6.453 Quantum Optical Communication
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6.453 Quantum Optical Communication

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Polarization entanglement, qubit teleportation, quadrature entanglement and continuous-variable teleportation.

Introduction

In Lecture 1, we exhibited three remarkable quantum optical phenomena that defied classical explanation: the squeezed-state waveguide tap, polarization entanglement, and qubit teleportation. Also in Lecture 1 was the promise that, before the semester was over, you would have a complete quantum-mechanical understanding of these examples (and others as well). So far, we have delivered on the squeezed state waveguide tap. Last time, we got our first real look at polarization entanglement. We'll reprise that at the start of today's lecture, but we won't complete our treatment of the singlet state's non-classical nature until later this term. Thus, our main goal in today's lecture will be teleportation. We'll start by building on the entanglement embodied by the singlet state, and show how it enables the qubit teleportation protocol that was described in Lecture 1 and mentioned at the very end of Lecture 13. Then we'll return to entanglement, but this time look at entanglement of field quadratures. This type of entanglement will serve as the foundation for another approach to teleportation, known as continuous-variable teleportation, whose characteristics we will begin to study today.

Reprise of Polarization Entanglement

Slide 3 summarizes a setup for demonstrating singlet-state polarization entanglement. We have two single-mode quantum fields whose joint state is the singlet.

$$|\psi^-\rangle_{12} = \frac{|x\rangle_1|y\rangle_2 - |y\rangle_1|x\rangle_2}{\sqrt{2}}, \tag{1}$$

where $|u\rangle_k$ for $u = x, y$ and $k = 1, 2$ denotes the single-photon state of the kth field in which that single photon is u-polarized. It follows that the reduced density operators for the two fields are

$$\hat{\rho}_1 = \mathrm{tr}_2(|\psi^-\rangle_{12\,12}\langle\psi^-|) = \frac{|x\rangle_{11}\langle x| + |y\rangle_{11}\langle y|}{2}, \tag{2}$$

and

$$\hat{\rho}_2 = \mathrm{tr}_1(|\psi^-\rangle_{12\,12}\langle\psi^-|) = \frac{|x\rangle_{22}\langle x| + |y\rangle_{22}\langle y|}{2}. \qquad (3)$$

These reduced density operators imply that polarization analysis performed on either field individually will yield completely random results, viz.,

$$\Pr(N_{k_{\mathbf{i}}} = 1) = {}_k\langle \mathbf{i}|\hat{\rho}_k|\mathbf{i}\rangle_k = 1/2 \quad \text{and} \quad \Pr(N_{k_{\mathbf{i}'}} = 1) = {}_k\langle \mathbf{i}'|\hat{\rho}_k|\mathbf{i}'\rangle_k = 1/2, \qquad (4)$$

for $k = 1, 2$ and *all* polarization bases $\{\mathbf{i}, \mathbf{i}'\}$, where $|\mathbf{i}\rangle_k$ and $|\mathbf{i}'\rangle_k$ denote $\mathbf{i}$- and $\mathbf{i}'$-polarized single-photon states of field k, respectively.

The individual-measurement behavior we have just found from quantum theory is not hard to replicate in a classical setting with rigid particle photons for fields 1 and 2. Suppose that these photons have polarization states characterized by Poincaré-sphere unit vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, that are random and uniformly distributed over the sphere. Using $\mathbf{r} \leftrightarrow \mathbf{i}$ and $-\mathbf{r} \leftrightarrow \mathbf{i}'$ to denote the Poincaré-sphere equivalent to the $\{\mathbf{i}, \mathbf{i}'\}$ basis, we get the following rigid-particle photon counting theory results:

$$\Pr(N_{k_{\mathbf{i}}} = 1) = \left\langle \frac{1 + \mathbf{r}^T\mathbf{r}_k}{2} \right\rangle = 1/2 \quad \text{and} \quad \Pr(N_{k_{\mathbf{i}'}} = 1) = \left\langle \frac{1 - \mathbf{r}^T\mathbf{r}_k}{2} \right\rangle = 1/2, \quad (5)$$

for $k = 1, 2$.

