6.453 Quantum Optical Communication
Spring 2009

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Lecture Number 18
Fall 2008

Jeffrey H. Shapiro

Date: Tuesday, November 18, 2008

Reading: For random processes:

- J.H. Shapiro, *Optical Propagation, Detection, and Communication*, chapter 4.

Reading: For continuous-time photodetection:

- J.H. Shapiro, H.P. Yuen, and J.A. Machado Mata, "Optical communication with two-photon coherent states—Part II: photoemissive detection and structured receiver performance," IEEE Trans. Inform. Theory **IT-25**, 179–192 (1979).
- H.P. Yuen and J.H. Shapiro, "Optical communication with two-photon coherent states—Part III: quantum measurements realizable with photoemissive detectors," IEEE Trans. Inform. Theory **IT-26**, 78–92 (1980).
- J.H. Shapiro, "Quantum noise and excess noise in optical homodyne and heterodyne receivers," IEEE J. Quantum Electron. **QE-21**, 237–250 (1985).
- L. Mandel and E. Wolf *Optical Coherence and Quantum Optics,* (Cambridge University Press, Cambridge, 1995) sections 9.1–9.8, 12.1–12.4, 12.9, 12.10.

Introduction

Today we begin a two-lecture treatment of semiclassical versus quantum photodetection theory in a continuous-time setting. We'll build these theories in a slightly simplified framework, i.e., scalar fields[1] with no (x, y) dependence illuminating the active region of a photodetection that lies within a region $\mathcal{A}$ of area A in a constant-z plane.[2] Also, we'll focus our attention on almost-ideal photodetection, i.e., we will allow for sub-unity quantum efficiency ($\eta < 1$), but otherwise our detector will be the continuous-time version of the ideal photodetector that we treated earlier this

[1]These scalar fields may be regarded as being linearly polarized (for the classical case) or only being excited in one linear polarization (for the quantum case).

[2]The absence of transverse dependence means that only the normally-incident plane-wave component of the electromagnetic field is non-zero (in the classical case) or excited (in the quantum case).

semester within the single-mode construct. The particular tasks we have set for today's lecture are to develop the semiclassical and quantum photodetection statistical models for direct detection, and to exhibit some continuous-time signatures of nonclassical light. Because this work will require elements of random process theory, supplementary notes covering the background we shall assume have been provided. We will begin our treatment by reprising some remarks, from Lecture 8, about the relationship between real photodetectors and the idealized model that we shall employ today and next time.

A Real Photodetector

Slide 3 shows a theorist's cartoon of a real photodetector. The two large blocks on this slide are the photodetector and the post-detection preamplifier. The smaller blocks within the two large blocks are phenomenological, i.e., they do not represent discrete components out of which the larger entities are constructed. Nevertheless, it is instructive to walk our way through this photodetection system by means of these phenomenological blocks. Incoming light—whether we model it in classical or quantum terms—illuminates an optical filter that models the wavelength dependence of the photodetector's sensitivity. The light emerging from this filter then strikes the core of the photodetector, i.e., the block that converts light into a light-induced current, which we call the photocurrent. Photodetectors have some current flow in the absence of illumination, and this *dark* current adds to the photocurrent within the detector. High-sensitivity photodetectors—such as avalanche photodiodes and photomultiplier tubes—have internal mechanisms that amplify (multiply) the initial photocurrent (and the dark current), and we have shown that on Slide 3 as a current multiplication block.[3] This current multiplication in general has some randomness associated with it, imposing an excess noise on top of any noise already inherent in the photocurrent and dark current. The electrical filter that is next encountered models the electrical bandwidth of the photodetector's output circuit, and the thermal noise generator models the noise associated with the dissipative elements in the detector. Because the output current from a photodetector may not be strong enough to regard all subsequent processing as noiseless, we have included the preamplifier block in Slide 3. Its filter, noise generator, and gain blocks model the bandwidth characteristics, noise figure, and gain of a real preamplifier. Ordinarily, the output from such a preamplifier is strong enough that any further signal processing can be regarded as noise free.

[3]We have shown the photocurrent and dark current as undergoing the *same* multiplication process. In real detectors, these two currents may encounter different multiplication factors.

An Almost-Ideal Photodetector

Because we are interested in the *fundamental* limits of photodetection—be they represented in semiclassical or quantum terms—we will strip away *almost* all the non-ideal elements of the real photodetection system shown on Slide 3, and restrict our attention to the almost-ideal photodetector shown on Slide 4.[4] Be warned, however, that in experimental work we cannot always ignore the phenomena cited in our discussion of Slide 3. Nevertheless, it does turn out that there are photodetection systems that can approach the following almost-ideal behavior under some circumstances.

Our almost-ideal photodetector is a near-perfect version of the photocurrent generator block from Slide 3. In particular, our almost-ideal photodetector has these properties.

- Its optical sensitivity covers all frequencies.
- Its conversion of light into current has efficiency η.
- It does not have any dark current.
- It does not have any current multiplication.
- It has infinite electrical bandwidth.
- Its subsequent preamplifier has infinite bandwidth and no noise, so it need not be considered as it does not degrade the photodetection performance.

