**SMA5212/16.920J/2.097J - Numerical Methods for Partial Differential Equations**

Massachusetts Institute of Technology — Singapore - MIT Alliance

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**Problem Set 2 - Hyperbolic Equations**

Handed out: March 10, 2003 — Due: March 31, 2003

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# Problem 1 - Solitons (50p)

## Problem Statement

J. Scott Russell wrote in 1844:

> *"I believe I shall best introduce this phenomenon by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel."*

In 1895, Korteweg and de Vries formulated the equation

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1}$$

which models Russell's observation. The term $uu_x$ describes the sharpening of the wave and $u_{xxx}$ the dispersion (i.e., waves with different wave lengths propagate with different velocities). The balance between these two terms allows for a propagating wave with unchanged form. The primary application of solitons today are in optical fibers, where the linear dispersion of the fiber provides smoothing of the wave, and the non-linear properties give the sharpening. The result is a very stable and long-lasting pulse that is free from dispersion, which is a problem with traditional optical communication techniques.

## Questions

**1)** (5p) Show using direct substitution that the one-soliton solution

$$u_1(x,t) = -\frac{v}{2\cosh^2\left(\frac{1}{2}\sqrt{v}(x - vt - x_0)\right)} \tag{2}$$

solves the KdV equation (1). Here, $v > 0$ and $x_0$ are arbitrary parameters.

**2)** (10p) We will solve the KdV equation numerically using the method of lines and finite difference approximations for the space derivatives. Rewrite the equation as

$$\frac{\partial u}{\partial t} = 6uu_x - u_{xxx}, \tag{3}$$

and derive a second-order accurate difference approximation of the right-hand side.

**3)** (15p) For the time integration, we will use a fourth order Runge-Kutta scheme:

$$\alpha^1 = \Delta t\, f(u^i) \tag{4}$$
$$\alpha^2 = \Delta t\, f(u^i + \alpha^1/2) \tag{5}$$
$$\alpha^3 = \Delta t\, f(u^i + \alpha^2/2) \tag{6}$$
$$\alpha^4 = \Delta t\, f(u^i + \alpha^3) \tag{7}$$
$$u^{i+1} = u^i + \frac{1}{6}(\alpha^1 + 2\alpha^2 + 2\alpha^3 + \alpha^4). \tag{8}$$

The stability region for this scheme consists of all $z$ such that $|1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}| \leq 1$. In particular, all points on the imaginary axis between $\pm i2\sqrt{2}$ are included.

Our equation (1) is non-linear, and to make a stability analysis we first have to linearize it. In this case, it turns out that the stability will be determined by the discretization of the third-derivative term $u_{xxx}$. Therefore, consider the simplified problem

$$\frac{\partial u}{\partial t} = -u_{xxx}, \tag{9}$$

and use von Neumann stability analysis to derive an expression for the maximum allowable time-step $\Delta t$ in terms of $\Delta x$.

**4)** (20p) Write a program that solves the equation using your discretization. Solve it in the region $-8 \leq x \leq 8$ with a grid size $\Delta x = 0.1$, and use periodic boundary conditions:

$$x(-8) = x(8). \tag{10}$$

Integrate from $t = 0$ to $t = 2$, using an appropriate time-step that satisfies the condition you derived above. For each of the initial conditions below, plot the solution at $t = 2$ and comment on the results.

a. To begin with, use a single soliton (2) as initial condition, that is, $u(x, 0) = u_1(x, 0)$. Set $v = 16$ and $x_0 = 0$.

b. The one-soliton solution looks almost like a Gaussian. Try $u(x, 0) = -8e^{-x^2}$.

c. Try the two-soliton solution $u(x, 0) = -6/\cosh^2(x)$.

d. Create "your own" two-soliton solution by superposing two one-soliton solutions with $v = 16$ and $v = 4$ (both with $x_0 = 0$).

e. Same as before, but with $v = 16, x_0 = 4$ and $v = 4, x_0 = -4$. Describe what happens when the two solitons cross (amplitudes, velocities), and after they have crossed.