The interesting—and non-classical—behavior of the singlet state is seen when we perform polarization analysis—using the *same* arbitrarily-chosen basis—on *both* fields. In this case we get

$$\Pr(N_{1_{\mathbf{i}}} = 1, N_{2_{\mathbf{i}'}} = 1) = |{}_1\langle \mathbf{i}|_2\langle \mathbf{i}'|\psi^-\rangle_{12}|^2 = 1/2 \qquad (6)$$

$$\Pr(N_{1_{\mathbf{i}'}} = 1, N_{2_{\mathbf{i}}} = 1) = |{}_1\langle \mathbf{i}'|_2\langle \mathbf{i}|\psi^-\rangle_{12}|^2 = 1/2, \qquad (7)$$

from quantum theory, which—together with the marginal probabilities we have already found—leads to the conditional probabilities

$$\Pr(\, N_{2_{\mathbf{i}'}} = 1 \mid N_{1_{\mathbf{i}}} = 1\,) = \Pr(\, N_{2_{\mathbf{i}}} = 1 \mid N_{1_{\mathbf{i}'}} = 1\,) = 1, \qquad (8)$$

for *all* bases $\{\mathbf{i}, \mathbf{i}'\}$. According to quantum theory, therefore, the two-photon singlet state has the following remarkable property. Its individual photon components are randomly polarized, but if one photon is detected in the $\mathbf{i}$ polarization then the other photon will *definitely* be found to be in the $\mathbf{i}'$ polarization, regardless of the basis choice.

The joint measurement behavior that we have just found from the quantum theory *cannot* be matched by classical physics. The best we can do is to say the each of the two rigid-particle photons is randomly polarized, but those polarizations are completely correlated, i.e., if the first photon has Poincaré-sphere unit vector $\mathbf{r}_1$ for its polarization, then the second photon has Poincaré-sphere unit vector $-\mathbf{r}_1$ for its polarization. In this case we find that[1]

$$\Pr(N_{1_{\mathbf{i}}} = 1, N_{2_{\mathbf{i}'}} = 1) \;=\; \left\langle \left(\frac{1+\mathbf{r}^T\mathbf{r}_1}{2}\right)^2 \right\rangle = 1/3 \tag{9}$$

$$\Pr(N_{1_{\mathbf{i}'}} = 1, N_{2_{\mathbf{i}}} = 1) \;=\; \left\langle \left(\frac{1-\mathbf{r}^T\mathbf{r}_1}{2}\right)^2 \right\rangle = 1/3, \tag{10}$$

from classical theory, which—together with the marginal probabilities we have already found—leads to the conditional probabilities

$$\Pr(\, N_{2_{\mathbf{i}'}} = 1 \mid N_{1_{\mathbf{i}}} = 1\,) = \Pr(\, N_{2_{\mathbf{i}}} = 1 \mid N_{1_{\mathbf{i}'}} = 1\,) = 2/3, \tag{11}$$

for *all* bases $\{\mathbf{i}, \mathbf{i}'\}$. In classical physics we just cannot say that photon 2 is *definitely* $\mathbf{i}'$-polarized when photon 1 has been detected in the $\mathbf{i}$ polarization.

Polarization Qubits and the Bell Basis

Before delving into qubit teleportation, it behooves us to say a few words about polarization qubits and the Bell basis. Consider the quantum fields,[2]

$$\hat{\mathbf{E}}_A(t) = \frac{(\hat{a}_{A_x}\mathbf{i}_x + \hat{a}_{A_y}\mathbf{i}_y)e^{-j\omega t}}{\sqrt{T}} \quad \text{and} \quad \hat{\mathbf{E}}_B(t) = \frac{(\hat{a}_{B_x}\mathbf{i}_x + \hat{a}_{B_y}\mathbf{i}_y)e^{-j\omega t}}{\sqrt{T}}, \tag{12}$$

for $0 \le t \le T$, where the usual "other modes" terms are unexcited and have been omitted. Now, assume that the states of the $\hat{\mathbf{E}}_A(t)$ and $\hat{\mathbf{E}}_B(t)$ fields *each* contain exactly one photon. Everything about these individual states is specified—by the forms we have taken for the field operators—*except* their polarizations. A general pure state of polarization for a single photon, however, is expressible as a superposition of x- and y-polarized single–photon states, i.e.,

$$|\psi\rangle_K \equiv \alpha_K|x\rangle_K + \beta_K|y\rangle_K, \quad \text{for } K = A, B \text{ and } |\alpha_K|^2 + |\beta_K|^2 = 1, \tag{13}$$

specifies all pure states of polarization for single photons of the $\hat{\mathbf{E}}_A(t)$ and $\hat{\mathbf{E}}_B(t)$ fields., respectively. An arbitrary unit-length superposition of two orthonormal quantum states is a quantum bit, or qubit. Thus, what we have just exhibited is how an abstract qubit can be coded into the polarization of a single-photon quantized electromagnetic field.