As a result, the photocurrent takes the form of a random train of area-q impulses, where q is the electron charge, and a counting circuit driven by this photocurrent will produce, as its output, a staircase function of unit-height steps which increments when each impulse occurs, i.e., the photocount record

$$N(t) = \frac{1}{q}\int_0^t \mathrm{d}u\, i(u), \quad \text{for } 0 \le t \le T, \tag{1}$$

as shown on Slide 4.

Classical Fields and Quantum Field Operators

The foundations for the semiclassical and quantum theories that we will present for continuous-time photodetection are the classical field and the quantum field operator, respectively. Given the assumptions that we made earlier, we can take the classical

[4] The sole non-ideality that we shall include is the sub-unity quantum efficiency, $\eta < 1$, of the photocurrent generator. This non-ideality is relatively easy to incorporate into our analysis, as we have already seen for the single-mode case. Moreover, $\eta < 1$ has a major impact on the utility of non-classical light, as we have seen for the single-mode analysis of the squeezed-state waveguide tap. Thus it is important to retain its effects in our continuous-time treatment.

field illuminating the active surface of the photodetector to be $E(t)e^{-j\omega_0 t}$ in the semiclassical theory, and the quantum field operator illuminating that surface to be $\hat{E}(t)e^{-j\omega_0 t}$ in the quantum theory. In both cases these are scalar, positive-frequency entities with units $\sqrt{\text{photons/s}}$. Moreover, from our treatment of field quantization, we will assume the following form for the quantum field's commutator with its adjoint,

$$[\hat{E}(t)e^{-j\omega_0 t}, \hat{E}^\dagger(u)e^{j\omega_0 u}] = [\hat{E}(t), \hat{E}^\dagger(u)] = \delta(t-u). \tag{2}$$

Physically, $\hat{E}(t)$ is an operator that annihilates a photon at time t, and $\hat{E}^\dagger(t)$ is an operator that creates a photon at time t.

We will assume that both the classical field and the state of the quantum field are quasimonochromatic (narrowband), as in our discussion of field quantization and the assumption of the delta-function field commutator. This means that $E(t)e^{-j\omega_0 t}$ is a passband process whose bandwidth $\Delta\omega$ is much smaller than the light beam's center frequency ω_0. In the quantum case we have that the only excited modes in $\hat{E}(t)e^{-j\omega_0 t}$ lie within a bandwidth $\Delta\omega$ of the center frequency ω_0 with $\Delta\omega \ll \omega_0$.[5]

Classical electromagnetic wave theory does not ordinarily work with photon-units fields, as we are doing. Instead, standard shot-noise analyses of photodetection quantify illumination strength in Watts, rather than photons/sec. We have chosen to use the latter units, for the classical case, to maximize the connection to the quantum theory. However, to ensure a proper linkage back to standard shot-noise treatments, we exploit the quasimonochromatic condition to relate $E(t)$ to the short-time average power $P(t)$ falling upon the detector's photosensitive region. Consider an x-polarized, $+z$-going quasimonochromatic (center frequency ω_0) plane wave whose real-valued electric (V/m units) and magnetic (A/m units) fields are

$$\vec{E}(\vec{r},t) = \text{Re}(\vec{\mathbf{E}}(t-z/c)e^{-j\omega_0 t}) \quad \text{and} \quad \vec{H}(\vec{r},t) = \text{Re}(\vec{\mathbf{H}}(t-z/c)e^{-j\omega_0 t}), \tag{3}$$

where c is the speed of light and the complex fields, $\vec{\mathbf{E}}(t)$ and $\vec{\mathbf{H}}(t)$, are

$$\vec{\mathbf{E}}(t) \equiv E(t)\vec{i}_x \quad \text{and} \quad \vec{\mathbf{H}}(t) \equiv H(t)\vec{i}_y. \tag{4}$$

Here, $E(t)$ and $H(t)$ are baseband (bandwidth $\Delta\omega \ll \omega_0$) complex fields obeying $H(t) = \sqrt{\epsilon_0/\mu_0}\, E(t)$ with ϵ_0 and μ_0 being the permittivity and permeability of free space, and $\vec{i}_x$ and $\vec{i}_y$ are unit vectors in the x and y directions, respectively. In terms of the complex-valued Poynting vector, $\vec{\mathbf{S}}(\vec{r},t) \equiv \vec{\mathbf{E}}(t) \times \vec{\mathbf{H}}^*(t)/2$ we then have that the short-time average (T_a-sec, with $\omega_0 T_a \gg 1$ and $\Delta\omega T_a \ll 1$) power flowing into a region $\mathcal{A}$ of area A in a constant-z plane at time t is

$$P(t) \approx \frac{1}{T_a}\int_{t-T_a}^{t} \mathrm{d}u \int_{\mathcal{A}} \mathrm{d}x\,\mathrm{d}y\,\text{Re}[\vec{\mathbf{S}}(\vec{r},u)]\cdot\vec{i}_z \tag{5}$$

$$\approx \frac{\text{Re}[E(t-z/c)H^*(t-z/c)]A}{2} = \sqrt{\frac{\epsilon_0}{\mu_0}}\frac{|E(t-z/c)|^2}{2}, \tag{6}$$

