Solutions to In-Class Problems Week 12, Mon.

The Four-Step Method

This is a good approach to questions of the form, "What is the probability that ——-?" Intuition *will* mislead you, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
   (a) Assign edge probabilities.
   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)

Problem 1. **[A Baseball Series]** The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. (In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends.) Assume that the Red Sox win each game with probability $3/5$, regardless of the outcomes of previous games.

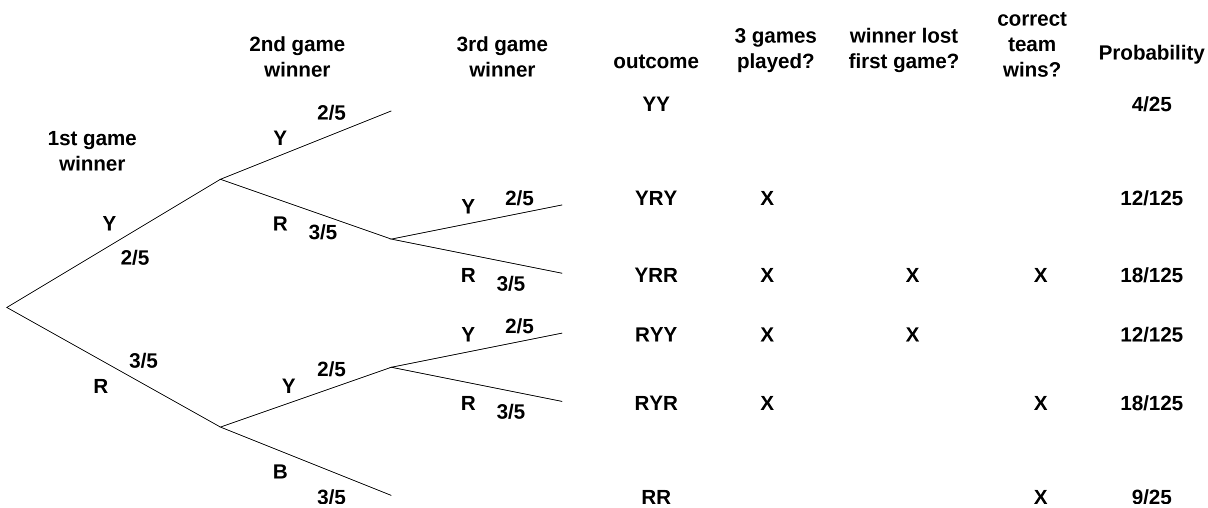
Answer the questions below using the four-step method. You can use the same tree diagram for all three problems.

(a) What is the probability that a total of 3 games are played?

(b) What is the probability that the winner of the series loses the first game?

(c) What is the probability that the *correct* team wins the series?

Solution. A tree diagram is worked out below.



From the tree diagram, we get:

$$\Pr\{3 \text{ games played}\} = \frac{12}{125} + \frac{18}{125} + \frac{12}{125} + \frac{18}{125} = \frac{12}{25}$$

$$\Pr\{\text{winner lost first game}\} = \frac{18}{125} + \frac{12}{125} = \frac{6}{25}$$

$$\Pr\{\text{correct team wins}\} = \frac{18}{125} + \frac{18}{125} + \frac{9}{25} = \frac{81}{125}$$

■

Problem 2. [The Four-Door Deal] Suppose that *Let's Make a Deal* is played according to different rules. Now there are **four** doors, with a prize hidden behind one of them. The contestant is allowed to pick a door. The host must then reveal a different door that has no prize behind it. The contestant is allowed to stay with his or her original door or to pick one of the other two that are still closed. If the contestant chooses the door concealing the prize in this second stage, then he or she wins.

