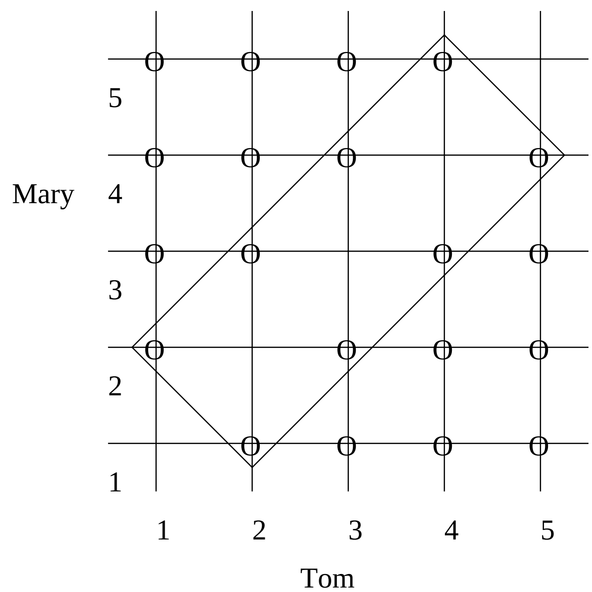
Problem Set 3: Solutions
Due: March 1, 2006

1. The problem did not explicitly state that two cars cannot share a parking space, but it was expected that you would assume this when doing the required counting.

   The figure below depicts the full outcome space for the case of $N = 5$. The 8 outcomes in the box (out of the total of 20 outcomes) are those for which Mary and Tom are parked adjacently.



   Extending this idea to a parking lot with N spaces, the desired probability is given by

$$\begin{aligned}\mathbf{P}(\text{parked adjacently}) &= \frac{\text{number of outcomes with adjacent parking}}{\text{total number of outcomes}} \\ &= \frac{2(N-1)}{N^2 - N} \\ &= \frac{2}{N}.\end{aligned}$$

2. (a) There are nine equally-likely ordered pairs (i, j), $i \in \{1, 2, 3\}$, $j \in \{1, 2, 3\}$. By looking at the five possible sums and their frequencies, we obtain

$$p_X(k) = \begin{cases} 1/9, & k = 1; \\ 2/9, & k = 2; \\ 3/9, & k = 3; \\ 2/9, & k = 4; \\ 1/9, & k = 5; \\ 0, & \text{otherwise.} \end{cases}$$

   (b) The fair price is $\mathbf{E}[5X]$ because then the net expected result is $\mathbf{E}[5X - a] = 0$.

$$\mathbf{E}[5X] = \frac{1}{9} \cdot 5 + \frac{2}{9} \cdot 10 + \frac{3}{9} \cdot 15 + \frac{2}{9} \cdot 20 + \frac{1}{9} \cdot 25 = 15$$

(c) The possible values for X are changed, but the probabilities are unchanged:

$$p_X(k) = \begin{cases} 1/9, & k = 1; \\ 2/9, & k = 4; \\ 3/9, & k = 9; \\ 2/9, & k = 16; \\ 1/9, & k = 25; \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{E}[5X] = \frac{1}{9} \cdot 5 + \frac{2}{9} \cdot 20 + \frac{3}{9} \cdot 45 + \frac{2}{9} \cdot 80 + \frac{1}{9} \cdot 125 = \frac{155}{3}$$

3. Denote the die rolls by W and Z. The sixteen equally-likely (W, Z) ordered pairs are depicted below, where the label in each cell is the (X, Y) pair.

	$Z = 1$	$Z = 2$	$Z = 3$	$Z = 4$
$W = 1$	(0,1)	(1,1)	(2,1)	(3,1)
$W = 2$	(1,1)	(1,2)	(2,2)	(3,2)
$W = 3$	(2,1)	(2,2)	(2,3)	(3,3)
$W = 4$	(3,1)	(3,2)	(3,3)	(3,4)