[5]Alternatively, we could say that $E(t)$ is a baseband field with bandwidth $\Delta\omega \ll \omega_0$ and that $\hat{E}(t)$ is a baseband field operator whose excitation is confined to a bandwidth $\Delta\omega \ll \omega_0$.

where the first approximate equality is due to $\omega_0 T_a \gg 1$ and the second approximate equality follows from $\Delta\omega T_a \ll 1$.[6] Our quasimonochromatic photon-units classical field will be chosen such that when it illuminates a photodetector whose active region lies in the $z = 0$ plane, the short-time average power flowing into that active region at time t is

$$P(t) = \hbar\omega_0 |E(t)|^2. \tag{7}$$

Comparing Eqs. (6) and (7) shows the simple rescaling that can be used to go from SI units to photon units for a linearly-polarized, quasimonochromatic, $+z$-going plane wave, viz.,

$$E(t)|_{\text{photon units}} = \sqrt{\frac{c\epsilon_0 A}{2\hbar\omega_0}}\, E(t)|_{\text{SI units}}, \tag{8}$$

where we have used $c = 1/\sqrt{\epsilon_0\mu_0}$. For the remainder of our work on continuous-time photodetection we will use the photon-units form for the classical baseband field $E(t)$. However, to connect with standard shot-noise theory, we shall employ $P(t)$ for the short-time average power (in Watts) illuminating the photodetector.

Semiclassical Photodetection versus Quantum Photodetection

The continuous-time theory for semiclassical photodetection is as follows. Suppose that the classical, quasimonochromatic, positive-frequency, photon-units field $E(t)e^{-j\omega_0 t}$ illuminates a photodetector that is located in the $z = 0$ plane. Then, given knowledge of the short-time average power, $P(t)$, falling on the detector's photosensitive region during the time interval $0 \le t \le T$,[7] we have that the photocount record, $N(t)$ for $0 \le t \le T$, is an inhomogeneous Poisson counting process (IPCP) with rate function $\lambda(t) = \eta P(t)/\hbar\omega_0$,[8] where $0 \le \eta \le 1$ is the detector's quantum efficiency.

The preceding semiclassical theory is to be contrasted with the following continuous-time theory for quantum photodetection. Suppose that the only excited field modes from the positive-frequency, photon-units field operator, $\hat{E}(t)e^{-j\omega_0 t}$, that illuminates the photodetector fall within a narrow bandwidth $\Delta\omega$ about the center frequency ω_0. Then, the photocurrent $i(t)$ produced by this detector is a classical random process whose statistics are equivalent to those of the photocurrent operator

$$\hat{i}(t) \equiv q\hat{E}'^{\dagger}(t)\hat{E}'(t), \tag{9}$$

[6] Physically, a quasimonochromatic field has baseband modulation of much lower bandwidth than its center frequency. The short-time averaging process integrates over many periods of the center frequency, but does not average out the baseband modulation.

[7] The duration, T, of the measurement interval is much *longer* than the short-time averaging period, T_a, from the previous section. In particular, for the modulated laser fields used in optical communications we will have $\Delta\omega T \gg 1$.

[8] Note that the symbol λ, used here, does *not* refer to wavelength; the rate function has units sec^{-1}.

where $0 \leq \eta \leq 1$ is the photodetector's quantum efficiency, and

$$\hat{E}'(t) \equiv \sqrt{\eta}\,\hat{E}(t) + \sqrt{1-\eta}\,\hat{E}_\eta(t), \tag{10}$$

with $\hat{E}_\eta(t)$ being the baseband operator for a field whose modes are all in their vacuum states. As a result, the photocount record, $N(t)$ versus t for $0 \leq t \leq T$, is a classical random process whose statistics are equivalent to those of the following operator,

$$\hat{N}(t) \equiv \frac{1}{q}\int_0^t \mathrm{d}u\,\hat{\imath}(u) = \int_0^t \mathrm{d}u\,\hat{E}'^\dagger(u)\hat{E}'(u). \tag{11}$$

Note that $\hat{E}'(t)$ is a proper field operator, i.e., we have $[\hat{E}'(t), \hat{E}'^\dagger(u)] = \delta(t-u)$.

These photodetection models are natural continuous-time generalizations of what we saw earlier this semester for the single-mode case. There, semiclassical photodetection led to a Poisson-distributed random variable for the photon count, given the illumination strength. Here, the photocount record versus time is a Poisson random process, given the illumination strength. There, quantum photodetection, with a sub-unity quantum efficiency detector, led to a photon count that realized the number operator measurement for the effective field mode obtained by mixing the incoming signal mode with a vacuum-state operator associated with having $\eta < 1$. Here, we get that the photocount record realizes the photon number operator measurement for the effective-field modes collected over the observation interval. The majority of our work on single-mode photodetection addressed the relationship between its semiclassical and quantum theories. There, we determined the conditions under which semiclassical theory yields the same quantitative predictions as those found from quantum theory, and we presented signatures of cases for which the semiclassical theory did not match what was found from quantum theory. Our primary concern will be the same for continuous-time photodetection, but here we will have to work a bit harder. We already have descriptions for multi-mode number states and multi-mode coherent states, and, without much difficulty, we can and will later develop a description for multi-mode squeezed states. Thus it will be straightforward to examine the statistics of continuous-time quantum photodetection for these interesting classes of field states. More work will be required, however, for the semiclassical case. In particular, because random processes are more complicated than single random variables, and we do *not* assume prior knowledge of Poisson processes, there is some foundational material we need to present before we're ready for a critical comparison of the two theories of continuous-time photodetection.