The natural basis for the joint state of the polarization qubits carried by $\hat{\mathbf{E}}_A(t)$ and $\hat{\mathbf{E}}_B(t)$ in the construct we have introduced above is the tensor-product basis. In

[1] The details of this calculation were presented in Lecture 1.

[2] We have chosen to use subscripts A and B here, instead of 1 and 2, to match up with "Alice" and "Bob" who appear in our block diagram of qubit teleportation on Slide 5.

particular, a pure state of the two fields—i.e., a pure two-qubit state in which each field carries one qubit in the polarization of its single photon—can be written in this basis as

$$|\psi\rangle_{AB} = \alpha_{xx}|x\rangle_A|x\rangle_B + \alpha_{xy}|x\rangle_A|y\rangle_B + \alpha_{yx}|y\rangle_A|x\rangle_B + \alpha_{yy}|y\rangle_A|y\rangle_B, \tag{14}$$

where

$$|\alpha_{xx}|^2 + |\alpha_{xy}|^2 + |\alpha_{yx}|^2 + |\alpha_{yy}|^2 = 1. \tag{15}$$

If the $\{\alpha_{jk} : j,k = x,y\}$ are such that this state factors into

$$|\psi\rangle_{AB} = (\alpha_A|x\rangle_A + \beta_A|y\rangle_A) \otimes (\alpha_B|x\rangle_B + \beta_B|y\rangle_B), \tag{16}$$

then the tensor product basis is the most convenient way to represent the joint state of the two modes. However, if the $\{\alpha_{jk} : j,k = x,y\}$ are such that a factorization of this type *cannot* be done, then the two qubits are entangled. In this case, it may be more convenient to work with the Bell basis.

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ denote the Hilbert spaces spanned by $\{|x\rangle_A, |y\rangle_A\}$ and $\{|x\rangle_B, |y\rangle_B\}$, respectively. These are the state spaces for the single-photon polarization qubits of the fields $\hat{\mathbf{E}}_A(t)$ and $\hat{\mathbf{E}}_B(t)$. The tensor product basis, $\{|u\rangle_J|v\rangle_K : u.v = x,y; J,K = A,B\}$ is one basis for $\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$. The Bell basis is another. To conform with the notation used in the slides, let's denote the four Bell states that comprise this basis as follows:[3]

$$|B_0\rangle_{AB} = \frac{|x\rangle_A|y\rangle_B - |y\rangle_A|x\rangle_B}{\sqrt{2}} \tag{17}$$

$$|B_1\rangle_{AB} = \frac{|x\rangle_A|y\rangle_B + |y\rangle_A|x\rangle_B}{\sqrt{2}} \tag{18}$$

$$|B_2\rangle_{AB} = \frac{|x\rangle_A|x\rangle_B - |y\rangle_A|y\rangle_B}{\sqrt{2}} \tag{19}$$

$$|B_3\rangle_{AB} = \frac{|x\rangle_A|x\rangle_B + |y\rangle_A|y\rangle_B}{\sqrt{2}}. \tag{20}$$

It is left as an exercise for you to verify that the Bell states are orthonormal. Then, because they all lie in $\mathcal{H}$ and $\mathcal{H}$ is a 4-D Hilbert space, it is evident that the Bell states form a basis. Unlike the tensor-product basis, the Bell basis is comprised of entangled states.

[3]More standard notation would be $|B_0\rangle = |\psi^-\rangle$, $|B_1\rangle = |\psi^+\rangle$, $|B_2\rangle = |\phi^-\rangle$, and $|B_3\rangle = |\phi^+\rangle$. The first of these states is the singlet that we have encountered already. The remaining three are known as triplet states.

