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Linear attenuators, phase-insensitive and phase-sensitive linear amplifiers

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Introduction

In this lecture will continue our quantum-mechanical treatment of linear attenuators and linear amplifiers. Among other things, we will distinguish between phase-insensitive and phase-sensitive amplifiers. We will also show that the attenuator and the phase-insensitive amplifier preserve classicality, i.e., their outputs are classical states when their inputs are classical states. Finally, we will use the transformation effected by the two-mode parametric amplifier to introduce the notion of entanglement.

Single-Mode Linear Attenuation and Phase-Insensitive Linear Amplification

Slide 3 shows the quantum models for linear attenuation and linear amplification that were presented in Lecture 11. In both cases we are concerned with single-mode quantum fields at the input and output, whose excited modes are as follows,[1]

$$\hat{E}_{\text{in}}(t) = \frac{\hat{a}_{\text{in}} e^{-j\omega t}}{\sqrt{T}} \quad \text{and} \quad \hat{E}_{\text{out}}(t) = \frac{\hat{a}_{\text{out}} e^{-j\omega t}}{\sqrt{T}}, \quad \text{for } 0 \le t \le T, \tag{1}$$

where

$$\hat{a}_{\text{out}} = \begin{cases} \sqrt{L}\,\hat{a}_{\text{in}} + \sqrt{1-L}\,\hat{a}_L, & \text{for the attenuator} \\ \sqrt{G}\,\hat{a}_{\text{in}} + \sqrt{G-1}\,\hat{a}_G^\dagger, & \text{for the amplifier,} \end{cases} \tag{2}$$

with $0 < L < 1$ being the attenuator's transmissivity and $G > 1$ being the amplifier's gain. The presence of the auxiliary-mode annihilation operators, $\hat{a}_L$ and $\hat{a}_G$, in these input-output relations, ensures that

$$[\hat{a}_{\text{out}}, \hat{a}_{\text{out}}^\dagger] = 1, \tag{3}$$

[1]For the sake of brevity, we have omitted the "other terms" that are needed to ensure that these field operators have the appropriate commutators for freely propagating fields. So long as the photodetection measurements that we make are not sensitive to these vacuum-state other modes, there is no loss in generality in using these compact single-mode expressions.

as is required for the $\hat{E}_{\text{out}}(t)$ expression to be a proper photon-units representation of a single-mode quantum field. Minimum noise is injected by the auxiliary modes when they are in their vacuum states, so, unless otherwise noted, we shall assume that they are indeed in these unexcited states.

It is easy to show that the annihilation operator input-output relation, (2), implies the following input-output relation for the θ-quadratures,

$$\hat{a}_{\text{out}_\theta} = \begin{cases} \sqrt{L}\,\hat{a}_{\text{in}_\theta} + \sqrt{1-L}\,\hat{a}_{L_\theta}, & \text{for the attenuator} \\ \sqrt{G}\,\hat{a}_{\text{in}_\theta} + \sqrt{G-1}\,\hat{a}_{G_{-\theta}}, & \text{for the amplifier,} \end{cases} \tag{4}$$

where $\hat{a}_\theta \equiv \text{Re}(\hat{a}e^{-j\theta})$ defines the θ-quadrature of an annihilation operator $\hat{a}$. Taking the expectation of these equations, with $\hat{a}_{\text{in}}$ being in an arbitrary quantum state, gives,

$$\langle \hat{a}_{\text{out}_\theta} \rangle = \begin{cases} \sqrt{L}\,\langle \hat{a}_{\text{in}_\theta} \rangle, & \text{for the attenuator} \\ \sqrt{G}\,\langle \hat{a}_{\text{in}_\theta} \rangle, & \text{for the amplifier.} \end{cases} \tag{5}$$

Because $\langle \hat{a}_{\text{out}_\theta} \rangle / \langle \hat{a}_{\text{in}_\theta} \rangle$ is *independent* of θ, for both the attenuator and the amplifier, we say that these systems are *phase-insensitive*, i.e., *all* the input quadratures undergo the same mean-field attenuation (for the attenuator) or gain (for the amplifier).