Poisson Processes and Their Properties

What does it mean to say that a photocount record, $N(t)$ for $t \geq 0$, is an inhomogenous Poisson counting process with rate function $\lambda(t)$? That is the first question we should answer in developing some understanding of Poisson processes and their properties. Here is the definition we need.

Definition: A random process $N(t)$ for $t \geq 0$ is an inhomogeneous Poisson counting process (IPCP) with rate function $\lambda(t)$ if it satisfies all of the following conditions:[9]

- $N(0) = 0$,
- $N(t)$ has statistically independent increments for $t \geq 0$,
- $\Pr(N(t) - N(u) = n) = \dfrac{\left(\int_u^t \mathrm{d}s\,\lambda(s)\right)^n \exp\left(-\int_u^t \mathrm{d}s\,\lambda(s)\right)}{n!}$, for $t \geq u \geq 0$, and $n = 0, 1, 2, \ldots,$

where $\lambda(t) \geq 0$ is a deterministic time function. The first and third conditions are easily visualized. The counting process starts at zero at time $t = 0$, and counts up so that at any later time it can only take on non-negative integer values, as shown for the photocount record on slide 4 (where the initial time is shown to be $t = t_0$ instead of $t = 0$). In fact, as shown on slide 4, the process counts up by unity-height steps. To see that this is so we can use the third condition in our definition—with the assumption that $\lambda(t)$ is finite and continuous—to deduce

$$\Pr(N(t + \Delta t) - N(t) = n) \approx \begin{cases} 1 - \lambda(t)\Delta t, & \text{for } n = 0 \\ \lambda(t)\Delta t, & \text{for } n = 1 \\ 0, & \text{for } n \geq 2, \end{cases} \tag{12}$$

as $\Delta t \to 0$. Hence the rate function $\lambda(t)$ is the probability per unit time that the process $N(t)$ increments by one at time t.[10]

The second condition in our Poisson process definition requires a more elaborate explanation. Consider the semiclosed time interval $(u, t]$, i.e., all times s obeying $0 \leq u < s \leq t$. The *increment* of the random process $N(t)$ associated with this semiclosed interval is defined to be $N(t) - N(u)$. For fixed u and t, this increment is a random variable, viz., it is the difference between the value of the process at time u and its value at a later time t. The third condition in our definition of the Poisson counting process shows that this increment is Poisson distributed with mean value

$$\langle N(t) - N(u) \rangle = \int_u^t \mathrm{d}s\,\lambda(s), \quad \text{for } t \geq u \geq 0. \tag{13}$$

[9]The process is said to be homogeneous if $\lambda(t)$ is a constant.

[10]Semiclassical photodetection theory for a deterministic field with short-time average power $P(t)$ illuminating the detector sets this rate function equal to $\eta P(t)/\hbar\omega_0$. This result makes intuitive sense in that: $P(t)$ is the number Joules per second illuminating the detector at time t; $\hbar\omega_0$ is the number of Joules per photon at the center frequency of the quasimonochromatic light; and η is the average number of detections per photon hitting the detector. Thus $\eta P(t)/\hbar\omega_0$ is the average number of detections at time t in response to the illumination power $P(t)$.

What does it mean to say that the process $N(t)$ has statistically independent increments? It's simple. Suppose that $(u_1, t_1], (u_2, t_2], \ldots, (u_K, t_K]$ are a set of K *non-overlapping* time intervals, i.e., $0 \leq u_1 \leq t_1 \leq u_2 \leq t_2 \leq \ldots \leq u_K \leq t_K$. Then their associated increments, $\{ N(t_k) - N(u_k) : 1 \leq k \leq K \}$, are statistically independent random variables. Note that because we are using semiclosed intervals, $(u_k, t_k]$ and $(u_{k+1}, t_{k+1}]$ are non-overlapping even if $u_{k+1} = t_k$, because $(u_k, t_k]$ includes its upper limit whereas $(u_{k+1}, t_{k+1}]$ does *not* include its lower limit.

The supplementary notes on random processes state that one way to view a random process $N(t)$ for $t \geq 0$ is as a collection of joint random variables that are indexed by the time parameter t. It follows that a complete statistical characterization of that random process must be able to provide the joint statistics of any set of its time samples. The definition we have given for the inhomogeneous Poisson counting process is quite succinct. Nevertheless, as we will now show, it *does* provide a complete statistical characterization. Because $N(t)$ only takes on non-negative integer values, a complete statistical characterization of this process must provide information that enables us to calculate the joint probability mass function $\Pr(N(t_1) = n_1, N(t_2) = n_2, \ldots, N(t_K) = n_K)$ for non-negative integers $\{ n_k : 1 \leq k \leq K \}$, an arbitrary set of non-negative time samples $\{ t_k : 1 \leq k \leq K \}$ and all positive integers K. Without loss of generality, we will assume that $0 \leq t_1 < t_2 < \cdots < t_K$, so that we can write the desired joint probability mass function in terms of increments of the process, viz.,

