Numerical Methods for PDEs

Integral Equation Methods, Lecture 4
Formulating Boundary Integral Equations

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April 30, 2003

Outline

Laplace Problems

- Exterior Radiation Condition
- Green's function

Ansatz or Indirect Approach

- Single and Double Layer Potentials
- First and Second Kind Equations

Greens Theorem Approach

- First and Second Kind Equations

3-D Laplace Problems

Differential Equation

Laplace's equation in 3-D

$$\nabla^2 u(\vec{x}) = \frac{\partial^2 u(\vec{x})}{\partial x^2} + \frac{\partial^2 u(\vec{x})}{\partial y^2} + \frac{\partial^2 u(\vec{x})}{\partial z^2} = 0$$

where

$$\vec{x} = x, y, z \in \Omega$$

and Ω is bounded by Γ.

3-D Laplace Problems

Boundary Conditions

Dirichlet Condition

$$u(\vec{x}) = u_{\Gamma}(\vec{x}) \quad \vec{x} \in \Gamma$$

OR

Neumann Condition

$$\frac{\partial u(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial u_{\Gamma}(\vec{x})}{\partial n_{\vec{x}}} \quad \vec{x} \in \Gamma$$

PLUS

A Radiation Condition

3-D Laplace Problems

Boundary Conditions

Radiation Condition

The Radiation Condition

$$lim_{\|\vec{x}\|\rightarrow\infty} u(\vec{x}) \rightarrow 0$$

not specific enough! Need

$$lim_{\|\vec{x}\|\rightarrow\infty} u(\vec{x}) \rightarrow O(\|\vec{x}\|^{-1})$$

OR

$$lim_{\|\vec{x}\|\rightarrow\infty} u(\vec{x}) \rightarrow O(\|\vec{x}\|^{-2})$$

3-D Laplace Problems

Greens Function

Laplace's Equation Greens Function

$$\nabla^2 G(\vec{x}) = 4\pi\delta(\vec{x})$$

$\delta(\vec{x}) \equiv$ impulse in 3-D

Defined by its behavior in an integral

$$\int \delta(\vec{x}')f(\vec{x}')d\Omega' = f(0)$$

Not too hard to show

$$G(\vec{x}) = \frac{1}{||\vec{x}||}$$

Ansatz (Indirect) Formulations

Single Layer Potential

Consider

$$u(\vec{x}) = \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

$u(\vec{x})$ automatically satisfies $\nabla^2 u = 0$ on Ω.

Must now enforce boundary conditions

Ansatz (Indirect) Formulations

Single Layer Potential

Boundary Conditions

Dirichlet Problem

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Neumann Problem

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial}{\partial n_{\vec{x}}} \int_\Gamma \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Ansatz (Indirect) Formulations

Single Layer Potential

Care Evaluating Integrals

On a smooth surface:

$$\lim_{x \to \Gamma} \frac{\partial}{\partial n_{\vec{x}}} \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

$$= 2\pi\sigma(\vec{x}') + \int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

Ansatz (Indirect) Formulations

Single Layer Potential

Neumann Problem 2nd Kind!

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}') + \int_\Gamma \frac{\partial}{\partial n_{\vec{x}}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}')d\Gamma'$$

Ansatz (Indirect) Formulations

Single Layer Potential

Radiation Condition

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) = \int_{\Gamma} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \to O(\|\vec{x}\|^{-1})$$

Unless

$$\int_{\Gamma} \sigma(\vec{x}') d\Gamma' = 0$$

Then

$$lim_{\|\vec{x}\| \to \infty} u(\vec{x}) \to O(\|\vec{x}\|^{-2})$$

Ansatz (Indirect) Formulations

Double Layer Potential

Consider

$$u(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \mu(\vec{x}') d\Gamma'$$

$u(\vec{x})$ automatically satisfies $\nabla^2 u = 0$ on Ω.

Must now enforce boundary conditions

Ansatz (Indirect) Formulations

Double Layer Potential

Boundary Conditions

Dirichlet Problem

$$u_\Gamma(\vec{x}) = \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Neumann Problem

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = \frac{\partial}{\partial n_{\vec{x}}} \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma' \quad \vec{x} \in \Gamma$$

Neumann Problem generates Hypersingular Integral

Ansatz (Indirect) Formulations

Double Layer Potential

Dirichlet Problem 2nd Kind!

$$\frac{\partial u_\Gamma(\vec{x})}{\partial n_{\vec{x}}} = 2\pi\sigma(\vec{x}') + \int_\Gamma \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x} - \vec{x}'\|} \sigma(\vec{x}') d\Gamma'$$

Ansatz (Indirect) Formulations

Double Layer Potential

Radiation Condition

$$lim_{\|\vec{x}\|\rightarrow\infty} u(\vec{x}) = \int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma' \rightarrow O(\|\vec{x}\|^{-2})$$

Add Extra Term to slow decay

$$u(\vec{x}) = \int_{\Gamma} \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{\|\vec{x}-\vec{x}'\|} \sigma(\vec{x}') d\Gamma' + \alpha G(\vec{x}^{*}) \quad \vec{x}^{*} \ni \Omega$$

Green's Theorem Approach

Green's Second Identity

$$\int_{\Omega} \left[u \nabla^2 w - w \nabla^2 u\right] d\Omega = \int_{\Gamma} \left[w \frac{\partial u}{\partial n} - u \frac{\partial w}{\partial n} d\Gamma\right]$$

Now let $w = \frac{1}{||\vec{x} - \vec{x}'||}$

$$2\pi u(\vec{x}) = \int_{\Gamma} \left[\frac{1}{||\vec{x} - \vec{x}'||} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n_{\vec{x}'}} \frac{1}{||\vec{x} - \vec{x}'||} d\Gamma\right]$$

Easy to implement any boundary conditions!