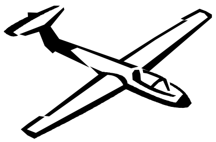
LECTURE 2

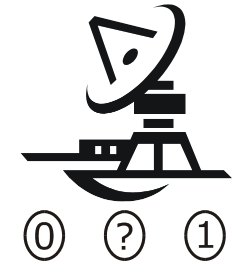
- Readings: Sections 1.3, 1.4

Lecture outline

- Review
- Conditional Probability
- Three important tools:
  - Total probability theorem
  - Bayes' rule
  - Multiplication rule



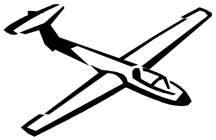
Example 0: Radar



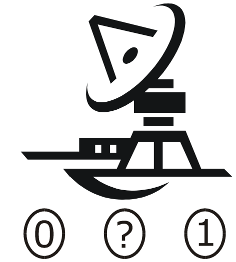
- Radar device, with 3 readings:
  - Low (0), Medium (?), High (1)

- Probabilistic Modeling:
  - Sample Space / Outcomes:
    - Airplane Presence + Radar Reading
  - Probability Law:

Airplane \ Radar	Low(0)	Medium(?)	High(1)
Absent	0.45	0.20	0.05
Present	0.02	0.08	0.20



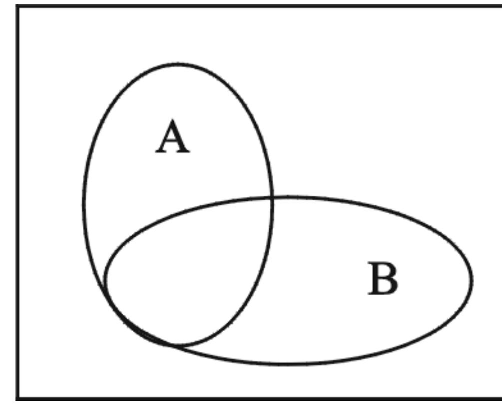
Example 0: Radar
(continued)



- Questions:
  - What is the probability that the radar reads a medium level (?) if there are no airplanes?
  - What is the probability of having an airplane?
  - What is the probability of the airplane being there if the radar reads low (0)?
  - When should we decide there is an airplane, and when should we decide there is none?

Airplane \ Radar	Low(0)	Medium(?)	High(1)
Absent	0.45	0.20	0.05
Present	0.02	0.08	0.20

Conditional Probability



- $\mathrm{P}(A|B)$ = probability of *A* given that *B* occurred.
  - *B* becomes our universe

- **Definition**: Assuming $\mathrm{P}(B) \neq 0$, we have:

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

- **Consequences**: If $\mathrm{P}(A) \neq 0$, $\mathrm{P}(B) \neq 0$ then

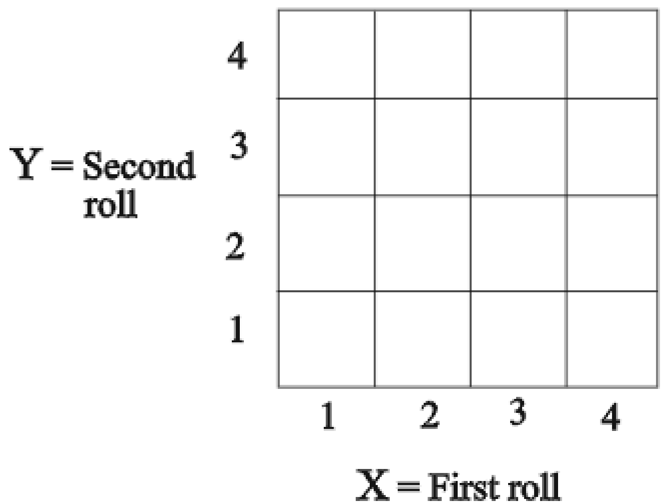
$$\mathrm{P}(A \cap B) = \mathrm{P}(B) \cdot \mathrm{P}(A \mid B) = \mathrm{P}(A) \cdot \mathrm{P}(B \mid A)$$

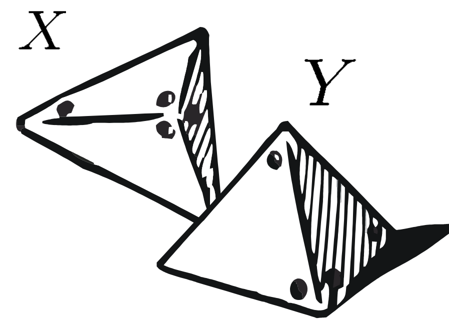
Example 0: Radar
(continued)

