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Continuous-time theories of coherent detection: semiclassical and quantum

Introduction

Today we will complete our two-lecture treatment of semiclassical versus quantum photodetection theory in continuous time, focusing our attention on the coherent detection scenarios of homodyne and heterodyne detection. As we did last time for direct detection, we will build these theories in a scalar-wave, quasimonochromatic framework in which there is no (x, y) dependence for the fields illuminating the active region of the photodetector.[1] The particular tasks we have set for today's lecture are like those we pursued last time: develop the semiclassical and quantum photodetection statistical models for homodyne and heterodyne detection, and exhibit some continuous-time signatures of non-classical light. However, because the signatures that we will examine rely on noise spectral densities, it will be useful for us to back up and elaborate on the direct-detection photocurrent noise spectrum that we considered briefly in the Lecture 18.

Semiclassical versus Quantum Photocurrent Statistics

For the almost-ideal photodetector—perfect, except for its $0 < \eta \leq 1$ quantum efficiency—the semiclassical theory of photodetection states that, given the illumination power $\{\, P(t) : -\infty < t < \infty \,\}$, the photocurrent $\{\, i(t) : -\infty < t < \infty \,\}$ is an inhomogeneous Poisson impulse train. In particular, if $\{\, N(t) : t_0 \leq t \,\}$ is the photocount record starting at time t_0, then

$$i(t) = q\frac{\mathrm{d}N(t)}{\mathrm{d}t}, \quad \text{for } t \geq t_0. \tag{1}$$

The photocount record is a staircase function, $\sum_n u(t - t_n)$, comprised of unit height steps located at the photodetection event times, $\{\, t_n : 1 \leq n < \infty \,\}$. Thus the photocurrent is a train of area-q impulses, $q\sum_n \delta(t - t_n)$, that are located at those

[1] For the quantum case, this means that only the normally-incident plane wave components of the incident field operator have non-vacuum states.

photodetection event times. For both processes, it is the photodetection event times that provide all the information. So, because these times are Poisson distributed in the semiclassical theory, given the illumination power, $N(t)$ is a Poisson counting process and $i(t)$ is a Poisson impulse train. In both cases the rate function is $\lambda(t) = \eta P(t)/\hbar\omega_0$, where $P(t) = \hbar\omega_0|E(t)|^2$ gives the short-time average power of the quasimonochromatic illumination in terms of the classical, photon-units, baseband complex field $E(t)$.

The quantum theory for the photocurrent produced our almost-ideal detector is as follows. The observed classical $i(t)$ has statistics that are identical to those of the photocurrent operator

$$\hat{i}(t) \equiv q\hat{E}'^{\dagger}(t)\hat{E}'(t), \tag{2}$$

where

$$\hat{E}'(t) \equiv \sqrt{\eta}\,\hat{E}(t) + \sqrt{1-\eta}\,\hat{E}_\eta(t). \tag{3}$$

Here, $\hat{E}(t)$ and $\hat{E}_\eta(t)$ are baseband field operators representing the illumination and the effect of sub-unity quantum efficiency, respectively. They commute with each other and with each other's adjoint and satisfy the canonical commutation relations

$$\left[\hat{E}(t), \hat{E}^{\dagger}(u)\right] = \delta(t-u) \quad \text{and} \quad \left[\hat{E}_\eta(t), \hat{E}_\eta^{\dagger}(u)\right] = \delta(t-u). \tag{4}$$

The modes associated with $\hat{E}(t)$ may be in arbitrary states, but those associated with $\hat{E}_\eta(t)$ are in their vacuum states. When $\hat{E}(t)$ is in the coherent state $|E(t)\rangle$, the photocurrent becomes an inhomogeneous Poisson impulse train with rate function $\lambda(t) = \eta|E(t)|^2$, and we recover the semiclassical theory by identifying the coherent-state eigenfunction $\{\, E(t) : -\infty < t < \infty \,\}$ as the classical baseband field, in keeping with[2]

$$\langle E(t)|\hat{E}(u)|E(t)\rangle = E(u), \quad \text{for } -\infty < u < \infty. \tag{5}$$

