MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Departments of Electrical Engineering, Mechanical Engineering, and the Harvard-MIT Division of Health Sciences and Technology

6.022J/2.792J/BEH.371J/HST542J: Quantitative Physiology: Organ Transport Systems

PROBLEM SET 6

Assigned: March 18, 2004

Due: April 1, 2004

Problem 1

A pressure wave, P_i, incident on an arterial (or bronchial) bifurcation will suffer a reflection, P_r, of magnitude

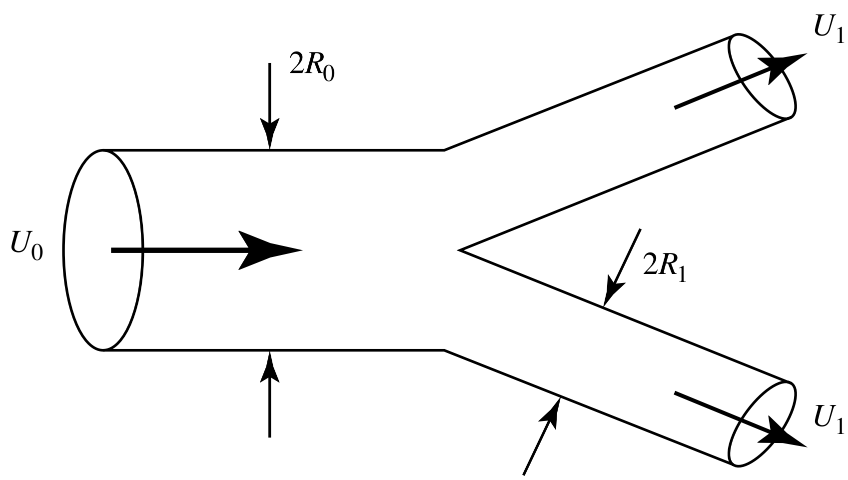
$$\frac{P_r}{P_i} \equiv \Gamma = \frac{Z_L - Z_o}{Z_L + Z_o}$$

where Z_o is the upstream arterial impedance, and Z_L is the impedance of the bifurcation. The impedance, Z, of a vessel of radius R may be calculated from the formulæderived in the notes:

$$Z^2 = \frac{\rho}{AC_u} \qquad C_u \approx \frac{2\pi R^3}{hE}$$

where ρ is the fluid density, A is the vessel cross-section, h is the vessel wall thickness, and E is the vessel modulus of elasticity.

Figure 1:



A. Assuming a symmetric bifurcation, ρ = constant, $E_0 = E_1$, $\frac{h_1}{R_1} = \frac{h_0}{R_0}$, and no increase in total cross-sectional area across the bifurcation, calculate the reflection coefficient.

B. Suppose the vessels distal to the bifurcation are severely calcified, so that $E_1 \gg E_0$. What will the reflection coeffecient be? Does this suggest a noninvasive method of detecting the presence of severe arterial disease?

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Problem 2

This problem deals with the estimation of the pressure drop to be expected across a vascular stenosis. Figure 2 is a sketch of the cross-section of a stenotic artery with a concentric plaque. An idealized model is shown in Figure 3, demonstrating a narrowed region followed by a sudden expansion where the fluid will generally exhibit turbulent flow. Energy will be lost in two ways: in viscous flow in the narrow region and in turbulent loss in the expansion. In this problem we will concentrate on the latter.

Figure 2:

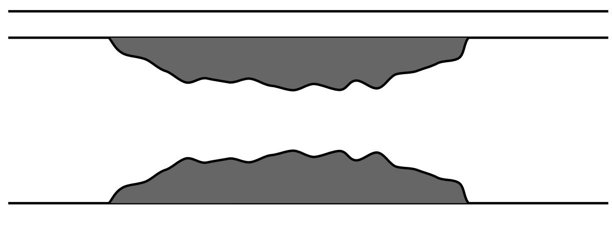
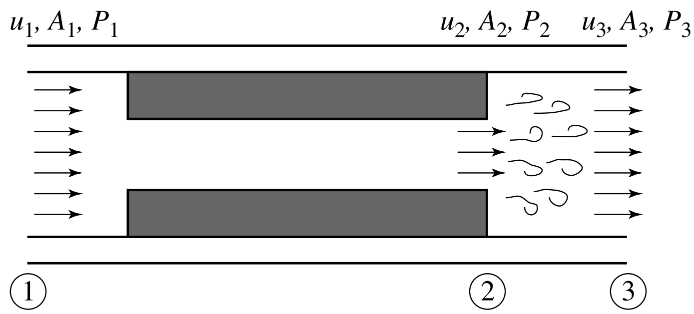


Figure 3:



The first task will be to estimate the pressure drop from point 2 to point 3. Position 3 is chosen far enough downstream to be in a region of uniform flow, of velocity u_3. In order to solve this problem we must make use of the "linear momentum theorem".