$$\Pr(N(t_1) = n_1, N(t_2) = n_1, \ldots, N(t_K) = n_K) =$$

$$\Pr(N(t_1) = n_1, \Delta N(t_2, t_1) = m_{2,1}, \Delta N(t_3, t_2) = m_{3,2}, \ldots, \Delta N(t_K, t_{K-1}) = m_{K,K-1}),$$

where $\Delta N(t, u) \equiv N(t) - N(u)$ is the increment associated with the interval $(u, t]$, and $m_{k+1,k} \equiv n_{k+1} - n_k$. Now, because the time intervals $(0, t_1]$, $(t_1, t_2]$,..., $(t_{K-1}, t_K]$ are non-ovelapping and $N(0) = 0$ we can exploit the statistical independence and Poisson distributions for their associated increments to get

$$\Pr(N(t_1) = n_1, N(t_2) = n_1, \ldots, N(t_K) = n_K)$$

$$= \prod_{k=1}^{K} \Pr(N(t_k) - N(t_{k-1}) = n_k - n_{k-1}) \tag{14}$$

$$= \prod_{k=1}^{K} \frac{\left(\int_{t_{k-1}}^{t_k} \mathrm{d}s \, \lambda(s) \right)^{n_k - n_{k-1}} \exp\left(- \int_{t_{k-1}}^{t_k} \mathrm{d}s \, \lambda(s) \right)}{(n_k - n_{k-1})!}, \tag{15}$$

with $t_0 = 0$ and $n_0 = 0$, completing our demonstration that the definition we have given for the Poisson process provides a complete statistical characterization of that process.

First and Second Moments of Semiclassical Photodetection

The complete statistical characterization of a random process is a powerful tool for analyzing the performance of communication and measurement systems in which this process appears as an input. Very often, as discussed in our work with random variables, we content ourselves with the useful but more limited information provided by first and second moments. In the case of a single random variable, these are its mean value and variance. For random processes, this will involve mean functions and covariance functions. Thus, to enable our comparison of the first and second moments predicted by the semiclassical and quantum theories of continuous-time photodetection, we now derive the semiclassical versions of these moment functions for both the photocount process $N(t)$, and the photocurrent process[11]

$$i(t) = q\frac{\mathrm{d}N(t)}{\mathrm{d}t}. \tag{16}$$

In the section that follows, we will do the same for the quantum theory.

Let us first assume that the illumination power, $P(t)$, is known at all times; we will return to allow for randomness in $P(t)$ after we handle the deterministic case. The mean function of $N(t)$ is $m_N(t) \equiv \langle N(t)\rangle$. It is a deterministic function of time that, at each time instant, gives the mean value of the random variable obtained by sampling the counting process at that time instant. From our definition of the IPCP, we have that

$$m_N(t) \equiv \langle N(t)\rangle = \langle [N(t) - N(0)]\rangle = \int_0^t \mathrm{d}s\,\lambda(s) = \frac{\eta}{\hbar\omega_0}\int_0^t \mathrm{d}s\,P(s), \quad \text{for } t \geq 0, \tag{17}$$

where the second equality follows from $N(0) = 0$ and the third equality follows from the Poisson distribution for the increment $N(t) - N(0)$. The mean function of the photocurrent $i(t)$ is easily found from the preceding result. We have that

$$m_i(t) \equiv \langle i(t)\rangle = \left\langle q\frac{\mathrm{d}N(t)}{\mathrm{d}t}\right\rangle = q\frac{\mathrm{d}\langle N(t)\rangle}{\mathrm{d}t} = q\lambda(t) = \frac{q\eta P(t)}{\hbar\omega_0}, \tag{18}$$

where the third equality follows by interchanging the order of the two linear operations, viz., ensemble averaging (integration in a probability space) and differentiation in time. We see that the mean value of the photocurrent at time t equals the electron charge multiplied by the average number of photodetections at that time. Note that we have not, and will not, put a $t \geq 0$ condition on the average photocurrent. Under constant illumination power, the counting process must start at some finite t_0, because if it were begun at $t_0 = -\infty$ then an infinite number of counts would have accumulated by any finite time. The photocurrent, on the other hand, is not subject

[11] If $N(t)$ is an inhomogeneous Poisson counting process on $t_0 \leq t$, with counts at times $t_1 < t_2 < t_3 < \cdots$, it can be written as $N(t) = \sum_{n=1}^{\infty} u(t - t_n)$, where $u(t)$ is the unit-step function. Hence, $i(t) = q\sum_{n=1}^{\infty}\delta(t - t_n)$ is a train of area-q impulses, as shown on slide 3.

to this problem, cf. $\sum_n u(t-t_n)$ for a counting process with $\sum_n \delta(t-t_n)$ for its derivative when the counts can occur over the entire infinite interval $-\infty < t_n < \infty$.