Output State of the Attenuator

In Lecture 11 we derived the means and variances of photon number and quadrature measurements made on the output of the linear attenuator. Today we will obtain the complete statistical characterization of this output, and use our result to determine when semiclassical theory can be employed for photodetection measurements made on the attenuator's output. Our route to these results will be through characteristic functions.[2]

We know that the output mode density operator, $\hat{\rho}_{\text{out}}$, is completely characterized by its associated anti-normally ordered characteristic function,

$$\chi_A^{\rho_{\text{out}}}(\zeta^*, \zeta) = \langle e^{-\zeta^* \hat{a}_{\text{out}}} e^{\zeta \hat{a}_{\text{out}}^\dagger} \rangle. \tag{6}$$

Substituting in from (2) and using the fact that the $\hat{a}_{\text{in}}$ and $\hat{a}_L$ modes are in a product state, with the latter being in its vacuum state, gives

$$\begin{aligned} \chi_A^{\rho_{\text{out}}}(\zeta^*, \zeta) &= \langle e^{-\zeta^*(\sqrt{L}\,\hat{a}_{\text{in}} + \sqrt{1-L}\,\hat{a}_L)} e^{\zeta(\sqrt{L}\,\hat{a}_{\text{in}}^\dagger + \sqrt{1-L}\,\hat{a}_L^\dagger)} \rangle & (7) \\ &= \langle e^{-\zeta^* \sqrt{L}\,\hat{a}_{\text{in}}} e^{\zeta \sqrt{L}\,\hat{a}_{\text{in}}^\dagger} \rangle \langle e^{-\zeta^* \sqrt{1-L}\,\hat{a}_L} e^{\zeta \sqrt{1-L}\,\hat{a}_L^\dagger} \rangle & (8) \\ &= \chi_A^{\rho_{\text{in}}}(\zeta^* \sqrt{L}, \zeta \sqrt{L}) e^{-|\zeta|^2(1-L)}. & (9) \end{aligned}$$

[2]This should not be surprising. We are dealing with a linear quantum transformation. In classical probability theory it is well known that characteristic function techniques are very convenient for dealing with linear classical transformations. So, we are going to see that the same is true in the quantum case.

We won't use the operator-valued inverse transform to find $\hat{\rho}_{\text{out}}$ from this result, but we will examine what happens when the $\hat{a}_{\text{in}}$ mode is in the coherent state $|\alpha_{\text{in}}\rangle$. Here, our known expression for the anti-normally ordered characteristic function of the coherent state leads to

$$\chi_A^{\rho_{\text{out}}}(\zeta^*, \zeta) = e^{-\zeta^* \sqrt{L}\,\alpha_{\text{in}} + \zeta \sqrt{L}\,\alpha_{\text{in}}^*} e^{-|\zeta|^2}, \tag{10}$$

which we recognize as being equal to $\langle \sqrt{L}\,\alpha_{\text{in}}| e^{-\zeta^* \hat{a}_{\text{out}}} e^{\zeta \hat{a}_{\text{out}}^\dagger} |\sqrt{L}\,\alpha_{\text{in}}\rangle$. This shows that a coherent-state input $|\alpha_{\text{in}}\rangle$ to the attenuator results in a coherent-state output $|\sqrt{L}\,\alpha_{\text{in}}\rangle$ from the attenuator.[3] Moreover, if the input mode is in a classical state, i.e., its density operator has a P-representation

$$\hat{\rho}_{\text{in}} = \int \mathrm{d}^2\alpha\, P_{\text{in}}(\alpha, \alpha^*) |\alpha\rangle\langle\alpha|, \tag{11}$$

with $P_{\text{in}}(\alpha, \alpha^*)$ being a joint probability density function for $\alpha_1 = \text{Re}(\alpha)$ and $\alpha_2 = \text{Im}(\alpha)$, then it follows that the output mode is *also* in a classical state, with a proper P-function given by

$$P_{\text{out}}(\alpha, \alpha^*) = \frac{1}{L}\, P_{\text{in}}\left(\frac{\alpha}{\sqrt{L}}, \frac{\alpha^*}{\sqrt{L}}\right). \tag{12}$$

The derivation of this scaling relation—which coincides with the like result from classical probability theory—is left as an exercise for the reader. The essential message, however, is not the derivation; it is that linear attenuation (with a vacuum-state auxiliary mode) preserves classicality.

Output State of the Phase-Insensitive Linear Amplifier

Turning to the phase-insensitive linear amplifier, we will determine its output-state behavior by the same characteristic function technique that we just used for the linear attenuator. Substituting in from (2) and using the fact that the $\hat{a}_{\text{in}}$ and $\hat{a}_G$ modes are in a product state, with the latter being in its vacuum state, gives

$$\chi_A^{\rho_{\text{out}}}(\zeta^*, \zeta) = \langle e^{-\zeta^*(\sqrt{G}\,\hat{a}_{\text{in}} + \sqrt{G-1}\,\hat{a}_G^\dagger)} e^{\zeta(\sqrt{G}\,\hat{a}_{\text{in}}^\dagger + \sqrt{G-1}\,\hat{a}_G)}\rangle \tag{13}$$

