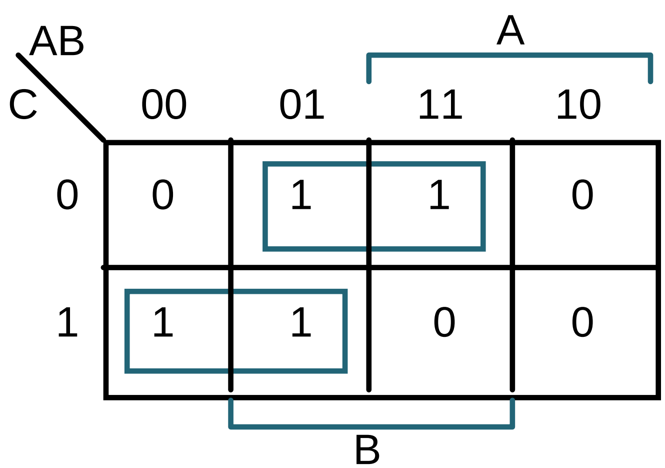


F(A,B,C) =



F(A,B,C) = Σm(0,4,5,7)

F =

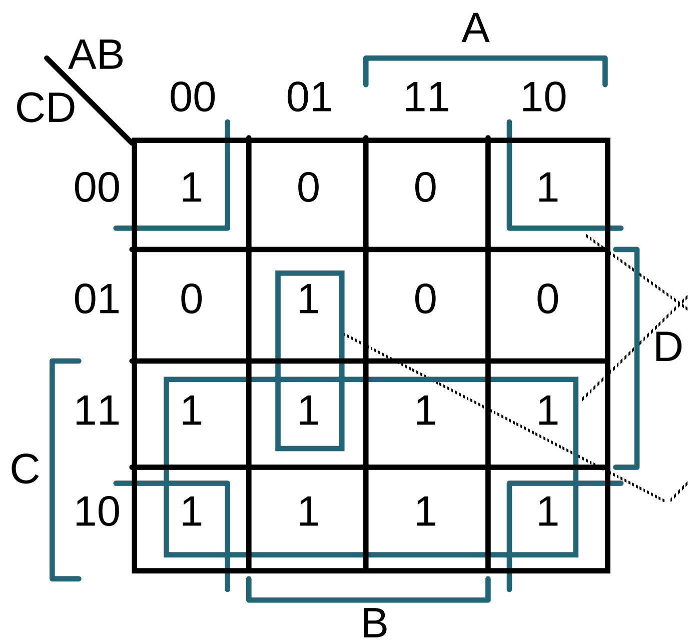


F' simply replace 1's with 0's and vice versa

F'(A,B,C) = Σm(1,2,3,6)