When $\hat{E}(t)$ is in a classically-random mixture of coherent states—so that its density operator has a proper P representation—the quantum theory again reduces to the semiclassical theory with $E(t)$ being a random process whose statistics are given by the P function. We call such states *classical*; all other states are therefore non-classical. It turns out that all non-classical states exhibit quantum photodetection statistics in at least one of the three basic configurations—direct, homodyne, or heterodyne detection—that cannot be explained by semiclassical theory.[3] In the rest of this lecture we shall limit our attention to coherent detection, and, moreover, focus on

[2]This equation reveals a subtle defect in our coherent-state notation. It would be better, but much less compact, to write the coherent state as $|\{\, E(t) : -\infty < t < \infty \,\}\rangle$, to indicate that it is an eigenstate of the field operator at *all* times with an eigenvalue, at time u, that is given by sampling its associated eigenfunction, $\{\, E(t) : -\infty < t < \infty \,\}$, at time $t = u$.

[3]We proved this statement for the single-mode case by showing that the statistics of heterodyne detection determine the density operator. The same can be shown to be true for the continuous-time case, e.g., by means of a modal expansion and our previous proof, but we will not supply the details.

the photocurrent noise spectrum that is observed when the illumination is statistically stationary.[4]

Photocurrent Statistics for Statistically Stationary Sources

The notion of statistical stationarity has to do with invariance to shifts in the time origin. When a real-valued, classical random process $x(t)$ is stationary (to at least second order), its mean function will be a constant,

$$\langle x(t)\rangle = \text{constant} \equiv \langle x\rangle, \tag{6}$$

and its covariance function will depend only on the time difference between the two noise samples,

$$\langle \Delta x(t+\tau)\Delta x(t)\rangle = \text{function of } \tau \text{ only} \equiv K_{xx}(\tau), \tag{7}$$

where $\Delta x(t) \equiv x(t) - \langle x\rangle$ is the noise part of the process, i.e., a zero-mean random process equal to the difference between the original process $x(t)$ and its mean $\langle x\rangle$.[5]

In semiclassical photodetection, the photocurrent $i(t)$ will be stationary if the illumination power $P(t)$ is stationary, in which case we get

$$\langle i\rangle = \frac{q\eta\langle P\rangle}{\hbar\omega_0} \quad \text{and} \quad K_{ii}(\tau) = \underbrace{q\langle i\rangle\delta(\tau)}_{\text{shot noise}} + \underbrace{\frac{q^2\eta^2 K_{PP}(\tau)}{(\hbar\omega_0)^2}}_{\text{excess noise}}, \tag{8}$$

where our identification of the noise contributions was justified in Lecture 18. In quantum photodetection theory, the corresponding results for a statistically stationary field state[6] are as follows:

$$\langle i\rangle = q\eta\langle \hat{E}^\dagger(0)\hat{E}(0)\rangle, \tag{9}$$

and

$$K_{ii}(\tau) = q\langle i\rangle\delta(\tau) + q^2\eta^2\left[\langle \hat{E}^\dagger(\tau)\hat{E}^\dagger(0)\hat{E}(\tau)\hat{E}(0)\rangle - \langle \hat{E}^\dagger(\tau)\hat{E}(0)\rangle^2\right], \tag{10}$$

where we exploited stationarity in the bracketed term, cf. the general result for the non-stationary case given in Lecture 18.

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[4]Statistical stationarity, of either classical stochastic processes or a mixed quantum state, should not be confused with the notion of stationary quantum states for a system governed by a given Hamiltonian.

[5]Processes that obey these two properties are said to exhibit wide-sense stationarity, which is a weaker property that second-order stationarity. Processes that violate Eq. (6) but satisfy Eq. (7) are said to be covariance stationary.

[6]For our purposes, a statistically stationary field state is one that yields a quantum photodetection theory photocurrent whose mean and covariance satisfy Eqs. (6) and (7), respectively.