$$= \langle e^{-\zeta^* \sqrt{G}\,\hat{a}_{\text{in}}} e^{\zeta\sqrt{G}\,\hat{a}_{\text{in}}^\dagger}\rangle \langle e^{-\zeta^*\sqrt{G-1}\,\hat{a}_G^\dagger} e^{\zeta\sqrt{G-1}\,\hat{a}_G}\rangle \tag{14}$$

$$= \chi_A^{\rho_{\text{in}}}(\zeta^*\sqrt{G}, \zeta\sqrt{G}). \tag{15}$$

Once again, we will not try to get an explicit general result for $\hat{\rho}_{\text{out}}$, but only pursue that result for coherent-state inputs. When the $\hat{a}_{\text{in}}$ mode is in the coherent

[3] This same result can be gleaned from the homework, where the characteristic function approach is used to show that coherent-state inputs to a beam splitter produce coherent-state outputs from the beam splitter.

state $|\alpha_{\rm in}\rangle$, we find that

$$\chi_A^{\rho_{\rm out}}(\zeta^*,\zeta) = e^{-\zeta^*\sqrt{G}\,\alpha_{\rm in}+\zeta\sqrt{G}\,\alpha_{\rm in}^*}e^{-G|\zeta|^2} = \langle\sqrt{G}\,\alpha_{\rm in}|e^{-\zeta^*\hat{a}_{\rm out}}e^{\zeta\hat{a}_{\rm out}^\dagger}|\sqrt{G}\,\alpha_{\rm in}\rangle e^{-(G-1)|\zeta|^2}. \tag{16}$$

Because the classical state

$$\hat{\rho} = \int \mathrm{d}^2\alpha\,\frac{e^{-|\alpha|^2/(G-1)}}{\pi(G-1)}|\alpha\rangle\langle\alpha|, \tag{17}$$

has an anti-normally ordered characteristic function equal to $e^{-G|\zeta|^2}$, it follows that the classical *pure-state* input $|\alpha_{\rm in}\rangle$ produces a classical *mixed-state* output whose P-representation is

$$\hat{\rho}_{\rm out} = \int \mathrm{d}^2\alpha\,\frac{e^{-|\alpha-\sqrt{G}\,\alpha_{\rm in}|^2/(G-1)}}{\pi(G-1)}|\alpha\rangle\langle\alpha|. \tag{18}$$

Moreover, if the input state is a classical state with proper P-function $P_{\rm in}(\alpha,\alpha^*)$, we then find that the output state is also classical, with its P-function being given by

$$P_{\rm out}(\alpha,\alpha^*) = \frac{1}{G}\,P_{\rm in}\!\left(\frac{\alpha}{\sqrt{G}},\frac{\alpha}{\sqrt{G}}\right)\star\frac{e^{-|\alpha|^2/(G-1)}}{\pi(G-1)}, \tag{19}$$

where $\star$ denotes 2-D convolution. The derivation, which we have omitted, is a straightforward classical probability theory exercise. The key statement to be made here is that the phase-insensitive linear amplifier (with a vacuum-state auxiliary mode) preserves classicality.

Semiclassical Models for the Linear Attenuator and the Phase-Insensitive Linear Amplifier

We have just seen that the linear attenuator and the phase-insensitive linear amplifier—both with vacuum-state auxiliary modes—preserve classicality. That means if we restrict the $\hat{a}_{\rm in}$ mode to be in a classical state, then we can use semiclassical theory to find the statistics of photodetection measurements that are made on the $\hat{a}_{\rm out}$ mode. Let us explore that semiclassical theory now. From the P-representation transformation that we found above for the linear attenuator, its *classical* single-mode input and output fields,

$$E_{\rm in}(t) = \frac{a_{\rm in}e^{-j\omega t}}{\sqrt{T}} \quad\text{and}\quad E_{\rm out}(t) = \frac{a_{\rm out}e^{-j\omega t}}{\sqrt{T}} \quad \text{for } 0\le t\le T, \tag{20}$$

are related by

$$a_{\rm out} = \sqrt{L}\,a_{\rm in}. \tag{21}$$

This implies that semiclassical photodetection theory applies for the linear attenuator with the output field as given above, i.e.,

- Direct detection of the attenuator's output field yields a final count that is a Poisson-distributed random variable with mean $L|a_{\text{in}}|^2$, given knowledge of a_{in}.

- Homodyne detection of the attenuator output's θ-quadrature yields a variance-1/4 Gaussian-distributed random variable with mean $\sqrt{L}\, a_{\text{in}_\theta}$, given knowledge of a_{in}.

