6.453 Quantum Optical Communication
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6.453 *Quantum Optical Communication* Lecture 17

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6.453 *Quantum Optical Communication* - Lecture 17

- Announcements
  - Pick up graded mid-term exam, lecture notes, slides
- Quantization of the Electromagnetic Field
  - Maxwell's equations
  - Plane-wave mode expansions
  - Multi-mode number states and coherent states

Classical Electromagnetic Fields in Free Space

- Maxwell's Equations in Differential Form:

$$\nabla \times \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\vec{H}(\vec{r},t), \qquad \nabla \cdot \epsilon_0 \vec{E}(\vec{r},t) = 0$$

$$\nabla \times \vec{H}(\vec{r},t) = \epsilon_0 \frac{\partial}{\partial t}\vec{E}(\vec{r},t), \qquad \nabla \cdot \mu_0 \vec{H}(\vec{r},t) = 0$$

- Vector Potential $\vec{A}(\vec{r},t)$ in Coulomb Gauge, $\nabla \cdot \vec{A}(\vec{r},t) = 0$:

$$\vec{E}(\vec{r},t) \equiv -\frac{\partial}{\partial t}\vec{A}(\vec{r},t), \quad \vec{H}(\vec{r},t) \equiv \frac{1}{\mu_0}\nabla \times \vec{A}(\vec{r},t)$$

- 3-D Vector Wave Equation:

$$\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}(\vec{r},t) = \vec{0}$$

Classical Electromagnetic Waves in Free Space

- Separation of Variables in the 3-D Vector Wave Equation:

$$\vec{A}(\vec{r},t) = \frac{1}{2\sqrt{\epsilon_0}} \sum_{\vec{l},\sigma} q_{\vec{l},\sigma}(t)\vec{u}_{\vec{l},\sigma}(\vec{r}) + \mathrm{cc}$$

- Separation Condition and Separation Constant:

$$\frac{\nabla^2 \vec{u}_{\vec{l},\sigma}(\vec{r})}{\vec{u}_{\vec{l},\sigma}(\vec{r})} = \frac{1}{c^2}\frac{\mathrm{d}^2 q_{\vec{l},\sigma}(t)/\mathrm{d}t^2}{q_{\vec{l},\sigma}(t)} \equiv -\frac{\omega_{\vec{l}}^2}{c^2}$$

- Helmholtz Equation and Harmonic Oscillator Equation:

$$\nabla^2 \vec{u}_{\vec{l},\sigma}(\vec{r}) + \frac{\omega_{\vec{l}}^2}{c^2}\vec{u}_{\vec{l},\sigma}(\vec{r}) = \vec{0}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} q_{\vec{l},\sigma}(t) + \omega_{\vec{l}}^2 q_{\vec{l},\sigma}(t) = 0$$

Periodic Boundary Conditions $\rightarrow$ Plane Waves

- Periodic Boundary Conditions for $L \times L \times L$ Cube:

$$\vec{u}_{\vec{l},\sigma}(\vec{r}) = \vec{u}_{\vec{l},\sigma}(\vec{r} + n_x L \vec{i}_x + n_y L \vec{i}_y + n_z L \vec{i}_z)$$

- Plane Wave Solutions:

$$\vec{u}_{\vec{l},\sigma}(\vec{r}) = \frac{1}{L^{3/2}} e^{j\vec{k}_{\vec{l}}\cdot\vec{r}}\, \vec{e}_{\vec{l},\sigma} \rightarrow \text{ plane waves}$$

$$\vec{e}_{\vec{l},\sigma} \cdot \vec{k}_{\vec{l}} = 0, \text{ for } \sigma = 0,1 \rightarrow \text{transversality}$$

$$\vec{k}_{\vec{l}} = \frac{2\pi}{L}[\ l_x \quad l_y \quad l_z\ ]^T, \quad \frac{\omega_{\vec{l}}^2}{c^2} = \vec{k}_{\vec{l}} \cdot \vec{k}_{\vec{l}}$$

Dimensionless Reformulation and the Hamiltonian

- Define: $a_{\vec{l},\sigma}(t) = \sqrt{\frac{\omega_{\vec{l}}}{2\hbar}}\, q_{\vec{l},\sigma}(t) = \text{dimensionless}$

- Electric and Magnetic Fields:

$$\vec{E}(\vec{r},t) = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar\omega_{\vec{l}}}{2\epsilon_0 L^3}}\, a_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\, \vec{e}_{\vec{l},\sigma} + \text{cc}$$

$$\vec{H}(\vec{r},t) = \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar c^2}{2\mu_0\omega_{\vec{l}} L^3}}\, a_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t - \vec{k}_{\vec{l}}\cdot\vec{r})}\, \vec{k}_{\vec{l}} \times \vec{e}_{\vec{l},\sigma} + \text{cc}$$

- Hamiltonian:

$$H = \int_{L\times L\times L} \mathrm{d}^3\vec{r} \left[\frac{1}{2}\epsilon_0 \vec{E}(\vec{r},t)\cdot\vec{E}(\vec{r},t) + \frac{1}{2}\mu_0 \vec{H}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right]$$

$$= \sum_{\vec{l},\sigma} \hbar\omega_{\vec{l}}\, a_{\vec{l},\sigma}^* a_{\vec{l},\sigma}$$