Qubit Teleportation

We are now ready to describe and analyze the qubit teleportation system shown on Slide 5. Here, Charlie has a polarization qubit—a quantum message—that he wishes to send to Bob. Because Bob is quite far away from Charlie, there is no reliable way for Charlie to directly send this qubit to Bob. For example, suppose that Bob is 50 km away from Charlie and they are connected by a strand of low-loss (0.2 dB/km) optical fiber. Then, the probability that Charlie can successfully send a single photon down the line to Bob is 0.1, and this already assumes that Charlie's photon is at the 1.55 μm wavelength where the loss is lowest. Suppose that Charlie's qubit represents part of a quantum computation or quantum communication protocol, so that it can be any possible single-photon state, Charlie cannot determine its value by polarization analysis, nor can he perfectly clone his unknown qubit. Thus he is loath to take the chance that his photon will get to Bob unharmed. Luckily, Charlie is located very close to Alice, who has already shared a singlet state with Bob. In particular we shall assume that there is a source of polarization-entangled photons, and that one of these photons has been stored by Alice and the other by Bob.[4] Entanglement sharing is Step 1 of the qubit teleportation protocol. In Step 2, Alice makes a joint measurement on her stored photon—from the singlet she shared with Bob—and the message-qubit photon that Charlie has entrusted to her. The result of this quantum measurement—whose detailed description will be given below—turns out to be two classical bits, i.e., it is either 0, 1, 2, or 3. For Step 3 of the protocol, Alice sends her measurement result to Bob over a classical communication channel. Unlike sending a single-photon qubit down a long optical fiber, which *cannot* be assumed to have a high probability of success, there is no problem in assuming that Alice's two-bit classical message can be perfectly communicated to Bob. Upon receipt of this information, Bob performs Step 4, the final step of the teleportation protocol. He does this by applying the wave-plate transformation to his qubit—the part of the singlet state that he had stored in advance—chosen to be the one (of four possibilities) labeled by Alice's message. The result of this transformation will be that Bob's qubit is in the state that Charlie delivered to Alice at the start.

Let us go through the analysis of this protocol one step at a time.

Step 1 When Alice and Bob have successfully shared a singlet state, they each have a single-photon polarization qubit such that the joint state for the photons they have stored is

$$|\psi^-\rangle_{AB} = \frac{|x\rangle_A|y\rangle_B - |y\rangle_A|x\rangle_B}{\sqrt{2}}. \tag{21}$$

Charlie's message qubit is arbitrary,

$$|\psi\rangle_C = \alpha|x\rangle_C + \beta|y\rangle_C, \quad \text{where } |\alpha|^2 + |\beta|^2 = 1. \tag{22}$$

[4]The efficiency of the sharing process may be very low—i.e., Alice and Bob may have had to try many times in order to successfully create their shared entanglement—but that inefficiency does not endanger Charlie's precious message qubit.

The joint state of Alice, Bob, and Charlie is therefore $|\psi^-\rangle_{AB}|\psi\rangle_C$. Multiplying out and rearranging terms it can be shown—you should verify this—that the joint Alice-Bob-Charlie state can be written as follows in terms of the Alice-Charlie Bell states:

$$|\psi^-\rangle_{AB}|\psi\rangle_C = |\psi^-\rangle_{AB}(\alpha|x\rangle_C + \beta|y\rangle_C) \tag{23}$$

$$\begin{aligned} &= \frac{1}{2}[|B_0\rangle_{AC} \otimes (\alpha|x\rangle_B + \beta|y\rangle_B) - |B_1\rangle_{AC} \otimes (\alpha|x\rangle_B - \beta|y\rangle_B) \\ &+ |B_2\rangle_{AC} \otimes (\alpha|y\rangle_B + \beta|x\rangle_B) + |B_3\rangle_{AC} \otimes (\alpha|y\rangle_B - \beta|x\rangle_B)] \end{aligned} \tag{24}$$