Airplane \ Radar	Low(0)	Medium(?)	High(1)
Absent	0.45	0.20	0.05
Present	0.02	0.08	0.20

- Event “Present” = Plane is present.
- P(Medium|Present) =

Example 1: Die Roll
(Modeled in Lecture 1 using joint probability law)





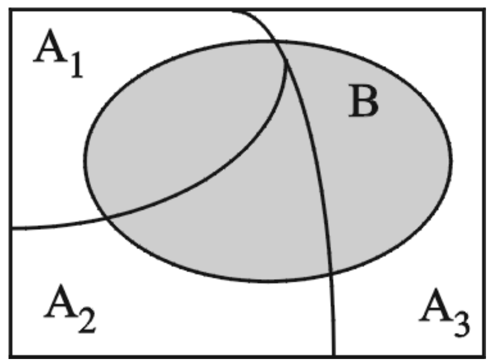
- Let B be the event: min(X, Y) = 2
- Let M = max(X, Y)

$$\mathbf{P}(M = 1|B) =$$

$$\mathbf{P}(M = 2|B) =$$

Total Probability Theorem

- Divide and conquer.
- Partition of sample space into $A_1,\ A_2,\ and\ A_3.$



- One way of computing $\mathbf{P}(B)$:

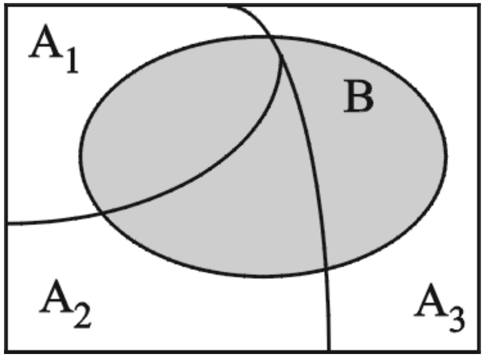
$$\begin{aligned}\mathbf{P}(B) = & \quad \mathbf{P}(A_1)\mathbf{P}(B|A_1) \\ & + \mathbf{P}(A_2)\mathbf{P}(B|A_2) \\ & + \mathbf{P}(A_3)\mathbf{P}(B|A_3)\end{aligned}$$

Radar Example: $\mathbf{P}(\text{Present}) =$

Bayes' Rule

- Rules for combining evidence ("inference").
- We have "prior" probabilities: $\mathbf{P}(A_i)$
- For each i, we know: $\mathbf{P}(B|A_i)$
- We wish to compute: $\mathbf{P}(A_i|B)$

$$\mathbf{P}(A_i \mid B) = \frac{\mathbf{P}(A_i \cap B)}{P(B)} = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A_i)\mathbf{P}(B \mid A_i)}{\sum_j \mathbf{P}(A_j)\mathbf{P}(B \mid A_j)}$$

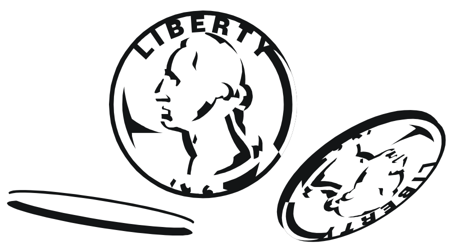


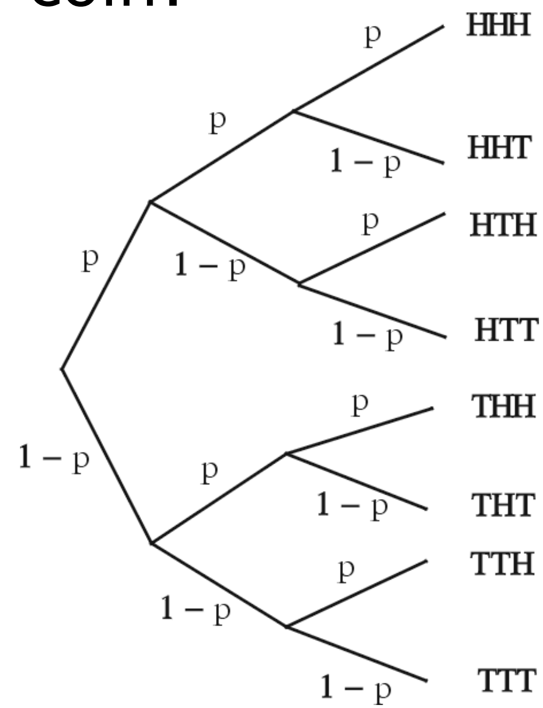
Radar Example: $\mathbf{P}(\text{Present}|\text{Low}) =$

Example 2: Coin Tosses
(Modeled using conditional probabilities)