Consider a flow field of fluid with a superimposed control volume (CV). Newton's second law for the system included within the CV is

$$\frac{\mathrm{d}\vec{M}}{\mathrm{d}t} = \sum \vec{F}_{system} \tag{1}$$

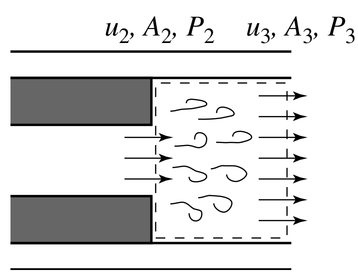
This equation states that the rate of change of the system momentum equals the sum of forces acting on the mass inside the control volume. Equation 1 may be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_{\mathrm{CV}}} \rho\vec{U}\,\mathrm{d}V + \oint_{A_{\mathrm{CV}}} \rho\vec{U}\left(\vec{U}\cdot\vec{n}\right)\mathrm{d}A = \sum \vec{F} \tag{2}$$

The first term is the rate of accumulation of momentum within the control volume; the second term is the net rate of momentum flux *out* of the CV.

Let us establish a CV for our model of the sudden expansion in Figure 3. (The volume within the dashed line in Figure 4.)

Figure 4:



The fluid is incompressible, and there is no net acceleration of the mass within the CV, so the first term of Equation 2 disappears. How do we evaluate the second term?

A. Calculate the momentum flux *into* the CV. (Note that flux is the rate of momentum passing into the CV per unit time.) Do you agree that the answer is:

$$\rho u_2(u_2 A_2) = \rho A_2 u_2^2$$

B. What is the momentum flux *out* of the CV at point 3?

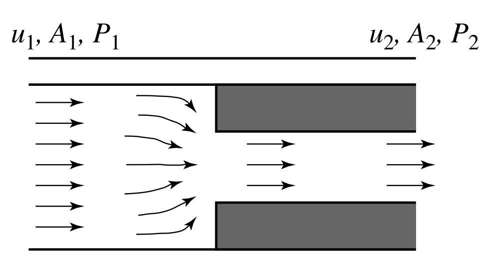
C. Use conservation of mass to relate u_3 to u_2 and the cross-sectional areas A_2 and A_3.

D. Next we need to calculate the right hand side of equation 2. The forces acting on the CV are *pressure* forces. Assume that the pressure at the entrance orifice, P_2, is equal across the entire face of the CV at position 2. What is the *net* force acting on the CV?

E. What is the pressure drop $P_2 - P_3$ as a function of the velocity u_2, the areas A_2 and A_3, and the fluid density?

F. Now let us focus on the entrance portion of the stenosis shown below, and consider inviscid flow (Figure 5).

Figure 5:



Use the Bernoulli principle and conservation of mass to relate the pressure drop $P_1 - P_2$ to the characteristics of the fluid, the entrance velocity (u_1), and the geometry.

G. If we neglect viscous losses in the region of the stenosis, what will be the total pressure drop $P_1 - P_3$? Your derivation should end up with the following:

$$P_1 - P_3 = \frac{1}{2}\rho u_1^2 \left[\frac{A_1}{A_2} - 1\right]^2$$

H. In a vessel where u_1 is 30 cm/sec, and the ratio A_2/A_1 is 0.1, estimate the pressure drop in mmHg. (Remember that 1 mmHg=1330 dynes/cm^2.) What pressure drop would be expected if the lumen *diameter* is reduced to 25% of its original size?