- Heterodyne detection of the attenuator's output field gives real and imaginary quadrature measurements that are statistically independent, variance-1/2 Gaussian random variables with mean values $\sqrt{L}\, a_{\text{in}_1} = \sqrt{L}\,\text{Re}(a_{\text{in}})$ and $\sqrt{L}\, a_{\text{in}_2} = \sqrt{L}\,\text{Im}(a_{\text{in}})$, respectively, given knowledge of a_{in}.

Physically, this makes perfect classical sense. Think of the attenuator as a beam splitter with transmissivity L for the input mode, and a zero-field input for its auxiliary mode. In that case it is obvious that $a_{\text{out}} = \sqrt{L}\, a_{\text{in}}$ will prevail. Things are different, however, for the amplifier, as we will now show.

From the P-representation transformation that we found above for the phase-insensitive linear amplifier, its *classical* single-mode input and output fields,

$$E_{\text{in}}(t) = \frac{a_{\text{in}} e^{-j\omega t}}{\sqrt{T}} \quad \text{and} \quad E_{\text{out}}(t) = \frac{a_{\text{out}} e^{-j\omega t}}{\sqrt{T}} \quad \text{for } 0 \le t \le T, \tag{22}$$

are related by

$$a_{\text{out}} = \sqrt{G}\, a_{\text{in}} + n. \tag{23}$$

In the last expression, n is a complex-valued random variable—statistically independent of a_{in}—whose real and imaginary parts are statistically independent, zero-mean, variance $(G-1)/2$ Gaussian random variables. This implies that semiclassical photodetection theory applies for the linear amplifier with the output field as given above, i.e.,

- Direct detection of the amplifier's output field yields a final count that is a Poisson-distributed random variable with mean $|\sqrt{G}\, a_{\text{in}} + n|^2$, given knowledge of a_{in} *and* n.

- Homodyne detection of the amplifier output's θ-quadrature yields a variance-1/4 Gaussian-distributed random variable with mean $\sqrt{G}\, a_{\text{in}_\theta} + n_\theta$, given knowledge of a_{in} *and* n, where $n_\theta \equiv \text{Re}(ne^{-j\theta})$.

- Heterodyne detection of the attenuator's output field gives real and imaginary quadrature measurements that are statistically independent, variance-1/2 Gaussian random variables with mean values $\sqrt{G}\, a_{\text{in}_1} + n_1 = \sqrt{G}\,\text{Re}(a_{\text{in}}) + \text{Re}(n)$ and $\sqrt{G}\, a_{\text{in}_2} + n_2 = \sqrt{G}\,\text{Im}(a_{\text{in}}) + \text{Im}(n)$, respectively, given knowledge of a_{in} *and* n, where $n_1 \equiv \text{Re}(n)$ and $n_2 \equiv \text{Im}(n)$.

The conditioning on n should be removed, as we will never have prior knowledge of its value. The direct detection result can be derived from a related case that will be treated on the homework. The homodyne and heterodyne results are trivial to obtain, because n_1 and n_2 are statistically independent, identically distributed, zero-mean, variance-$(G-1)/2$ random variables. We then have that

- Direct detection of the amplifier's output field yields a final count that, given knowledge of a_{in}, is a Laguerre-distributed random variable,

$$\Pr(\, N = n \mid a_{\text{in}}\,) = \frac{(G-1)^n}{G^{n+1}}\, e^{-|a_{\text{in}}|^2} L_n\!\left(-\frac{|a_{\text{in}}|^2}{(G-1)}\right), \text{ for } n = 0, 1, 2, \ldots, \tag{24}$$

  where $L_n(\cdot)$ is the nth Laguerre polynomial.

- Homodyne detection of the amplifier output's θ-quadrature yields a variance-$(2G-1)/4$ Gaussian-distributed random variable with mean $\sqrt{G}\, a_{\text{in}_\theta}$, given knowledge of a_{in}

- Heterodyne detection of the attenuator's output field gives real and imaginary quadrature measurements that are statistically independent, variance-$G/2$ Gaussian random variables with mean values $\sqrt{G}\, a_{\text{in}_1} = \sqrt{G}\,\text{Re}(a_{\text{in}})$ and $\sqrt{G}\, a_{\text{in}_2} = \sqrt{G}\,\text{Im}(a_{\text{in}})$, respectively, given knowledge of a_{in}.