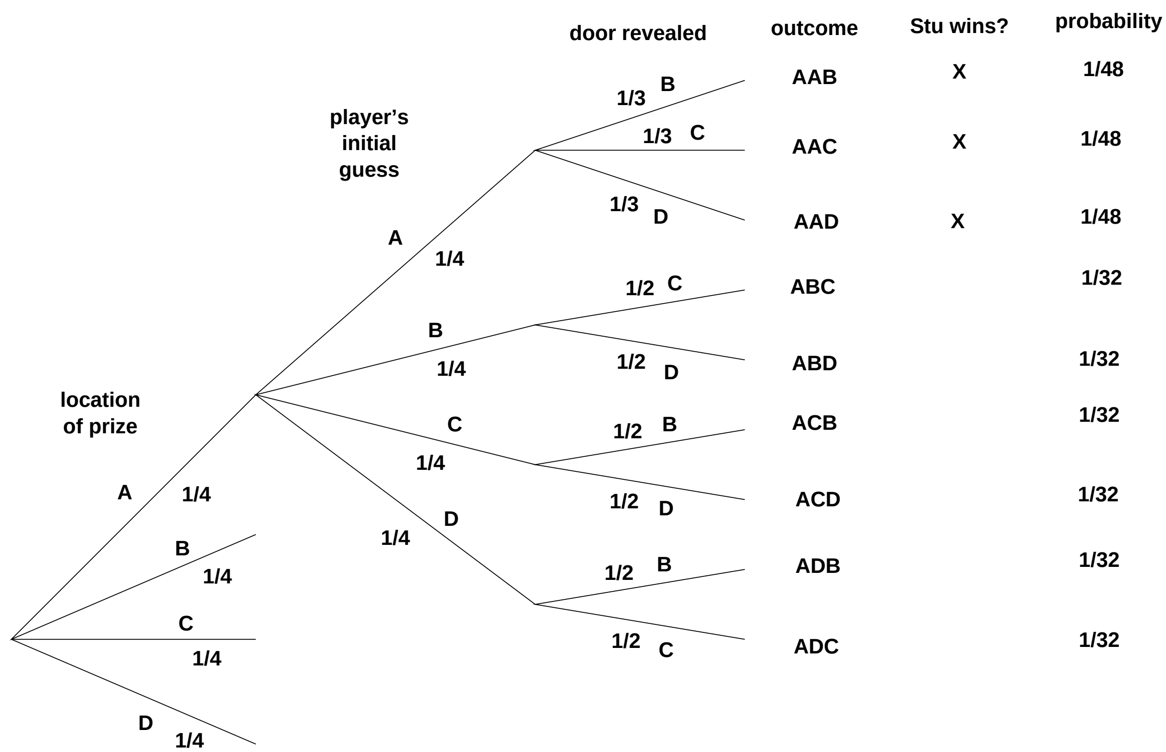
(a) Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that he wins the prize?

The tree diagram is awkwardly large. This often happens; in fact, sometimes you'll encounter *infinite* tree diagrams! Try to draw enough of the diagram so that you understand the structure of the remainder.

Solution. Let's make the same assumptions as in the original problem:

1. The prize is equally likely to be behind each door.
2. The contestant is equally likely to pick each door initially, regardless of the prize's location.
3. The host is equally likely to reveal each door that does not conceal the prize and was not selected by the player.

A partial tree diagram is shown below. The remaining subtrees are symmetric to the full-expanded subtree.



The probability that Stu wins the prize is:

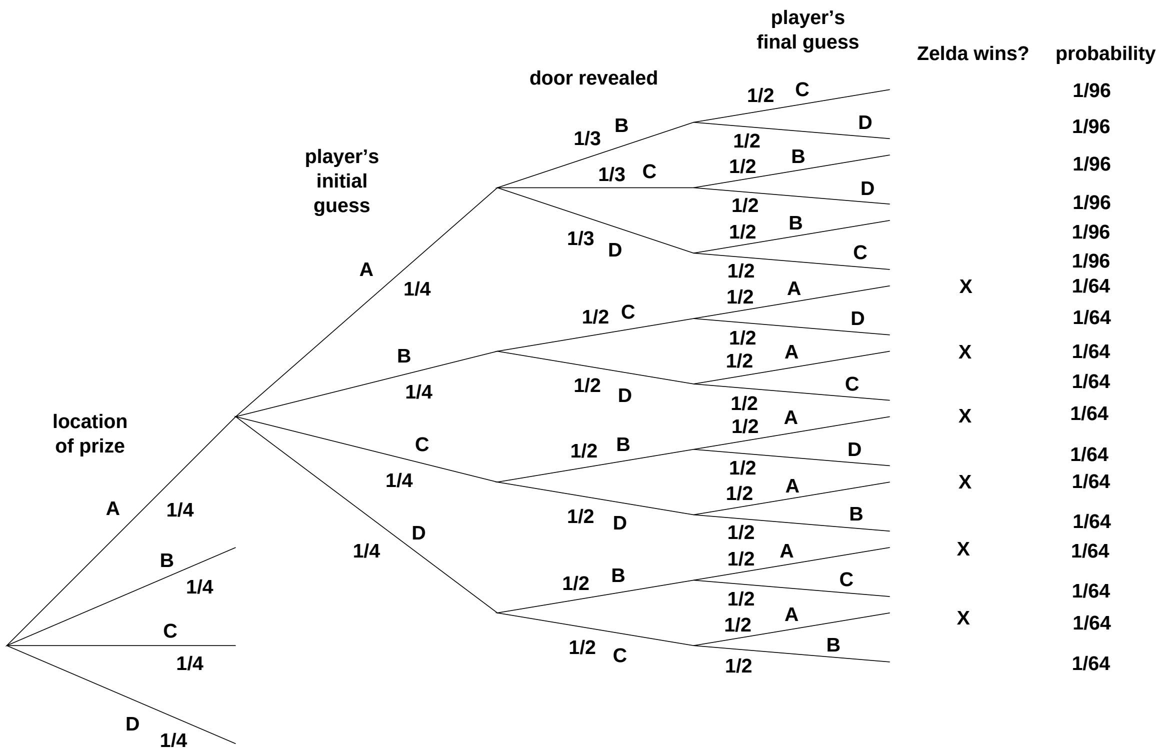
$$\Pr\{\text{Stu wins}\} = 4 \cdot \left(\frac{1}{48} + \frac{1}{48} + \frac{1}{48}\right) = \frac{1}{4}$$