Quantized Electromagnetic Field

- Field Operators:

$$\hat{\vec{E}}(\vec{r},t) = \underbrace{\sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar\omega_{\vec{l}}}{2\epsilon_0 L^3}}\,\hat{a}_{\vec{l},\sigma}\,e^{-j(\omega_{\vec{l}}t-\vec{k}_{\vec{l}}\cdot\vec{r})}\,\vec{e}_{\vec{l},\sigma}}_{\hat{\vec{E}}^{(+)}(\vec{r},t)} + \text{hc}$$

$$\hat{\vec{H}}(\vec{r},t) = \underbrace{\sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar c^2}{2\mu_0\omega_{\vec{l}}L^3}}\,\hat{a}_{\vec{l},\sigma}\,e^{-j(\omega_{\vec{l}}t-\vec{k}_{\vec{l}}\cdot\vec{r})}\,\vec{k}_{\vec{l}}\times\vec{e}_{\vec{l},\sigma}}_{\hat{\vec{H}}^{(+)}(\vec{r},t)} + \text{hc}$$

- Commutators: $\left[\hat{a}_{\vec{l},\sigma}, \hat{a}^{\dagger}_{\vec{l}',\sigma'}\right] = \delta_{\vec{l}\vec{l}'}\delta_{\sigma\sigma'}$ and $\left[\hat{a}_{\vec{l},\sigma}, \hat{a}_{\vec{l}',\sigma'}\right] = 0$
- Hamiltonian: $\hat{H} = \sum_{\vec{l},\sigma} \hbar\omega_{\vec{l}}\left[\hat{a}^{\dagger}_{\vec{l},\sigma}\hat{a}_{\vec{l},\sigma} + \frac{1}{2}\right]$

Multi-Mode Number States and Coherent States

- Modal Number Operators: $\hat{N}_{\vec{l},\sigma} \equiv \hat{a}^{\dagger}_{\vec{l},\sigma}\hat{a}_{\vec{l},\sigma}$
- Modal Number States: $\hat{N}_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma} = n_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}$
- Multi-Mode Number States: $|\mathbf{n}\rangle \equiv \otimes_{\vec{l},\sigma}|n_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}$
- Modal Coherent States: $\hat{a}_{\vec{l},\sigma}|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma} = \alpha_{\vec{l},\sigma}|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}$
- Multi-Mode Coherent States: $|\boldsymbol{\alpha}\rangle \equiv \otimes_{\vec{l},\sigma}|\alpha_{\vec{l},\sigma}\rangle_{\vec{l},\sigma}$

Coherent States are Field-Operator Eigenkets

- Classical Positive-Frequency Field Associated with $|\boldsymbol{\alpha}\rangle$:

$$\vec{E}^{(+)}(\vec{r},t) \equiv \sum_{\vec{l},\sigma} j\sqrt{\frac{\hbar\omega_{\vec{l}}}{2\epsilon_0 L^3}}\,\alpha_{\vec{l},\sigma}\, e^{-j(\omega_{\vec{l}}t-\vec{k}_{\vec{l}}\cdot\vec{r})}\,\vec{e}_{\vec{l},\sigma}$$

- Field Operator Eigenket Relation:

$$|\vec{E}^{(+)}(\vec{r},t)\rangle \equiv |\boldsymbol{\alpha}\rangle$$

$$\widehat{\vec{E}}^{(+)}(\vec{r},t)|\vec{E}^{(+)}(\vec{r},t)\rangle = \vec{E}^{(+)}(\vec{r},t)|\vec{E}^{(+)}(\vec{r},t)\rangle$$

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Simplified Model: Photodetection Theory Prelude

- Assumption 1: Only one polarization is excited
- Assumption 2: Only $+z$-going plane wave is excited
- Assumption 3: Only narrow bandwidth about ω_o is excited
- Assumption 4: Work with photon-units baseband operator
- Assumption 5: Quantization interval $\rightarrow t \in (-\infty, \infty)$
- Fourier-integral field operator relationships

$$\hat{E}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\,\hat{\mathcal{E}}(\omega)e^{-j\omega t}$$

$$\hat{\mathcal{E}}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\,\hat{E}(t)e^{j\omega t}$$

- Field-Operator Commutators:

$$\left[\hat{E}(t), \hat{E}^{\dagger}(u)\right] = \delta(t-u) \quad \text{and} \quad \left[\hat{\mathcal{E}}(\omega), \hat{\mathcal{E}}^{\dagger}(\omega')\right] = 2\pi\delta(\omega-\omega')$$

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Coming Attractions: Mid-Term + Lectures 18, 19

- Lectures 18, 19:
  Continuous-Time Photodetection
  - Semiclassical theory: Poisson-distributed shot noise
  - Quantum theory: Photon-flux operator measurement
  - Continuous-time signatures of non-classical light