The covariance function of $N(t)$ is

$$K_{NN}(t,u) \equiv \langle \Delta N(t)\Delta N(u)\rangle, \quad \text{for } t,u \geq 0, \tag{19}$$

where $\Delta N(s) \equiv N(s) - \langle N(s)\rangle$ is the noise part of the process at time s. When $t = u$ the covariance function gives the variance of the counting process at time t. For $t \neq u$, the covariance function gives a measure of the statistical dependence between $N(t)$ and $N(u)$, see the supplementary notes on random processes for more information. The covariance function for an IPCP is easily found from the independence and Poisson distribution of its increments. Specifically, for $t, u \geq 0$ we have that

$$K_{NN}(t,u) \equiv \langle \Delta N(t)\Delta N(u)\rangle = \langle \Delta N[\max(t,u)]\Delta N[\min(t,u)]\rangle \tag{20}$$

$$= \langle (\Delta N[\max(t,u)] - \Delta N[\min(t,u)])\Delta N[\min(t,u)]\rangle + \langle (\Delta N[\min(t,u)])^2\rangle \tag{21}$$

$$= \langle \Delta N[\max(t,u)] - \Delta N[\min(t,u)]\rangle\langle \Delta N[\min(t,u)]\rangle + \langle (\Delta N[\min(t,u)])^2\rangle \tag{22}$$

$$= \langle (\Delta N[\min(t,u)])^2\rangle = \langle N[\min(t,u)]\rangle \tag{23}$$

$$= \int_0^{\min(t,u)} \mathrm{d}s\,\lambda(s) = \frac{\eta}{\hbar\omega_0}\int_0^{\min(t,u)} \mathrm{d}s\,P(s), \tag{24}$$

where the fourth equality follows from the independence of the increments, the fifth equality follows from the zero-mean nature of $\Delta N(s)$ for all s, and the sixth equality follows from the increments being Poisson distributed. Note that when $t = u$ our covariance result reduces to

$$K_{NN}(t,t) = \langle [\Delta N(t)]^2\rangle = \langle N(t)\rangle, \quad \text{for } t \geq 0, \tag{25}$$

as it must, because $N(t) - N(0) = N(t)$ is Poisson distributed given knowledge of $\{\, P(s) : 0 \leq s \leq t\,\}$.

The covariance of the photocurrent process—$K_{ii}(t,u) \equiv \langle \Delta i(t)\Delta i(u)\rangle$, where $\Delta i(s) \equiv i(s) - \langle i(s)\rangle$ is the noise part of the process at time s—is found by differentiating the result we have obtained for $K_{NN}(t,u)$. The details are omitted, but the result is as follows,

$$K_{ii}(t,u) = q^2\frac{\partial^2 K_{NN}(t,u)}{\partial t\,\partial u} = q^2\frac{\eta P(t)}{\hbar\omega_0}\delta(t-u) = q\langle i(t)\rangle\delta(t-u), \tag{26}$$

where we shall use the final result without the restriction to $t, u \geq 0$, cf. our discussion of the mean photocurrent.

Because it is seldom the case that the power illuminating the photodetector will be precisely known, i.e., completely deterministic, it will be important for us to find the mean functions and covariance functions for the photocount record and the photocurrent when $P(t)$ is a random process. We shall not present the steps involved—suffice it to say the procedure of iterated expectation is employed—but content ourselves with the answers, viz.,

$$m_N(t) = \frac{\eta}{\hbar\omega_0}\int_0^t \mathrm{d}s\, P(s), \quad \text{for } t \geq 0, \tag{27}$$

$$m_i(t) = \frac{q\eta\langle P(t)\rangle}{\hbar\omega_0}, \quad \text{for all } t, \tag{28}$$

for the mean functions and

$$K_{NN}(t,u) = \underbrace{\langle \Delta N[\min(t,u)]\rangle}_{\text{shot noise}} + \underbrace{\int_0^t \mathrm{d}s \int_0^u \mathrm{d}s'\, \frac{\eta^2 K_{PP}(s,s')}{(\hbar\omega_0)^2}}_{\text{excess noise}}, \quad \text{for } t,u \geq 0, \tag{29}$$

$$K_{ii}(t,u) = \underbrace{q\langle i(t)\rangle\delta(t-u)}_{\text{shot noise}} + \underbrace{\frac{q^2\eta^2 K_{PP}(t,u)}{(\hbar\omega_0)^2}}_{\text{excess noise}}, \quad \text{for all } t,u. \tag{30}$$

In these covariance expressions, $\langle P(t)\rangle$ is the mean illumination power and $K_{PP}(t,u) \equiv \langle \Delta P(t)\Delta P(u)\rangle$, with $\Delta P(s) \equiv P(s) - \langle P(s)\rangle$ for $s = t,u$, is the covariance function of the illumination power. That the first terms in the photocount record and photocurrent covariance functions represent shot noise and the second terms in these expressions represent excess noise are easily justified. When the illumination power is deterministic (non-random), we have that $\Delta N(s) = 0$ with probability one for all s, hence $K_{PP}(t,u) = 0$ for all t,u and the photocount record and photocurrent covariance functions collapse to the terms we have labeled as being due to shot noise. But when the illumination power is non-random, the *only* source of photocount and photocurrent noise—in the semiclassical theory—is shot noise, so our shot noise labels are indeed appropriate. Moreover, any other noise, viz., the second terms in the preceding photocount and photocurrent covariances, must be due to excess noise, i.e., the randomness in the illumination power. Thus our excess noise labels also make sense.