We multiply by 4 to account for the four subtrees, of which we've only drawn one.

Notice that we expanded the tree out to the third ("door revealed") level to spell out the outcomes, but in this case we could, in fact, have stopped at the second level ("player's initial guess). This follows because the win/lose outcome is determined by the prize location and Stu's selected door, regardless of what happens after that. ■

(b) Contestant Zelda, an alien abduction researcher from Helena, Montana, switches to one of the remaining two doors with equal probability. What is the probability that she wins the prize?

Solution. A partial tree diagram is worked out below.



The probability that Zelda wins the prize is:

$$\Pr\{\text{Zelda wins}\} = 4 \cdot \left(\frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64} + \frac{1}{64}\right) = \frac{3}{8}$$

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Problem 3. [Simulating a fair coin] Suppose you need a fair coin to decide which door to choose in the 6.042 Monty Hall game. After making everyone in your group empty their pockets, all you managed to turn up is some crumpled bubble gum wrappers, a few used tissues, and one penny. However, the penny was from Prof. Rubinfeld's pocket, so it is **not** safe to assume that it is a fair coin.

How can we use a coin of unknown bias to get the same effect as a fair coin of bias $1/2$? Draw the tree diagram for your solution, but since it is infinite, draw only enough to see a pattern.

Suggestion: A neat trick allows you to sum all the outcome probabilities that cause you to say "Heads": Let s be the sum of all "Heads" outcome probabilities in the whole tree. Notice that *you can write the sum of all the "Heads" outcome probabilities in certain subtrees as a function of* s. Use this observation to write an equation in s and then solve.

Solution. Flip Prof. Rubinfeld's coin twice, if you see HT output Heads, if you see TH output Tails, and if you see HH or TT start over.

In the tree diagram in Figure 1, the small triangles represent subtrees that are themselves complete copies of the whole tree.

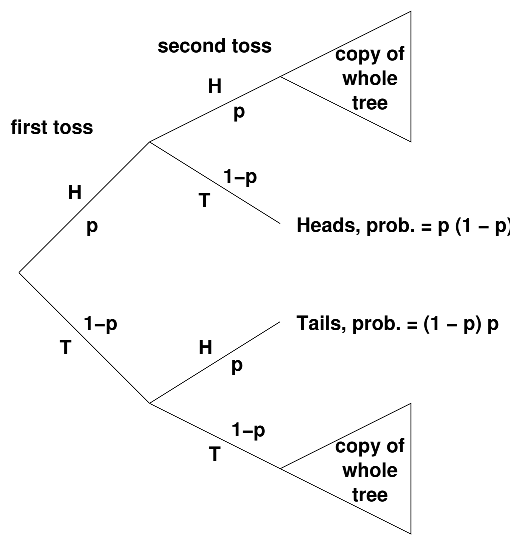

Figure 1: Simulating a Fair Coin.

Let s equal the sum of all "Heads" probabilities in the whole tree. There are two extra edges with probability p on the path to each outcome in the top subtree. Therefore, the sum of "Heads" probabilities in the upper tree is $p^2 s$. Similarly, the sum of "Heads" probabilities in the lower subtree is $(1-p)^2 s$. This gives the equation:

$$s = p^2 s + (1-p)^2 s + p(1-p)$$

The solution to this equation is $s = 1/2$, for all p between 0 and 1. ∎