# Problem 2 - Traffic Flow (50p)

## Problem Statement

Consider the traffic flow problem, described by the non-linear hyperbolic equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0 \tag{11}$$

with $\rho = \rho(x, t)$ the density of cars (vehicles/km), and $u = u(x, t)$ the velocity. Assume that the velocity $u$ is given as a function of $\rho$:

$$u = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{12}$$

With $u_{\max}$ the maximum speed and $0 \leq \rho \leq \rho_{\max}$. The flux of cars is therefore given by:

$$f(\rho) = \rho u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{13}$$

We will solve this problem using a first order finite volume scheme:

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}\right). \tag{14}$$

For the numerical flux function, we will consider two different schemes:

### a) Roe's Scheme

The expression of the numerical flux is given by:

$$F^R_{i+\frac{1}{2}} = \frac{1}{2}\left[f(\rho_i) + f(\rho_{i+1})\right] - \frac{1}{2}\left|a_{i+\frac{1}{2}}\right|(\rho_{i+1} - \rho_i) \tag{15}$$

with

$$a_{i+\frac{1}{2}} = u_{\max}\left(1 - \frac{\rho_i + \rho_{i+1}}{\rho_{\max}}\right). \tag{16}$$

Note that $a_{i+\frac{1}{2}}$ satisfies

$$f(\rho_{i+1}) - f(\rho_i) = a_{i+\frac{1}{2}}(\rho_{i+1} - \rho_i). \tag{17}$$

### b) Godunov's Scheme

In this case the numerical flux is given by:

$$F^G_{i+\frac{1}{2}} = f\left(\rho\left(x_{i+\frac{1}{2}}, t^{n+}\right)\right) = \begin{cases} \min_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i < \rho_{i+1} \\ \max_{\rho \in [\rho_i, \rho_{i+1}]} f(\rho), & \rho_i > \rho_{i+1}. \end{cases} \tag{18}$$

## Questions

**1)** (25p) For both Roe's Scheme and Godunov's Scheme, look at the problem of a traffic light turning green at time $t = 0$. We are interested in the solution at $t = 2$ using both schemes. What do you observe for each of the schemes? Explain briefly why the behavior you get arises.

Use the following problem parameters:

$$\begin{aligned} \rho_{\max} &= 1.0, \quad \rho_L = 0.8 \\ u_{\max} &= 1.0 \\ \Delta x &= \frac{4}{400}, \quad \Delta t = \frac{0.8\Delta x}{u_{\max}} \end{aligned} \tag{19}$$

The initial condition at the instant when the traffic light turns green is

$$\rho(0) = \begin{cases} \rho_L, & x < 0 \\ 0, & x \geq 0 \end{cases} \tag{20}$$

**For the rest of this problem use only the scheme(s) which are valid models of the problem.**

**2)** (25p) Simulate the effect of a traffic light at $x = -\frac{\Delta x}{2}$ which has a period of $T = T_1 + T_2 = 2$ units. Assume that the traffic light is $T_1 = 1$ units on red and $T_2 = 1$ units on green. Assume a sufficiently high flow density of cars (e.g. set $\rho = \frac{\rho_{\max}}{2}$ on the left boundary – giving a maximum flux), and determine the average flow, or capacity of cars over a time period $T$.

The average flow can be approximated as

$$\dot{q} = \frac{1}{N_T} \sum_{n=1}^{N_T} f^n = \frac{1}{N^T} \sum_{n=1}^{N_T} \rho^n u^n, \tag{21}$$

where $N_T$ is the number of time steps for each period $T$. You should run your computation until $\dot{q}$ over a time period does not change. Note that by continuity $\dot{q}$ can be evaluated over any point in the interior of the domain (in order to avoid boundary condition effects, we consider only those points on the interior domain).

**Note:** A red traffic light can be modeled by simply setting $F_{i+\frac{1}{2}} = 0$ at the position where the traffic light is located.

**3) BONUS**: (Possible +10p[1]) Assume now that we simulate two traffic lights, one located at $x = 0$, and the other at $x = 0.15$, both with a period $T$. Calculate the road capacity (= average flow) for different delay factors. That is if the first light turns green at time $t$, then the second light will turn green at $t + \tau$. Solve for $\tau = k\frac{T}{10}$, $k = 0, \ldots, 9$. Plot your results of capacity vs $\tau$ and determine the optimal delay $\tau$.

[1] Only applied to gain a maximum of 100%. Additional bonus points are not carried over to future assignments.