As as final exercise, in this semiclassical analysis, let us use the conditional direct detection statistics—i.e., those given both a_{in} *and* n—to find the mean and variance of the final photocount N. Define $\mathcal{N} = \sqrt{G}\, a_{\text{in}} + n$. From the Poisson distribution we know that the conditional mean and conditional variance of N are

$$E(\, N \mid \mathcal{N}\,) = \mathcal{N} \quad \text{and} \quad \text{var}(\, N \mid \mathcal{N}\,) = E(\, N^2 \mid \mathcal{N}\,) - [E(\, N \mid \mathcal{N}\,)]^2 = \mathcal{N}, \tag{25}$$

where $E(\,\cdot \mid \mathcal{N}\,)$ denotes expectation with respect to the probability mass function $\Pr(\, N = n \mid \mathcal{N}\,)$, i.e., the Poisson distribution with mean $\mathcal{N}$. To remove the conditioning on n that is implicit in the preceding mean and variance formulas, we average them over the conditional statistics of $\mathcal{N}$ given *only* knowledge of a_{in}. The mean is easily found,

$$E(\, N \mid a_{\text{in}}\,) = E_n(|\sqrt{G}\, a_{\text{in}} + n|^2) \tag{26}$$

$$= G|a_{\text{in}}|^2 + 2\sqrt{G}\,[a_{\text{in}} E_n(n^*) + \sqrt{G}\, a_{\text{in}}^* E_n(n)] + E_n(|n|^2) \tag{27}$$

$$= G|a_{\text{in}}|^2 + (G-1), \tag{28}$$

where $E(\,\cdot \mid a_{\text{in}}\,)$ denotes expectation with respect to the probability mass function $\Pr(\, N = n \mid a_{\text{in}}\,)$ and $E_n(\cdot)$ denotes expectation with respect to the Gaussian probability density function for n, i.e., the joint probability density function for n_1 and n_2.

For the mean-squared value of N, conditioned only on knowledge of $a_{\rm in}$, we proceed as follows. From the Poisson distribution we know that

$$E(\,N^2 \mid a_{\rm in}\,) = E_n(|\sqrt{G}\,a_{\rm in} + n|^4 + |\sqrt{G}\,a_{\rm in} + n|^2). \tag{29}$$

Working on the first term on the right-hand side leads to

$$\begin{aligned} E_n(|\sqrt{G}\,a_{\rm in} + n|^4) &= G^2|a_{\rm in}|^4 + 2G^{3/2}|a_{\rm in}|^2[a_{\rm in}E_n(n^*) + a_{\rm in}^*E_n(n)] \\ &+ 4G|a_{\rm in}|^2E_n(|n|^2) + G[a_{\rm in}^2E_n(n^{*2}) + a_{\rm in}^{*2}E_n(n^2)] \\ &+ 2\sqrt{G}\,[a_{\rm in}E_n(|n|^2n^*) + a_{\rm in}^*E_n(|n|^2n)] + E(|n|^4). \end{aligned} \tag{30}$$

From the given Gaussian statistics of n, we have

$$E_n(n) = E_n(n^*) = E(n^2) = E(n^{*2}) = E_n(|n|^2n) = E_n(|n|^2n^*) = 0, \tag{31}$$

and

$$E_n(|n|^2) = G - 1 \quad \text{and} \quad E_n(|n|^4) = 2(G-1)^2. \tag{32}$$

Putting everything together gives us

$$E(\,N^2 \mid a_{\rm in}\,) = G|a_{\rm in}|^2 + (G-1) + G^2|a_{\rm in}|^4 + 4G(G-1)|a_{\rm in}|^2 + 2(G-1)^2, \tag{33}$$

from which the conditional variance readily follows,

$$\text{var}(\,N \mid a_{\rm in}\,) = G|a_{\rm in}|^2 + (G-1) + 2G(G-1)|a_{\rm in}|^2 + (G-1)^2. \tag{34}$$

Now suppose that $a_{\rm in}$ is a complex-valued classical random variable with known values of $\langle |a_{\rm in}|^2\rangle$, $\langle |a_{\rm in}|^4\rangle$, and $\text{var}(|a_{\rm in}|^2)$. We can now remove the $a_{\rm in}$ conditioning, and we find that

$$\langle N\rangle = G\langle |a_{\rm in}|^2\rangle + (G-1) \tag{35}$$

$$\langle N^2\rangle = G^2\langle |a_{\rm in}|^4\rangle + 4G(G-1)\langle |a_{\rm in}|^2\rangle + 2(G-1)^2 \tag{36}$$

$$\begin{aligned} \langle \Delta N^2\rangle &= [G\langle |a_{\rm in}|^2\rangle + (G-1)] \\ &+ [G^2\text{var}(|a_{\rm in}|^2) + 2G(G-1)\langle |a_{\rm in}|^2\rangle + (G-1)^2], \end{aligned} \tag{37}$$

give the *unconditional* mean, mean-square, and variance of the photon count N.