First and Second Moments of Quantum Photodetection

In comparison with the work we performed to obtain the first and second moments of semiclassical photodetection, very little is required for us to get the comparable expressions for quantum photodetection. We have that the mean functions obey

$$m_N(t) = \int_0^t \mathrm{d}s\, q\langle \hat{E}'^\dagger(s)\hat{E}'(s)\rangle = \int_0^t \mathrm{d}s\, q\eta\langle \hat{E}^\dagger(s)\hat{E}(s)\rangle, \quad \text{for } t \geq 0, \tag{31}$$

$$m_i(t) = q\langle \hat{E}'^\dagger(t)\hat{E}'(t)\rangle = q\eta\langle \hat{E}^\dagger(t)\hat{E}(t)\rangle, \quad \text{for all } t. \tag{32}$$

For the covariance functions we find that

$$
\begin{aligned}
K_{NN}(t,u) &= \langle N[\min(t,u)]\rangle \\
&+ \int_0^t \mathrm{d}s \int_0^u \mathrm{d}s'\,\eta^2 \left[\langle \hat{E}^\dagger(s)\hat{E}^\dagger(s')\hat{E}(s)\hat{E}(s')\rangle - \langle \hat{E}^\dagger(s)\hat{E}(s)\rangle\langle \hat{E}^\dagger(s')\hat{E}(s')\rangle\right], \\
&\quad \text{for } t,u \ge 0, \qquad (33) \\
K_{ii}(t,u) &= q\langle i(t)\rangle\delta(t-u) \\
&+ q^2\eta^2 \left[\langle \hat{E}^\dagger(t)\hat{E}^\dagger(u)\hat{E}(t)\hat{E}(u)\rangle - \langle \hat{E}^\dagger(t)\hat{E}(t)\rangle\langle \hat{E}^\dagger(u)\hat{E}(u)\rangle\right], \\
&\quad \text{for all } t,u. \qquad (34)
\end{aligned}
$$

These quantum mean and covariance formulas bear a striking resemblance to their semiclassical counterparts, as can better be seen by using $P(t) = \hbar\omega_0 E^*(t)E(t)$, where $E(t)$ is the classical photon-units positive-frequency field, in Eqs. (27)–(30). Despite their formal similarities, there *are*, however, striking differences between the behavior of the semiclassical and quantum covariance functions, as we will now demonstrate.

Direct Detection Signatures of Non-Classical Light

In our study of single-mode photodetection we showed that any illumination whose density operator had a proper P representation—i.e., the illumination state was a coherent state or a random mixture of coherent states—gave quantum photodetection statistics that coincided exactly with those of the semiclassical theory. Thus we called such states *classical*, and deemed all other states to be non-classical. We also exhibited some signatures of single-mode non-classical light, viz., quantum photodetection statistics for such states that cannot be explained by the semiclassical theory. We will conclude today's lecture by extending that work, at least for direct detection, to continuous-time photodetection.

Suppose the light that falls on the active region of the photodetector is in the coherent state $|E(t)\rangle$.[12] Then, from the results of the previous section we have that

$$
m_N(t) = \eta \int_0^t \mathrm{d}s\,\langle \hat{E}(s)^\dagger \hat{E}(s)\rangle = \eta \int_0^t \mathrm{d}s\,|E(s)|^2 = \int_0^t \mathrm{d}s\,\frac{\eta P(s)}{\hbar\omega_0}, \quad \text{for } t \ge 0, \qquad (35)
$$

[12]This means that $\hat{E}(u)|E(t)\rangle = E(u)|E(t)\rangle$, for all t,u, where $E(s)$ versus s is a deterministic baseband classical photon-units field.

and

$$K_{NN}(t,u) = \eta \int_0^{\min(t,u)} \mathrm{d}s \, \langle \hat{E}(s)^\dagger \hat{E}(s) \rangle = \eta \int_0^{\min(t,u)} \mathrm{d}s \, |E(s)|^2 \tag{36}$$

$$= \int_0^{\min(t,u)} \mathrm{d}s \, \frac{\eta P(s)}{\hbar\omega_0}, \quad \text{for } t, u \geq 0, \tag{37}$$

where $P(s) = \hbar\omega_0 |E(s)|^2$ is the short-time average power at time s. Not surprisingly, these quantum results coincide with those from the semiclassical theory when the illumination is deterministic. It follows that if the input state is a random mixture of coherent states, we will get the *same* mean function and covariance function for the photocount record regardless of whether we use the quantum theory or the semiclassical theory where, in the latter case, the quantum theory's P function is used to give the statistics of the classical stochastic process $\{ E(s) : s \geq 0 \}$. Similar results can be developed for the relation between the quantum and semiclassical theories for the photocurrent's statistics, but we will omit them. Likewise, we will not try to be explicit about the P function for a general classical state in continuous time as it involves more advanced stochastic processes material, i.e., characteristic functionals, than we care to discuss here. Instead, we shall dwell on two non-classical light signatures that can appear in direct detection: sub-Poissonian photocount records and sub-shot-noise photocurrent spectra.