(a) From the table, we can read off the PMFs

$$p_X(k) = \begin{cases} 1/16, & k = 0; \\ 3/16, & k = 1; \\ 5/16, & k = 2; \\ 7/16, & k = 3; \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad p_Y(k) = \begin{cases} 7/16, & k = 1; \\ 5/16, & k = 2; \\ 3/16, & k = 3; \\ 1/16, & k = 4; \\ 0, & \text{otherwise,} \end{cases}$$

and thus compute the expectations

$$\mathbf{E}[X] = \frac{1}{16} \cdot 0 + \frac{3}{16} \cdot 1 + \frac{5}{16} \cdot 2 + \frac{7}{16} \cdot 3 = \frac{17}{8}$$

and

$$\mathbf{E}[Y] = \frac{7}{16} \cdot 1 + \frac{5}{16} \cdot 2 + \frac{3}{16} \cdot 3 + \frac{1}{16} \cdot 4 = \frac{15}{8}.$$

We get by linearity of the expectation that $\mathbf{E}[X - Y] = \mathbf{E}[X] - \mathbf{E}[Y] = \frac{1}{4}$.

(b) Using the PMFs in part (a), we can compute

$$\mathbf{E}[X^2] = \frac{1}{16} \cdot 0^2 + \frac{3}{16} \cdot 1^2 + \frac{5}{16} \cdot 2^2 + \frac{7}{16} \cdot 3^2 = \frac{43}{8}$$

and

$$\mathbf{E}[Y^2] = \frac{7}{16} \cdot 1^2 + \frac{5}{16} \cdot 2^2 + \frac{3}{16} \cdot 3^2 + \frac{1}{16} \cdot 4^2 = 30.$$

Thus, $\text{var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{55}{64}$ and $\text{var}(Y) = \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 = \frac{1695}{64}$.

Since X and Y are *not* independent, the variance of X and Y is not any simple combination of previous results. Instead, let $Z = X - Y$ and find the PMF of Z as

$$p_Z(k) = \begin{cases} 4/16, & k = -1; \\ 6/16, & k = 0; \\ 4/16, & k = 1; \\ 2/16, & k = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Now

$$\mathbf{E}[Z^2] = \frac{4}{16} \cdot (-1)^2 + \frac{6}{16} \cdot 0^2 + \frac{4}{16} \cdot 1^2 + \frac{2}{16} \cdot 2^2 = 1,$$

and $\text{var}(Z) = \mathbf{E}[Z^2] - (\mathbf{E}[Z])^2 = 1 - (1/4)^2 = \frac{15}{16}$. ($\mathbf{E}[Z]$ was computed in part (a) and can also be double-checked with the PMF above.)

We will use the formula

$$\text{var}(Y) = \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2$$

for the variance of a random variable Y. Let $Y = (X - \hat{x})$. Then

$$e(\hat{x}) = \mathbf{E}[(X - \hat{x})^2] = \text{var}(X - \hat{x}) + (\mathbf{E}[X - \hat{x}])^2 = \text{var}(X) + (\mathbf{E}[X] - \hat{x})^2,$$

where the last equality follows from the fact that shifting a random variable by a constant (in this case $\hat{x}$) does not change its variance. Since the first term is not dependent on $\hat{x}$ and the second is always nonnegative, we see that this expression is minimized when $\mathbf{E}[X] - \hat{x} = 0$. This is equivalent to the desired result of $\hat{x} = \mathbf{E}[X]$.

5. (a) From the joint PMF, there are six (x, y) coordinate pairs with nonzero probabilities of occurring. These pairs are (1, 1), (1, 3), (2, 1), (2, 3), (4, 1), and (4, 3). The probability of a pair is proportional to the product of the x and y coordinate of the pair. Because the probability of the entire sample space must equal 1, we have:

$$(1 \cdot 1)c + (1 \cdot 3)c + (2 \cdot 1)c + (2 \cdot 3)c + (4 \cdot 1)c + (4 \cdot 3)c = 1.$$

Solving for c, we get $c = \boxed{\frac{1}{28}}$

(b) There are three sample points for which $Y < X$.

$$\mathbf{P}(Y < X) = \mathbf{P}(\{(2,1)\}) + \mathbf{P}(\{(4,1)\}) + \mathbf{P}(\{(4,3)\}) = \frac{2 \cdot 1}{28} + \frac{4 \cdot 1}{28} + \frac{4 \cdot 3}{28} = \boxed{\frac{18}{28}}$$

(c) There are two sample points for which $Y > X$.

$$\mathbf{P}(Y > X) = \mathbf{P}(\{(1,3)\}) + \mathbf{P}(\{(2,3)\}) = \frac{1 \cdot 3}{28} + \frac{2 \cdot 3}{28} = \boxed{\frac{9}{28}}$$

(d) There is only one sample point for which $Y = X$.

$$\mathbf{P}(Y = X) = \mathbf{P}(\{(1,1)\}) = \frac{1 \cdot 1}{28} = \boxed{\frac{1}{28}}$$

Notice that, using the above two parts:

$$\mathbf{P}(Y < X) + \mathbf{P}(Y > X) + \mathbf{P}(Y = X) = \frac{18}{28} + \frac{9}{28} + \frac{1}{28} = 1$$

as expected.