Step 2 Because $\{|B_n\rangle_{AC} : 0 \leq n \leq 3\}$ is an orthonormal basis for the two-qubit Hilbert space $\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_C$, we know that

$$\hat{B}_{AC} \equiv \sum_{n=0}^{3} n|B_n\rangle_{AC} \tag{25}$$

is an observable on this Hilbert space. It is this observable that Alice measures in Step 2 of the qubit teleportation protocol. Her outcome will be one of the $\hat{B}_{AC}$ eigenvalues, $\{n : 0 \leq n \leq 3\}$. After Alice's measurement, according to the projection postulate, the two qubits for the pair of photons in her possession—her photon from the singlet that she shared with Bob plus the message-qubit photon that Charlie supplied—will be in the eigenstate (Bell state) associated with her measurement outcome. This assumes that the measurement can be done without annihilating these photons by photodetection. Either way, after the Bell-observable measurement, the states of Alice and Charlie's photons no longer carry any information about their states prior to the measurement. That this is so can be seen by noting that all four possible $\hat{B}_{AC}$ measurement outcomes are equally likely to occur. For example, we have that

$$\Pr(\hat{B}_{AC} \text{ outcome} = 0) = {}_{AC}\langle B_0|\hat{\rho}_{AC}|B_0\rangle_{AC} \tag{26}$$

$$= {}_{AC}\langle B_0|[\mathrm{tr}_B(|\psi^-\rangle_{AB}|\psi\rangle_{CC}\langle\psi|_{AB}\langle\psi^-|)]|B_0\rangle_{AC} \tag{27}$$

$$= \mathrm{tr}_B({}_{AC}\langle B_0|\psi^-\rangle_{AB}|\psi\rangle_{CC}\langle\psi|_{AB}\langle\psi^-|B_0\rangle_{AC}) \tag{28}$$

$$= \mathrm{tr}_B[(\alpha|x\rangle_B + \beta|y\rangle_B)(\alpha^*{}_B\langle x| + \beta^*{}_B\langle y|)]/4 = 1/4, \tag{29}$$

where the fourth equality follows from (24). You should take the time to verify that

$$\Pr(\hat{B}_{AC} \text{ outcome} = n) = 1/4, \quad \text{for } n = 1, 2, 3. \tag{30}$$

Step 3 Alice sends her measurement outcome, n, to Bob via a classical communication channel. Although Alice's four message possibilities are equally likely, they convey essential information to Bob. This can be seen by determining the state of Bob's qubit conditioned on the outcome of Alice's measurement. Looking at (24), it should be clear that: when Alice's outcome is $n = 0$, then Bob's qubit must be in the state $\alpha|x\rangle_B + \beta|y\rangle_B$; when Alice's outcome is $n = 1$, then Bob's qubit must be in the state $\alpha|x\rangle_B - \beta|y\rangle_B$; when Alice's outcome is $n = 2$, then Bob's qubit must be in the state $\alpha|y\rangle_B + \beta|x\rangle_B$; and when Alice's outcome is $n = 3$, then Bob's qubit must be in the state $\alpha|y\rangle_B - \beta|x\rangle_B$. We can easily prove that this is true. Suppose that Alice makes the $\hat{B}_{AC}$ measurement on her qubit and Charlie's, and that Bob does polarization analysis on his qubit in the $\mathbf{i} = \alpha\mathbf{i}_x + \beta\mathbf{i}_y$, $\mathbf{i}' = \beta^*\mathbf{i}_x - \alpha^*\mathbf{i}_y$ basis. Then we find that

$$\begin{aligned}&\Pr(\hat{B}_{AC}\text{ outcome} = 0, \text{Bob's photon clicks the } \mathbf{i} \text{ detector})\\ &\qquad = |{}_B\langle\mathbf{i}|_{AC}\langle B_0|\psi^-\rangle_{AB}|\psi\rangle_C|^2 = 1/4,\end{aligned} \tag{31}$$

where the second equality makes use of (24). Because we already know that $\Pr(\hat{B}_{AC}\text{ outcome} = 0) = 1/4$, it follows that, given this Bell measurement outcome, Bob's qubit must be in the $\alpha|x\rangle_B + \beta|y\rangle_B$ state, as its conditional probability of occurrence is 1. A similar derivation can be done for the other $\hat{B}_{AC}$ outcomes. Thus, we can rewrite (24) as

$$|\psi^-\rangle_{AB}|\psi\rangle_C = |\psi^-\rangle_{AB}(\alpha|x\rangle_C + \beta|y\rangle_C) \tag{32}$$