F' =

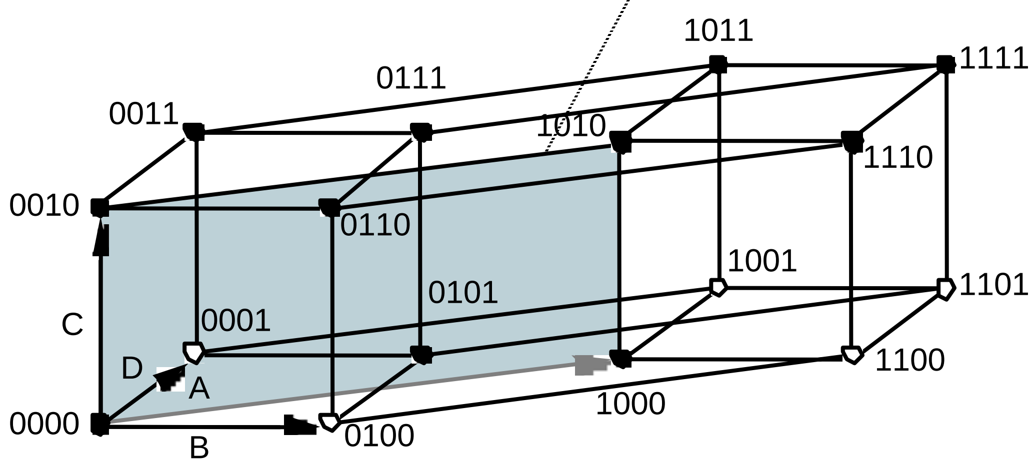
Four Variable Karnaugh Map



$F(A,B,C,D) = \Sigma m(0,2,3,5,6,7,8,10,11,14,15)$

$F = C + \overline{A}\,B\,D + \overline{B}\,\overline{D}$

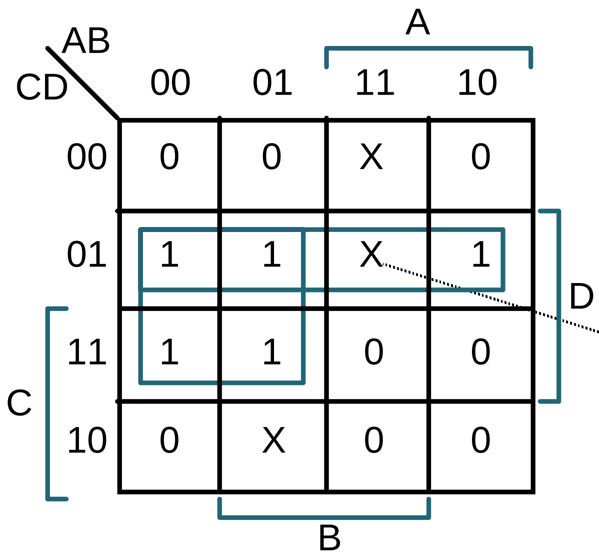
Find the smallest number of the largest possible subcubes that cover the ON-set



K-map Corner Adjacency Illustrated in the 4-Cube

K-Map Example: Don’t Cares

Don't Cares can be treated as 1's or 0's if it is advantageous to do so

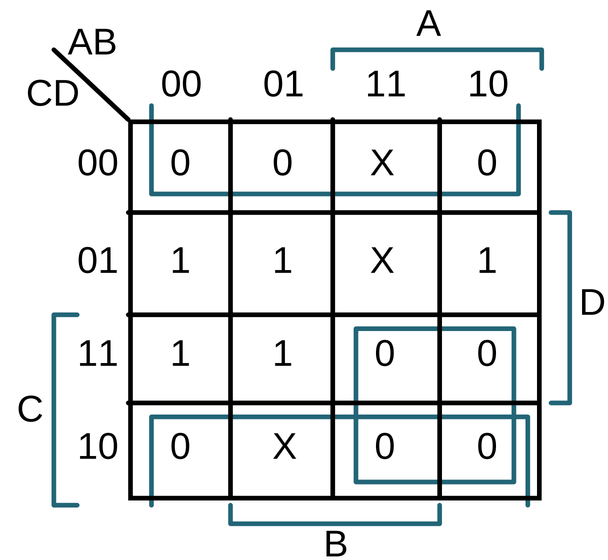


$F(A,B,C,D) = \Sigma m(1,3,5,7,9) + \Sigma d(6,12,13)$

$F = \overline{A}\,D + \overline{B}\,\overline{C}\,D$ w/o don't cares

$F = \overline{C}\,D + \overline{A}\,D$ w/ don't cares

By treating this DC as a "1", a 2-cube can be formed rather than one 0-cube



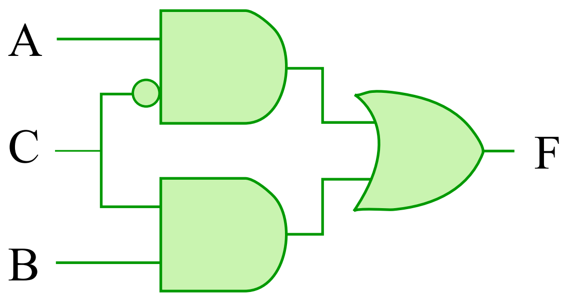
In PoS form: $F = D\,(\overline{A} + \overline{C})$

Equivalent answer as above, but fewer literals

Hazards

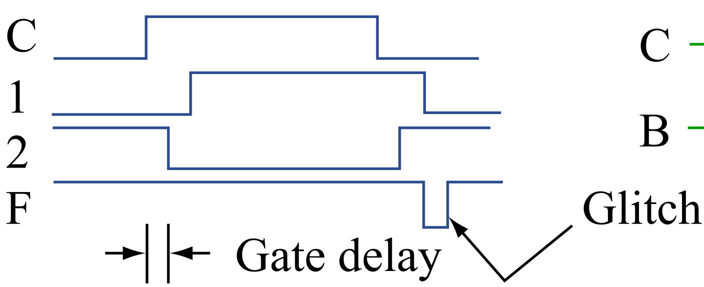
Static hazards: Consider this function:

$$F = A * \overline{C} + B * C$$



C \ AB	00	01	11	10
0	0	0	1	1
1	0	1	1	0

A = B = 1



Implemented with MSI gates:

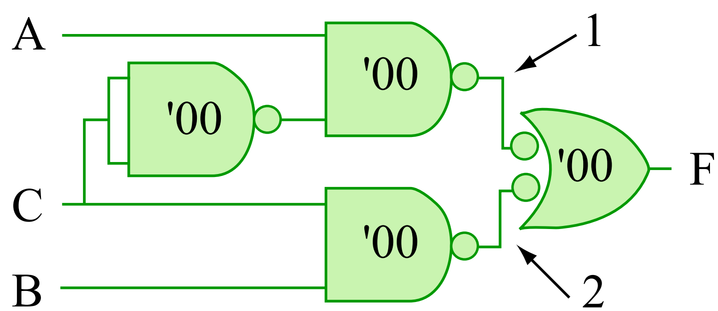


Figure by MIT OpenCourseWare.

Fixing Hazards

The glitch is the result of timing differences in parallel data paths. It is associated with the function jumping between groupings or product terms on the K-map. To fix it, cover it up with another grouping or product term!

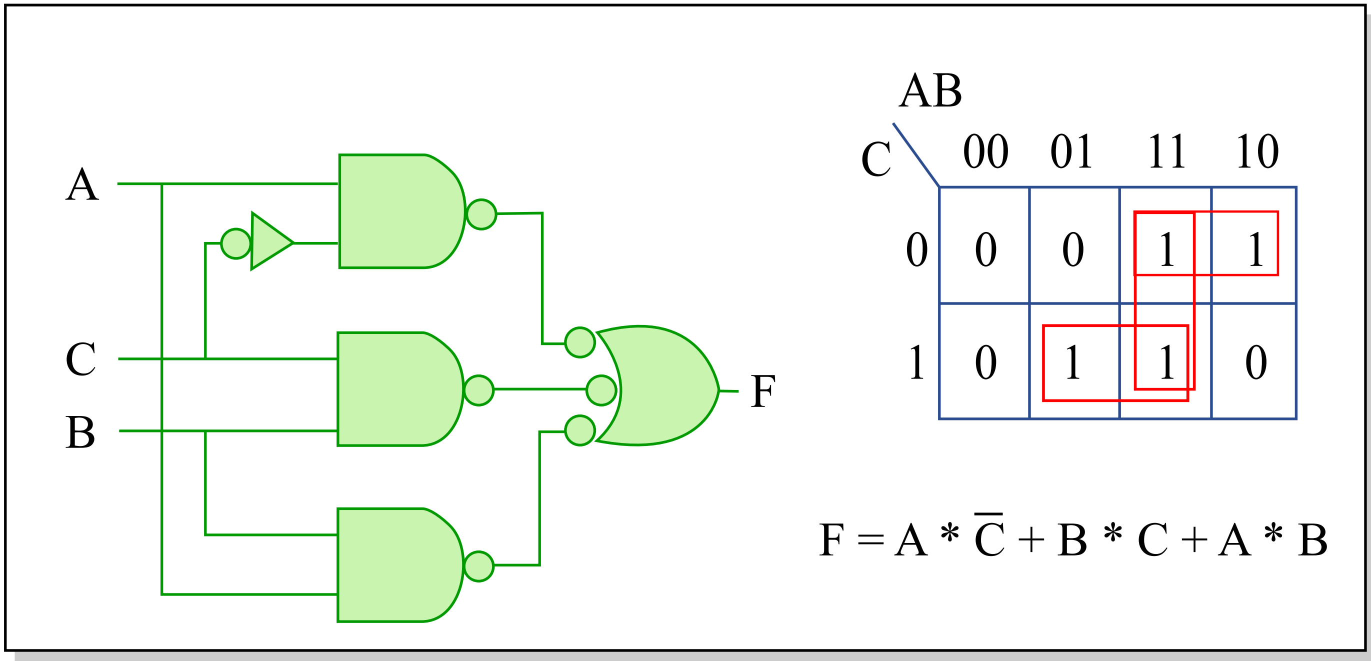


Figure by MIT OpenCourseWare.

- **In general, it is difficult to avoid hazards – need a robust design methodology to deal with hazards.**