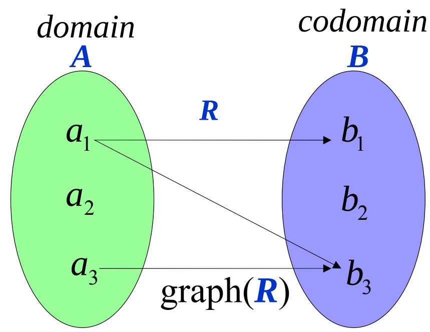
Mathematics for Computer Science

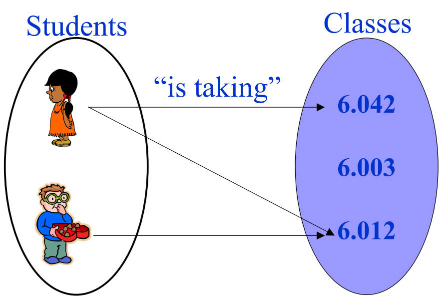
MIT 6.042J/18.062J

Relations

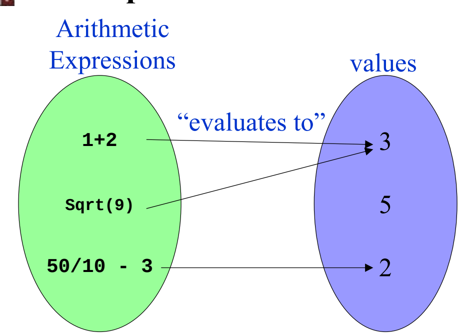
Binary relation *R* from *A* to *B*



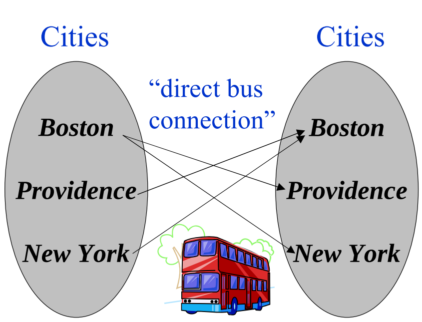
Example



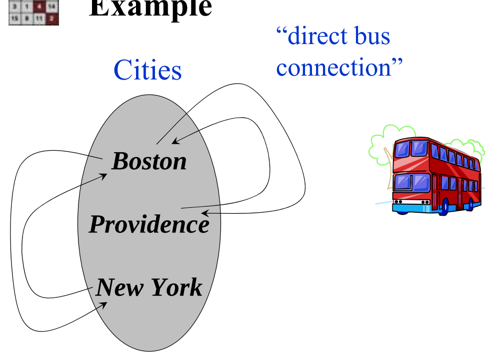
Example



Example

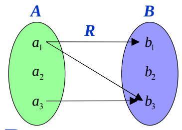


Example



Relation Abstraction

(Binary) Relation:
domain = set A
codomain = set B
graph = subset of $A \times B$

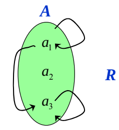


$\text{graph}(R) = \{\ (a_1,b_1),\ (a_1,b_3),\ (a_3,b_3)\ \}$

$A \times B = \{\ (a_1,b_1),\ (a_1,b_2),(a_1,b_3)$
$(a_2,b_1),\ (a_2,b_2),(a_2,b_3)$
$(a_3,b_1),\ (a_3,b_2),(a_3,b_3)\ \}$

Relation Abstraction

Relation on A:
domain = set A
codomain = set A



graph = $\{(a_1,a_1), (a_1,a_3),(a_3,a_3)\}$

Types of Binary Relations on A

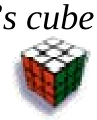
- **Equivalence**
- **Partial Orders**

Equivalence Relations

- Equivalence (mod 4):
  1 ≡ 5 (*same* remainder/4)
- Propositional equivalence:
  $\overline{P \wedge Q} \equiv \overline{P} \vee \overline{Q}$
  (*same* truth table)

Equivalence Relations

- Equivalent *code* (compilers):
  `x:=1;x:=x+1` ≡ `x:=2`?
  (*same* effect)
- *Rubik's cube* equivalence

 ≡ 

(*same* reachability group)

Def. of Equivalence on Set *A*

There is a function, f, on A such that

$$a\ R\ b \text{ iff } f(a) = f(b)$$

Equivalence Relations

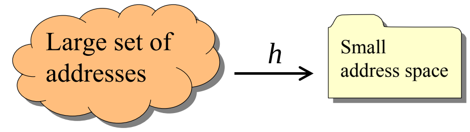
- Equivalence (mod 4):

  1 ≡ 5 (*same* remainder/4)

  $f(x) = x \bmod 4$

Hash Functions

How to map a large address space into a smaller address space?



So no collisions occur?

$\overbrace{h(\langle name1 \rangle) = h(\langle name2 \rangle)}$

Hash Collision Equivalence

$h(\langle name1 \rangle) = h(\langle name2 \rangle)$

Collides with is an equivalence relation

(on addresses in large space)

Athena Equivalence

Athena assigns user directories based on the first two letters of a username:

`rab` & `raej` in `r/a/`

Same User Directory

xterm

```
delhi:~$ ssh athena.dialup.mit.edu
Warning: Permanently added host key for IP address '18.7.16.69' to t
nown hosts.
~>pwd
/afs/athena.mit.edu/user/r/a/radhi
~>ls ..
index.html   raeburn/    raisa/      rallen63/   ranib/      rau
rab/         raej/       raisamg/    ralmeida/   ranthony/   rau
rabatin/     raf/        raj/        ralph/      ranying/    rau
rabbani/     rafaelm/    raj_ajay/   ralphons/   raoul/      rav
rabbib/      rafaeln/    raja/       ralphs/     rapa/       rav
rabi/        rafal/      rajappa/    rama/       raparker/   rav
rabi_u/      rafamit/    rajas/      ramamurt/   rapiwat/    rav
rabin/       rafel/      rajbansh/   raman/      raposo/     rav
rabino/      raffib/     rajdeep/    rameshs/    rappley/    rav
rabraff/     raffik/     rajeev/     rameson/    rapson/     rav
```

Athena Equivalence

- Names with

  same first 2 letters:

  Ben ≡ Betty

- *f(name)* = first two letters

Partitions

Theorem: An equivalence relation *partitions* its domain into collections of equivalent elements called

equivalence classes.