$$\begin{aligned}= \frac{1}{2}\Bigg[&|B_0\rangle_{AC} \otimes \underbrace{(\alpha|x\rangle_B + \beta|y\rangle_B)}_{\text{Bob's state if } \hat{B}_{AC}=0} - |B_1\rangle_{AC} \otimes \underbrace{(\alpha|x\rangle_B - \beta|y\rangle_B)}_{\text{Bob's state if } \hat{B}_{AC}=1}\\ &+ |B_2\rangle_{AC} \otimes \underbrace{(\alpha|y\rangle_B + \beta|x\rangle_B)}_{\text{Bob's state if } \hat{B}_{AC}=2} + |B_3\rangle_{AC} \otimes \underbrace{(\alpha|y\rangle_B - \beta|x\rangle_B)}_{\text{Bob's state if } \hat{B}_{AC}=3}\Bigg],\end{aligned} \tag{33}$$

where the statements in the underbraces follow from the orthogonality of the Bell states.[5]

Step 4 Once Bob has received Alice's message, he is ready to complete the teleportation protocol. If Alice sent him $n = 0$, then Bob does nothing, because his state, $\alpha|x\rangle_B + \beta|y\rangle_B$, is already a replica of Charlie's message. If Alice sent him $n = 1$, then Bob flips the phase of the y-polarization, because this will leave him with a replica of Charlie's state. If Alice sent him $n = 2$, then Bob swaps

[5]This rewriting leads to the interpretation that measuring $\hat{B}_{AC}$ *collapses* Bob's state to one of the four states identified by the underbraces. State vector collapse is a tricky thing. It can easily get one hung up on issues of causality. In reality, the singlet-state entanglement between Alice and Bob's qubits is such a strong dependence that the Bell measurement outcome implies that Bob's photon must be in the state associated with that outcome in (33).

the x and y polarizations in his qubit, because this will again leave him with a replica of Charlie's state. Finally, if Alice sent him $n = 3$, then Bob must both interchange the x and y polarizations and flip the phase of the y polarization to obtain a replica of Charlie's state.

The first thing to note here is that in every case Bob is performing a single-qubit rotation, which, for a polarization qubit, is easily accomplished with wave plates. The second thing to note here is that these transformations do *not* require Bob to know anything about Charlie's qubit, i.e., the values of α and β. This means that Bob has learned nothing about Charlie's qubit.

At this point it is germane to repeat the remarkable points about qubit teleportation that we listed in Lecture 1.

- Alice's measurement tells her *nothing* about the polarization state of Charlie's photon: her four possible measurement outcomes are always equally likely to have occurred.
- Alice's measurement destroys the polarization of both her photon and Charlie's, i.e., the teleportation protocol does not violate the no-cloning theorem, even though Bob will end up with a photon whose polarization state matches that of Charlie's, because by then there will not be a photon at Alice's location that contains any information about that polarization state.
- Causality is not violated, because the classical communication channel is light-speed limited.
- Bob learns nothing about the polarization state of Charlie's photon from this protocol, so teleportation does not violate the principle that the unknown polarization state of a single photon cannot be measured.

It is also worthwhile to repeat the comment made in Lecture 1 about the incredible power of entanglement that is revealed by the qubit teleportation protocol. Suppose that Charlie knows what polarization state—characterized by its Poincaré-sphere unit vector—that he wants Bob to have, i.e., Charlie has a specific but arbitrary $\mathbf{r}$ value in mind. In general, Charlie must send Bob an *infinite* number of classical bits—via a classical channel—for Bob to know the precise value of this real-valued, unit length, 3D vector. Yet, if Alice and Bob have shared an entangled photon pair, and Alice makes the appropriate joint measurement on her photon and Charlie's, she need only send Bob two bits of classical information to enable him to transform his photon into the $\mathbf{r}$ polarization.