(e) There are three sample points for which $y = 3$.

$$\mathbf{P}(Y = 3) = \mathbf{P}(\{(1,3)\}) + \mathbf{P}(\{(2,3)\}) + \mathbf{P}(\{(4,3)\}) = \frac{3}{28} + \frac{6}{28} + \frac{12}{28} = \boxed{\frac{21}{28}}$$

(f) In general, for two discrete random variables X and Y for which a joint PMF is defined, we have

$$p_X(x) = \sum_{y=-\infty}^{\infty} p_{X,Y}(x,y) \qquad \text{and} \qquad p_Y(y) = \sum_{x=-\infty}^{\infty} p_{X,Y}(x,y).$$

In this problem the number of possible (X, Y) pairs is quite small, so we can determine the marginal PMFs by enumeration. For example,

$$p_X(2) = \mathbf{P}(\{(2,1)\}) + \mathbf{P}(\{(2,3)\}) = \frac{8}{28}.$$

Overall, we get:

$$p_X(x) = \begin{cases} 4/28, & x = 1; \\ 8/28, & x = 2; \\ 16/28, & x = 4; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/7, & x = 1; \\ 2/7, & x = 2; \\ 4/7, & x = 4; \\ 0, & \text{otherwise} \end{cases}$$

and

$$p_Y(y) = \begin{cases} 7/28, & y = 1; \\ 21/28, & y = 3; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1/4, & y = 1; \\ 3/4, & y = 3; \\ 0, & \text{otherwise.} \end{cases}$$

(g) In general, the expected value of any discrete random variable X is given by

$$\mathbf{E}[X] = \sum_{x=-\infty}^{\infty} x p_X(x).$$

For this problem,

$$\mathbf{E}[X] = 1 \cdot \frac{1}{7} + 2 \cdot \frac{2}{7} + 4 \cdot \frac{4}{7} = \boxed{3}$$

and

$$\mathbf{E}[Y] = 1 \cdot \frac{1}{4} + 3 \cdot \frac{3}{4} = \boxed{\frac{5}{2}}$$

(h) The variance of a random variable X can be computed as $\mathbf{E}[X^2] - \mathbf{E}[X]^2$ or as $\mathbf{E}[(X - \mathbf{E}[X])^2]$. Here we use the second approach.

$$\text{var}(X) = (1-3)^2 \cdot \frac{1}{7} + (2-3)^2 \cdot \frac{2}{7} + (4-3)^2 \cdot \frac{4}{7} = \boxed{\frac{10}{7}}$$

$$\text{var}(Y) = \left(1 - \frac{5}{2}\right)^2 \frac{1}{4} + \left(3 - \frac{5}{2}\right)^2 \frac{3}{4} = \frac{9}{16} + \frac{1}{16} = \boxed{\frac{5}{8}}$$

G1†. Starting with the hint, we have

$$\mathbf{E}[(\alpha X + Y)^2] \geq 0,$$

which can be expanded to

$$\alpha^2 \mathbf{E}[X^2] + 2\alpha \mathbf{E}[XY] + \mathbf{E}[Y^2] \geq 0.$$

The lack of real solutions α to

$$\alpha^2 \mathbf{E}[X^2] + 2\alpha \mathbf{E}[XY] + \mathbf{E}[Y^2] = \beta$$

for any $\beta < 0$ implies that the discriminant of the above quadratic, $(2\mathbf{E}[XY])^2 - 4\mathbf{E}[X^2]\mathbf{E}[Y^2]$, must be nonpositive. Rearranging

$$(2\mathbf{E}[XY])^2 - 4\mathbf{E}[X^2]\mathbf{E}[Y^2] \leq 0$$

gives the desired result.

†Required for 6.431; optional for 6.041