We have used brackets to group terms in the variance expression (37) for an important reason. The first bracket on the right-hand side of this equation equals $\langle N\rangle$, and as such it is the variance contribution from the shot noise (Poissonian-variance noise) in N, i.e., the noise that would be present even if $|\sqrt{G}\,a_{\rm in} + n|^2$ were deterministic. The second term on the right-hand side of this equation equals $\text{var}(|\sqrt{G}\,a_{\rm in} + n|^2)$, and so it is the variance contribution of the excess noise in N, i.e., the noise that is due to randomness in the illumination $|\sqrt{G}\,a_{\text{in}}+n|^2$. Thus, we could also write (37) as

$$\langle \Delta N^2\rangle = \langle \mathcal{N}\rangle + \langle \Delta\mathcal{N}^2\rangle, \tag{38}$$

where the first term on the right is due to shot noise and the second term on the right is due to excess noise.

The Two-Mode Parametric Amplifier

We are now ready to tackle the phase-sensitive linear amplifier. To do so, we start with a two-mode model for a degenerate parametric amplifier, as shown on Slide 9. Limiting our field operators at the input and output to the two excited—x-polarized and y-polarized, frequency-ω—modes,[4]

$$\hat{\mathbf{E}}_{\text{in}}(t) = \frac{\hat{a}_{\text{in}_x}e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_x + \frac{\hat{a}_{\text{in}_y}e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_y \quad \text{and} \quad \hat{\mathbf{E}}_{\text{out}}(t) = \frac{\hat{a}_{\text{out}_x}e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_x + \frac{\hat{a}_{\text{out}_y}e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_y, \tag{39}$$

for $0 \le t \le T$, where $\mathbf{i}_x$ and $\mathbf{i}_y$ are x- and y-directed unit vectors, we write the input-output relation for this system as follows:

$$\hat{a}_{\text{out}_x} = \mu\hat{a}_{\text{in}_x} + \nu\hat{a}^\dagger_{\text{in}_y} \quad \text{and} \quad \hat{a}_{\text{out}_y} = \mu\hat{a}_{\text{in}_y} + \nu\hat{a}^\dagger_{\text{in}_x}, \quad \text{with } |\mu|^2 - |\nu|^2 = 1. \tag{40}$$

Equation (40) is a two-mode Bogoliubov transformation. The first thing that we must check is that this transformation—like the single-mode version we employed when we introduced the squeezed states—preserves commutator brackets. It is straightforward to show that this is so, e.g., we have that

$$[\hat{a}_{\text{out}_x}, \hat{a}^\dagger_{\text{out}_x}] = |\mu|^2[\hat{a}_{\text{in}_x}, \hat{a}^\dagger_{\text{in}_x}] + |\nu|^2[\hat{a}^\dagger_{\text{in}_y}, \hat{a}_{\text{in}_y}] = |\mu|^2 - |\nu|^2 = 1. \tag{41}$$

We leave it as an exercise for you to verify that

$$[\hat{a}_{\text{out}_y}, \hat{a}^\dagger_{\text{out}_y}] = 1, \tag{42}$$

and that

$$[\hat{a}_{\text{out}_x}, \hat{a}_{\text{out}_y}] = [\hat{a}_{\text{out}_x}, \hat{a}^\dagger_{\text{out}_y}] = 0. \tag{43}$$

The two-mode Bogoliubov transformation that characterizes this parametric amplifier embodies *both* phase-insensitive *and* phase-sensitive amplification. Suppose that μ and ν are real and positive. We can then make the identifications $\sqrt{G} = \mu$ and $\sqrt{G-1} = \nu$, where $G > 1$. So, if the $\hat{a}_{\text{in}_x}$ and $\hat{a}_{\text{out}_x}$ modes are regarded as input and output, we find that our input-output relation is

$$\hat{a}_{\text{out}_x} = \sqrt{G}\,\hat{a}_{\text{in}_x} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_y}. \tag{44}$$

[4]Here we are suppressing the "other terms" that are needed to make the field operators have proper commutators. As in what we have done earlier in this lecture, so long as our measurements are not sensitive to these vacuum-state other modes, there is no loss of generality in employing these simple two-mode field operator expressions.

Taking the $\hat{a}_{\text{in}_y}$ mode to be in its vacuum state makes this input-output relation identical to the phase-insensitive linear amplifier that we considered earlier today. Now, however, for the two-mode parametric amplifier, we have a physical locus for the auxiliary mode.