Quadrature Entanglement

Let us return to the two-mode parametric amplifier, shown on Slide 9, so that we can examine another form of entanglement—a continuous-variable entanglement of the quadratures—that will enable a different form of teleportation. We'll take the two-mode Bogoliubov transformation for the parametric amplifier to be

$$\hat{a}_{\text{out}_x} = \sqrt{G}\,\hat{a}_{\text{in}_x} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_y} \quad \text{and} \quad \hat{a}_{\text{out}_y} = \sqrt{G}\,\hat{a}_{\text{in}_y} + \sqrt{G-1}\,\hat{a}^\dagger_{\text{in}_x}, \tag{34}$$

where $G > 1$, and the input modes are assumed to be in their vacuum states. By direct calculation from these input-output relations, or by recourse to the complete statistics that we developed for this system in Lectures 12 and 13, we have that the quadrature variances of the individual output modes are all super-shot noise, i.e.,

$$\langle \Delta\hat{a}^2_{\text{out}_{xk}}\rangle = \langle \Delta\hat{a}^2_{\text{out}_{yk}}\rangle = \frac{(2G-1)}{4} > \frac{1}{4}, \quad \text{for } k = 1, 2. \tag{35}$$

On the other hand, because the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes are entangled, it turns out that

$$\left\langle \left(\frac{\Delta\hat{a}_{\text{out}_{x1}} - \Delta\hat{a}_{\text{out}_{y1}}}{\sqrt{2}} \right)^2 \right\rangle = \left\langle \left(\frac{\Delta\hat{a}_{\text{out}_{x2}} + \Delta\hat{a}_{\text{out}_{y2}}}{\sqrt{2}} \right)^2 \right\rangle \tag{36}$$

$$= \frac{(\sqrt{G} - \sqrt{G-1})^2}{4} \approx \frac{1}{16G} \ll \frac{1}{4}, \text{ for } G \gg 1. \tag{37}$$

So, the fluctuations in the $\hat{a}_{\text{out}_{x1}}$ quadrature are highly correlated with those of the $\hat{a}_{\text{out}_{y1}}$ quadrature, *and* the fluctuations in the $\hat{a}_{\text{out}_{x2}}$ quadrature are highly anti-correlated with those in the $\hat{a}_{\text{out}_{y2}}$ quadrature. Indeed these correlations exceed semi-classical limits for homodyne detection, because of the entangled nature of the joint state of the $\hat{a}_{\text{out}_x}$ and $\hat{a}_{\text{out}_y}$ modes. It is this non-classical linking of the quadrature fluctuations that we will exploit to teleport the state of message field mode from one location to another. Unlike qubit teleportation, which deals with 2D state spaces, continuous-variable teleportation that relies on quadrature entanglement transmits states that lie in the infinite-dimensional Hilbert space of a single field mode.

Continuous-Variable Teleportation

Slide 10 shows the transmitter setup ("Alice") for continuous-variable teleportation. It starts with a two-mode parametric amplifier—with vacuum-state input modes—as described in the previous section. The output modes $\hat{a}_x$ and $\hat{a}_y$ are separated by a polarizing beam splitter, with the former being sent to the receiver's location through a channel with transmissivity γ_x. The latter suffers propagation loss (transmissivity γ_y) en route to a 50/50 beam splitter where it is combined with the $\hat{a}$ mode, whose state, $|\psi\rangle$, is the message (from "Charlie") that is to be teleported to the receiver. The two outputs from this 50/50 beam splitter are sent to a pair of balanced homodyne detectors, with one having a local oscillator set to measure the real part quadrature of its illumination and the other having its local oscillator set to measure the imaginary part quadrature of its illumination. These homodyne detectors each have quantum efficiency $\eta < 1$. Their outputs, u and v, are sent over a classical communication channel to the receiver.

Slide 11 shows the receiver setup ("Bob") for continuous-variable teleportation. A strong coherent state $|\sqrt{N_L}\rangle$ is the input to an electro-optical (real and imaginary part) modulator that is driven by the u and v that were received from Alice over the classical channel. The output from this modulator is combined—on an asymmetric beam splitter whose transmissivity is $T \approx 1$—with the $\hat{a}'_x$ mode that was received from Alice. The output from this asymmetric beam splitter is Bob's replica of Charlie's message state.

The Road Ahead

Next time we will work through the details of both the transmitter and receiver for continuous-variable teleportation. We will use that analysis to evaluate the *fidelity* of continuous-wave teleportation, i.e., the degree to which the output state is a replica of the message state. Unlike the case of qubit teleportation, where it is possible to imagine approaching perfect teleportation, it turns out that continuous-variable teleportation is subject to *very* stringent conditions which make perfect teleportation utterly impractical.