- Look at 3 tosses of a biased coin:

  $\mathbf{P}(H) = p$, $\mathbf{P}(T) = 1 - p$

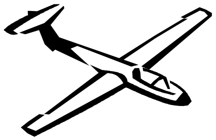




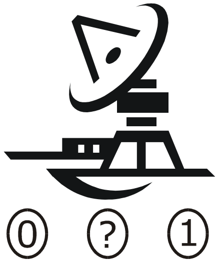
$\mathbf{P}(THT) =$

$\mathbf{P}(1 \text{ head}) =$

$\mathbf{P}(\text{first toss is } H \mid 1 \text{ head}) =$

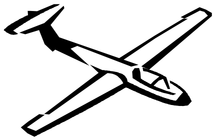


Example 0: Decision Rule



- Given the radar reading, what is the best decision about the plane?
- Criterion for decision:
  - Minimize “Probability of Error”
- Decision rules:
  - Decide absent or present for each reading.
- What is the optimal decision region?

Radar / Airplane	Low(0)	Medium(?)	High(1)
Absent	0.45	0.20	0.05
Present	0.02	0.08	0.20



Example 0: Decision Rule

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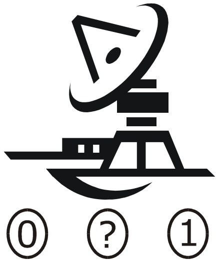
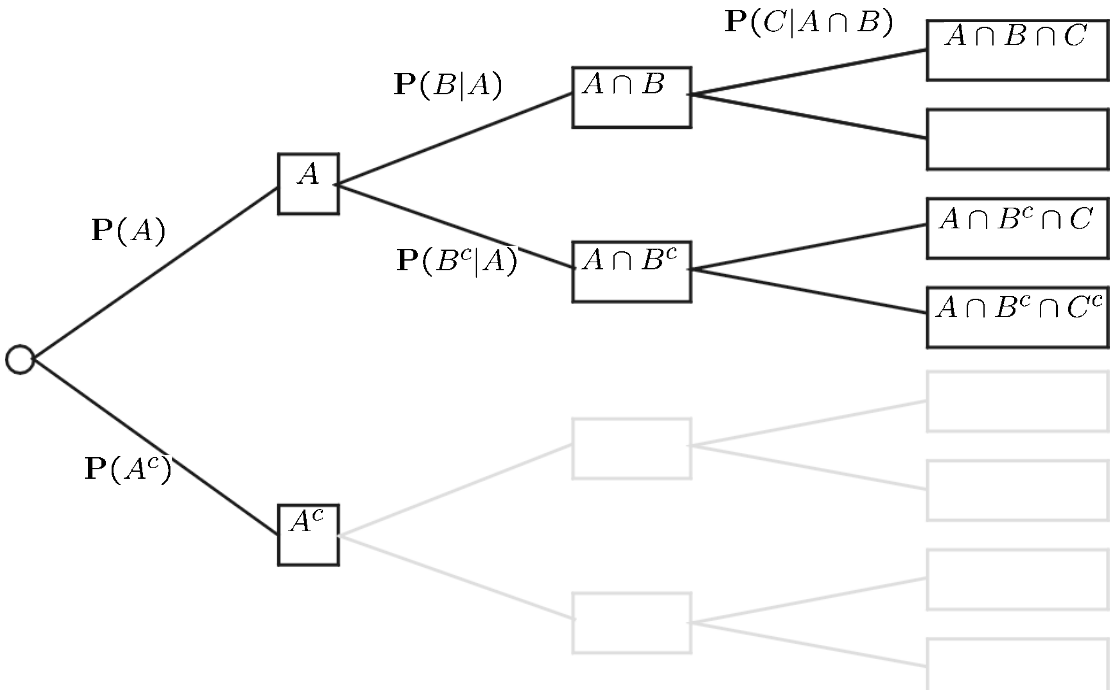


- P(Error)=?
- Error={Present and decision is absent} or {Absent and decision is present}
- Disjoint event!
- P(Error)=

Radar / Airplane	Low(0)	Medium(?)	High(1)
Absent	0.45	0.20	0.05
Present	0.02	0.08	0.20

Multiplication Rule

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\mathbf{P}(B|A)\mathbf{P}(C|A \cap B)$$



Example 3: Three cards are drawn from a 52-card deck. What's the probability that none of these cards is a heart?

Let A_i = i^{th} card not a heart. Then:

$$\mathbf{P}(A_1 \cap A_2 \cap A_3) = \mathbf{P}(A_1)\mathbf{P}(A_2|A_1)\mathbf{P}(A_3|A_1 \cap A_2)$$