What about phase-sensitive linear amplification? To see how this comes about, let us rewrite the input and output field operators as follows,

$$\hat{\mathbf{E}}_{\text{in}}(t) = \frac{\hat{a}_{\text{in}_+} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_+ + \frac{\hat{a}_{\text{in}_-} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_- \quad \text{and} \quad \hat{\mathbf{E}}_{\text{out}}(t) = \frac{\hat{a}_{\text{out}_+} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_+ + \frac{\hat{a}_{\text{out}_-} e^{-j\omega t}}{\sqrt{T}}\,\mathbf{i}_-, \tag{45}$$

where

$$\mathbf{i}_\pm = \frac{\mathbf{i}_x \pm \mathbf{i}_y}{\sqrt{2}}, \tag{46}$$

and

$$\hat{a}_{\text{in}_\pm} = \frac{\hat{a}_{\text{in}_x} \pm \hat{a}_{\text{in}_y}}{\sqrt{2}} \quad \text{and} \quad \hat{a}_{\text{out}_\pm} = \frac{\hat{a}_{\text{out}_x} \pm \hat{a}_{\text{out}_y}}{\sqrt{2}}. \tag{47}$$

Physically, what we have done here is to convert from writing the two field modes in the x-y basis to writing them the $\pm 45°$ basis. Trivial though this change may seem, it has a profound effect on the input-output behavior for a single mode, as we will now demonstrate.

Equation (40) leads to the following input-output relation for the $\pm 45°$ modes:

$$\hat{a}_{\text{out}_+} = \mu \hat{a}_{\text{in}_+} + \nu \hat{a}^\dagger_{\text{in}_+} \quad \text{and} \quad \hat{a}_{\text{out}_-} = \mu \hat{a}_{\text{in}_-} - \nu \hat{a}^\dagger_{\text{in}_-}, \quad \text{with } |\mu|^2 - |\nu|^2 = 1, \tag{48}$$

which is a pair of single-mode Bogoliubov transformations. Suppose that μ and ν are real and positive—with $\sqrt{G} = \mu$ and $\sqrt{G-1} = \nu$ for $G > 1$—and that we regard the $\hat{a}_{\text{in}_+}$ and $\hat{a}_{\text{out}_+}$ modes as the input and the output. We then have that the mean values of the $\hat{a}_{\text{out}_+}$ mode's θ-quadrature obeys

$$\begin{aligned} \langle \hat{a}_{\text{out}_{+\theta}} \rangle &= \text{Re}\{[(\sqrt{G} + \sqrt{G-1})\langle \hat{a}_{\text{in}_{+1}} \rangle + j(\sqrt{G} - \sqrt{G-1})\langle \hat{a}_{\text{in}_{+2}} \rangle] e^{-j\theta}\} && (49) \\ &= (\sqrt{G} + \sqrt{G-1})\langle \hat{a}_{\text{in}_{+1}} \rangle \cos(\theta) + (\sqrt{G} - \sqrt{G-1})\langle \hat{a}_{\text{in}_{+2}} \rangle \sin(\theta). && (50) \end{aligned}$$

where the 1 and 2 quadratures of $\hat{a}_{\text{in}_+}$ denote the real and imaginary parts of this annihilation operator. This mean-field input-output relation is linear, but it is phase sensitive, i.e., for the real-part quadrature ($\theta = 0$) we get amplification,

$$\langle \hat{a}_{\text{out}_{+1}} \rangle = (\sqrt{G} + \sqrt{G-1})\langle \hat{a}_{\text{in}_{+1}} \rangle, \quad \text{where } \sqrt{G} + \sqrt{G-1} > 1. \tag{51}$$

On the other hand, for the imaginary-part quadrature ($\theta = \pi/2$) we get attenuation,

$$\langle \hat{a}_{\text{out}_{+2}} \rangle = (\sqrt{G} - \sqrt{G-1})\langle \hat{a}_{\text{in}_{+2}} \rangle, \quad \text{where } 0 < \sqrt{G} - \sqrt{G-1} < 1. \tag{52}$$

Similar phase-sensitive behavior can be seen on the $\hat{a}_{\text{out}_-}$ mode, only now it is the real part quadrature that undergoes attenuation whereas the imaginary part quadrature enjoys amplification.