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Problem 3

Engineers for a medical products company have developed an inexpensive, hand-held device for monitoring peak expiratory flow rates in asthmatics. The device, shown in Figure 6, consists of a rigid cylinder with a long, narrow slit down the side and a close-fitting, spring-loaded piston diaphragm. Expired air enters the chamber of the peak flow meter from the left through the mouthpiece, then exits via the slit *at a uniform velocity.* The length of the slit available for flow, x, is equal to the displacement of the piston. Gas exits from the slit in the form of a *turbulent jet* into the atmosphere at pressure p_a.

Assuming the flow to be *quasi-steady and inviscid*, that the piston moves without friction, and that the displacement is linearly proportional to the force F acting on it:

$$F = kx$$

answer the following questions. You may take the following as given:

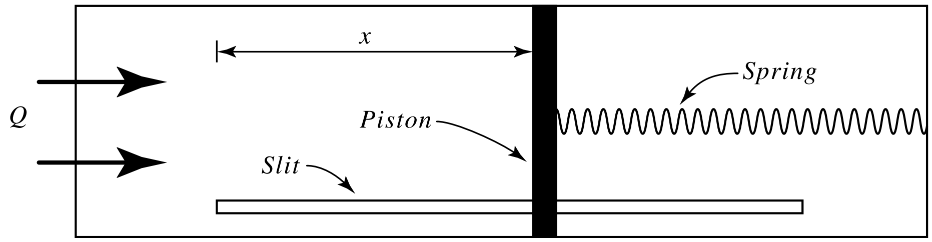
- density of expired air, ρ
- atmospheric pressure, p_a
- slit height, h ($h \ll D$)
- volume flow rate, Q
- spring constant, k
- piston diameter, D
- exposed slit length, x

A. Given the volume flow rate entering the device, Q, and assuming the piston to be displaced a distance x but *non-accelerating*, what is the velocity of gas exiting through the slit?

B. What is the pressure inside the chamber of the flowmeter, p_o?

C. What is the relationship between the displacement of the piston, x, and the other given parameters?

Figure 6:



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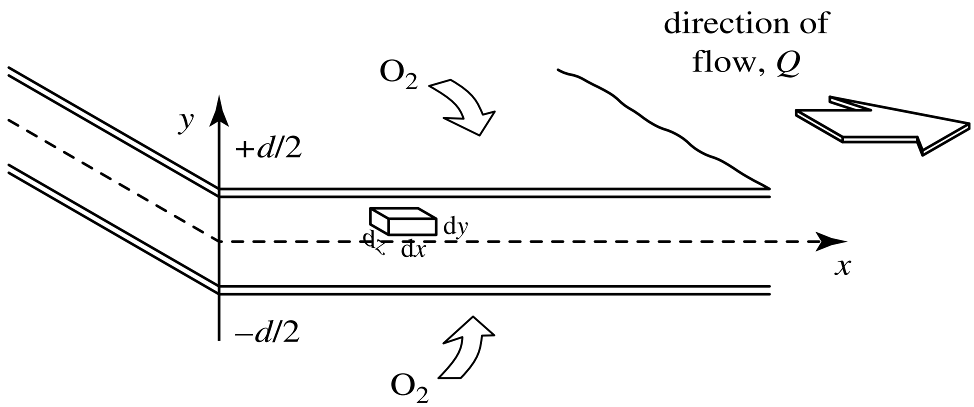
Problem 4

You are conducting experiments in your laboratory on a new artificial lung which consists of two parallel membranes separated by a small distance d. (See Figure 7.) Blood flows through the gap under steady laminar conditions due to a pressure gradient, $-\mathrm{d}p/\mathrm{d}x$, and oxygen transport occurs across the membranes. Assume blood to be a Newtonian fluid of viscosity μ.

Figure 7:



A. Using the coordinate system shown in Figure 7, what are the *boundary conditions* on the fluid velocity $v_x(y)$ and the shear rate $\dot{\gamma} = \dfrac{\partial v_x(y)}{\partial y}$?

B. Set up the differential equation of motion for the control volume $\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$.

C. Solve the equation (subject to the boundary conditions) for the blood velocity in the x-direction as a function of y. Sketch the result.

D. Using the result of (C), determine the flow rate per unit width, $Q/\mathrm{d}z$, in terms of the pressure gradient $-\mathrm{d}p/\mathrm{d}x$, the gap width d, and the blood viscosity μ.

E. What is the resistance, R, per unit width of the structure?

F. What is the shear rate at the membrane surface in terms of the average flow velocity, $\bar{V}$?

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