In semiclassical theory the variance of the photocount at time $t \geq 0$ is given by

$$\langle [\Delta N(t)]^2 \rangle = K_{NN}(t,t) = \int_0^t \mathrm{d}s \, \frac{\eta P(s)}{\hbar\omega_0} + \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \frac{\eta^2 K_{PP}(s,s')}{(\hbar\omega_0)^2}, \tag{38}$$

where, as noted earlier, the first term on the right is due to shot noise and the second term on the right is due to excess noise. In particular, the excess term *cannot* be negative, owing to fundamental properties of classical covariance functions. Thus semiclassical theory *always* predicts

$$\langle [\Delta N(t)]^2 \rangle \geq \langle N(t) \rangle = \int_0^t \mathrm{d}s \, \frac{\eta P(s)}{\hbar\omega_0}, \tag{39}$$

i.e., the photocount variance is at least equal to the Poissonian limit. In quantum photodetection theory, however, we get

$$\langle [\Delta N(t)]^2 \rangle = K_{NN}(t,t) = \eta \int_0^t \mathrm{d}s \, \langle \hat{E}^\dagger(s) \hat{E}(s) \rangle$$

$$+ \ \eta^2 \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \left[\langle \hat{E}^\dagger(s) \hat{E}^\dagger(s') \hat{E}(s) \hat{E}(s') \rangle - \langle \hat{E}^\dagger(s) \hat{E}(s) \rangle \langle \hat{E}^\dagger(s') \hat{E}(s') \rangle \right]. \tag{40}$$

Here, the shot-noise plus excess-noise interpretation of the terms on the right does *not*, in general, apply. Indeed, non-classical states can make the second term negative, giving rise to a photocount variance that is sub-Poissonian, i.e., less than the mean photocount. The multi-mode photon number state provides the simplest such example. Suppose, for a given $t \geq 0$, that the state contains exactly $n > 0$ photons within the time interval $0 \leq s \leq t$. Then it can be shown that the photocount $N(t)$ is sub-Poissonian for all $0 < \eta \leq 1$ because,

$$\langle N(t)\rangle = m_N(t) = \eta n \quad \text{and} \quad \langle[\Delta N(t)]^2\rangle = K_{NN}(t,t) = n\eta(1-\eta) < \langle N(t)\rangle. \tag{41}$$

To exhibit a second non-classical signature for direct detection, let us assume continuous-wave illumination—be it classical or quantum—that is statistically stationary to at least second order. In this case the photocurrent's mean function $m_i(t)$ is a constant (time independent), and its covariance function $K_{ii}(t,u)$ only depends on the time difference between the two samples. The latter condition permits us to define a photocurrent fluctuation spectrum by

$$\mathcal{S}_{ii}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, K_{ii}(\tau)e^{-j\omega\tau}, \tag{42}$$

where

$$K_{ii}(\tau) \equiv \langle \Delta i(t+\tau)\Delta i(t)\rangle \tag{43}$$

is, by assumption, independent of t. It is shown in the supplementary notes on random processes that $\mathcal{S}_{ii}(\omega)$ is the photocurrent's noise spectrum, viz., its mean-squared fluctuation strength per unit bilateral bandwidth at frequency ω. As such it must obey $\mathcal{S}_{ii}(-\omega) = \mathcal{S}_{ii}(\omega) \geq 0$ for all ω. Semiclassical photodetection, with statistically stationary illumination, gives

$$K_{ii}(\tau) = q\langle i\rangle\delta(\tau) + \frac{q^2\eta^2 K_{PP}(\tau)}{(\hbar\omega_0)^2}, \tag{44}$$

for its photocurrent covariance function, which leads to

$$\mathcal{S}_{ii}(\omega) = q\langle i\rangle + \frac{q^2\eta^2 \mathcal{S}_{PP}(\omega)}{(\hbar\omega_0)^2}, \tag{45}$$

for its photocurrent noise spectrum, where stationarity ensures that the mean photcurrent is time independent. The first term on the right in Eq. (45) is due to shot noise; it is a white noise, i.e., constant at all frequencies. The second term on the right in Eq. (45) is due to excess noise; it must be non-negative. In quantum photodetection, with statistically stationary illumination, the constraint on the photocurrent's noise spectrum becomes $\mathcal{S}_{ii}(-\omega) = \mathcal{S}_{ii}(\omega) \geq 0$ for all ω. Thus, when a photocurrent spectrum falls below the shot-noise level, viz., when

$$0 \leq \mathcal{S}_{ii}(\omega) < q\langle i\rangle, \tag{46}$$

we have an observation that cannot be explained by semiclassical theory. The continuous-time version of the single-mode squeezed state will be our key example for sub-shot-noise photocurrent spectra. This will be better seen in the next lecture, when we treat continuous-time homodyne detection, and even later this semester, when we examine how non-classical light can be generated through nonlinear optics.

The Road Ahead

In the next lecture we shall complete our quick trip through the semiclassical and quantum theories of continuous-time photodetection by studying coherent detection techniques, i.e., homodyne and heterodyne detection. We shall see that light beams which are in classical states—coherent states or their classically-random mixtures–give the same coherent-detection statistics in the semiclassical and quantum theories. We shall also exhibit some coherent-detection signatures of non-classical light.