Our final task for today will be to use characteristic functions to obtain the complete statistical characterization of the two-mode parametric amplifier when $\sqrt{G} = \mu > 0$ and $\sqrt{G-1} = \nu > 0$. Here, we choose to start from the Wigner characteristic function for the two output modes,[5]

$$\chi_W^{\rho_{\rm out}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = \langle e^{-\zeta_x^* \hat{a}_{{\rm out}_x} - \zeta_y^* \hat{a}_{{\rm out}_y} + \zeta_x \hat{a}_{{\rm out}_x}^\dagger + \zeta_y \hat{a}_{{\rm out}_y}^\dagger} \rangle. \tag{53}$$

By substituting in the two-mode Bogoliubov transformation that relates the output annihilation operators to the input annihilation operators it easily seen that

$$\chi_W^{\rho_{\rm out}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = \chi_W^{\rho_{\rm in}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y), \tag{54}$$

where

$$\xi_x = \sqrt{G}\,\zeta_x - \sqrt{G-1}\,\zeta_y^* \quad \text{and} \quad \xi_y = \sqrt{G}\,\zeta_y - \sqrt{G-1}\,\zeta_x^*. \tag{55}$$

Using the Baker-Campbell-Hausdorf theorem, we have that the anti-normally ordered characteristic function for $\hat{\rho}_{\rm out}$ satisfies

$$\chi_A^{\rho_{\rm out}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = \chi_W^{\rho_{\rm in}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y) e^{-(|\zeta_x|^2 + |\zeta_y|^2)/2}. \tag{56}$$

One important special case is worth exhibiting before we close. Let the input modes both be in their vacuum states. Then, because this implies

$$\chi_W^{\rho_{\rm in}}(\xi_x^*, \xi_y^*, \xi_x, \xi_y) = e^{-|\xi_x|^2/2 - |\xi_y|^2/2}, \tag{57}$$

we find

$$\chi_A^{\rho_{\rm out}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) = e^{-G(|\zeta_x|^2 + |\zeta_y|^2) + 2\sqrt{G(G-1)}\,{\rm Re}(\zeta_x \zeta_y)}. \tag{58}$$

Furthermore, because

$$\chi_A^{\rho_{{\rm out}_x}}(\zeta_x^*, \zeta_x) = \chi_A^{\rho_{\rm out}}(\zeta_x^*, 0, \zeta_x, 0) = e^{-G|\zeta_x|^2}, \tag{59}$$

and

$$\chi_A^{\rho_{{\rm out}_y}}(\zeta_y^*, \zeta_y) = \chi_A^{\rho_{\rm out}}(0, \zeta_y^*, 0, \zeta_y) = e^{-G|\zeta_y|^2}, \tag{60}$$

we see that $\hat{\rho}_{\rm out}$ is *not* a product state, viz.,[6]

$$\chi_A^{\rho_{\rm out}}(\zeta_x^*, \zeta_y^*, \zeta_x, \zeta_y) \neq \chi_A^{\rho_{{\rm out}_x}}(\zeta_x^*, \zeta_x) \chi_A^{\rho_{{\rm out}_y}}(\zeta_y^*, \zeta_y). \tag{61}$$

Instead, $\hat{\rho}_{\rm out}$ is an *entangled* state, whose properties be the subject of considerable attention in the next few lectures.[7]

[5] Below, we will convert the Wigner characteristic function to the anti-normally ordered characteristic function, from which the normally-ordered form of the density operator can be obtained by inverse Fourier transformation or the density operator itself can be found from an operator-valued inverse Fourier transform.

[6] Taking the inverse Fourier transform of both sides of this inequality shows that the normally-ordered form of $\hat{\rho}_{\rm out}$ does not factor into the product of the normally-ordered forms of $\hat{\rho}_{{\rm out}_x}$ and $\hat{\rho}_{{\rm out}_y}$, and so $\hat{\rho}_{\rm out}$ is *not* a product state.

[7] Strictly speaking, all we have shown, by demonstrating that $\hat{\rho}_{\rm out}$ is *not* a product state, is that

The Road Ahead

In the next lecture we shall continue our work on parametric amplification and entanglement. There we shall investigate the individual (marginal) output statistics of the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes, and show that their joint statistics exhibit a photon-twinning behavior. We shall then employ a dual parametric amplifier setup to produce polarization entangled photons, which will be the basis for qubit teleportation.

there is dependence (classical *or* quantum) between the states of the individual output modes $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$. However, as we will see next lecture, because the two-mode Bogoliubov transformation is unitary, a pure-state input leads to a pure-state output. Thus, when the input modes $\hat{a}_{\text{in}_x}$ and $\hat{a}_{\text{in}_y}$ are both in their vacuum states, the resulting $\hat{\rho}_{\text{out}}$ must be of the form $|\psi\rangle\langle\psi|$ for some pure state $|\psi\rangle$ on the joint state space of $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$. Because this pure state is *not* a product state it must be entangled.