Athena Partition

- All names starting with "aa"
- All names starting with "ab"
- All names starting with "ac"

⋮

- All names starting with "zz"

26×26 *equivalence classes*

Some properties of relations:

Relation R on set A *is*

Reflexive:

if aRa for all $a \in A$.

Symmetric:

if $aRb \rightarrow bRa$ for all $a,b \in A$.

Transitive:

if $[aRb \wedge bRc] \rightarrow aRc$ for all $a,b,c \in A$.

Equivalence Relation Properties

Equivalence Relation R on set A *is*

Reflexive: aRa

Symmetric: $aRb \rightarrow bRa$

Transitive: $[aRb \wedge bRc] \rightarrow aRc$

Equivalence Relation Properties

Theorem:

R is an equivalence relation

iff it is

Reflexive, Symmetric,

& Transitive

Team Problems

Problems 1 & 2

Ordering Relations

- $\leq$ on the Integers
- $<$ on the Reals
- $\subseteq$ on Sets (subset)
- $\subset$ on Sets (*proper* subset)

Partial Orders

$y << x$ (*much* less than)

(say, $y + 2 \leq x$)

$\neg\,[3 << 4]$ $\neg\,[4 << 3]$

incomparable

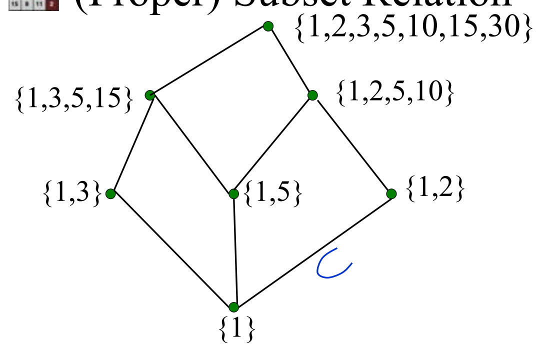
Partial Orders

The proper subset relation,

$\subset$,

on sets is the *canonical example.*

(Proper) Subset Relation

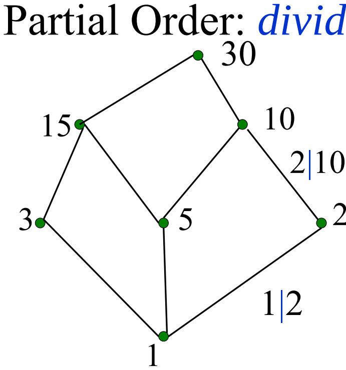


Partial Order: *divides*

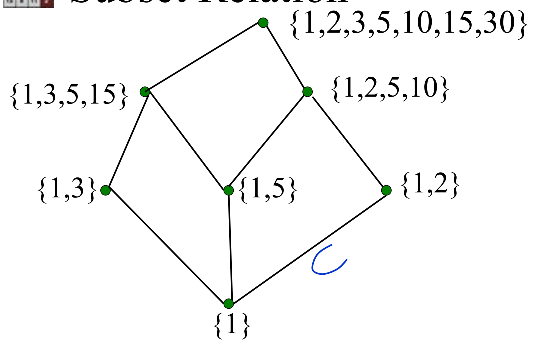
a *divides* b ($a \mid b$) iff

$ka = b$ for some $k \in \mathbb{N}$

Partial Order: *divides*



Subset Relation



Divides & Subset

same "shape"

Def. of Partial Order on Set A

There is a *set-valued* function, g, on A such that

$$a \; R \; b \text{ iff } g(a) \subset g(b)$$

$$\text{for } a \neq b$$

Divides & Subset

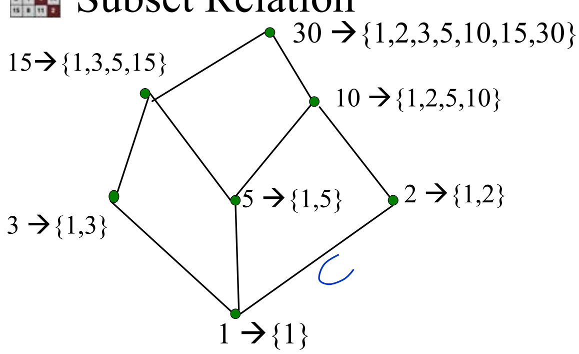
Let

$$g(n) ::= \text{divisors of } n$$

$$n \mid m \text{ iff } g(n) \subset g(m)$$

$$\text{for } n \neq m$$

Subset Relation



Properties of $\subset$

$[A \subset B$ and $B \subset C]$ implies $A \subset C$

Transitive

$A \subset B$ implies $\neg(B \subset A)$

for $A \neq B$

Antisymmetric

Axioms for Partial Order

Theorem: R is a partial order iff

Transitive & Antisymmetric

(Compare to Equivalence:

Reflexive, Transitive, ***Symmetric***.)

Total Order on A

Partial Order, R, such that

$$aRb \text{ or } bRa$$

for all $a \neq b \in A$

Total Orders

$$a < b \text{ or } b < a$$

(for numbers $a \neq b$)

Total Orders

$$a \leq b \text{ or } b \leq a$$

(for *all* a, b)

Team Problems

Problems 3 & 4