The photocurrent covariance function $K_{ii}(\tau)$ quantifies the noise strength in $i(t)$ at any time through its value at $\tau = 0$. This is because it gives us the noise variance (mean-squared fluctuation strength in $i(t)$):

$$\langle[\Delta i(t)]^2\rangle = K_{ii}(0). \tag{11}$$

The photocurrent covariance also provides a measure of the temporal behavior of the noise in $i(t)$ through the correlation coefficient,

$$\rho_{ii}(\tau) \equiv K_{ii}(\tau)/K_{ii}(0). \tag{12}$$

Indeed, when $i(t)$ is known, the linear estimator

$$\tilde{i}(t+\tau \mid t) \equiv \langle i\rangle + \rho_{ii}(\tau)[i(t) - \langle i\rangle], \tag{13}$$

has mean-squared error

$$\langle[\tilde{i}(t+\tau \mid t) - i(t+\tau)]^2\rangle = \langle[\Delta i(t)]^2\rangle[1 - \rho_{ii}^2(\tau)]. \tag{14}$$

So, when $|K_{ii}(\tau)| \approx K_{ii}(0)$, knowledge of $i(t)$ allows us to make a very low mean-squared error prediction about $i(t+\tau)$ by means of this linear estimator. Conversely, when $|K_{ii}(\tau)| \ll K_{ii}(0)$, our linear estimate of $i(t+\tau)$ based on knowledge of $i(t)$ is little better than guessing $i(t+\tau) = \langle i\rangle$.[7] More importantly, for what will follow today, the photocurrent covariance function provides, through its Fourier transform, information about the frequency content in the photocurrent fluctuations.

The Photocurrent-Noise Spectral Density

For statistically stationary illumination, the photocurrent-noise spectral density is defined to be the Fourier transform of its covariance function, i.e.,

$$\mathcal{S}_{ii}(\omega) \equiv \int_{-\infty}^{\infty} \mathrm{d}\tau\, K_{ii}(\tau)e^{-j\omega\tau}, \tag{15}$$

from which the covariance function may be recovered via the inverse transform integral,

$$K_{ii}(\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{S}_{ii}(\omega)e^{j\omega t}. \tag{16}$$

It is easily seen, from its definition, that $K_{ii}(\tau)$ must be a real-valued, even function of τ. It then follows that $\mathcal{S}_{ii}(\omega)$ must also be a real-valued, even function of its argument, ω. As detailed in the supplementary notes on random processes—and

[7]It can be shown that Eq. (13) is the minimum mean-squared error (MMSE) *linear* estimator of $i(t+\tau)$ given $i(t)$. If $i(t)$ is a Gaussian random process, then Eq. (13) provides the lowest mean-squared error of *any* estimator for $i(t+\tau)$ based on knowledge of $i(t)$.

shown schematically on slide 5—linear time-invariant filtering of $i(t)$ results in an output current

$$i'(t) \equiv \int_{-\infty}^{\infty} \mathrm{d}s\, i(s)h(t-s), \tag{17}$$

that is statistically stationary and has

$$\langle i' \rangle = H(0)\langle i \rangle \quad \text{and} \quad \mathcal{S}_{i'i'}(\omega) = \mathcal{S}_{ii}(\omega)|H(\omega)|^2, \tag{18}$$

where $h(t)$ is the filter's impulse response and $H(\omega)$ is its frequency response.[8] Taking $H(\omega)$ to be the ideal bandpass filter with unilateral bandwidth $\Delta\omega$ about center frequency ω_c, i.e.,

$$H(\omega) = \begin{cases} 1, & \text{for } |\omega \pm \omega_c| \leq \Delta\omega/2, \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

we can develop a physical interpretation for $\mathcal{S}_{ii}(\omega)$.

Suppose that $\mathcal{S}_{ii}(\omega)$ is a continuous function of frequency, and let us evaluate the variance (mean-squared fluctuation strength) in the output current $i'(t)$ obtained by passing $i(t)$ through the preceding bandpass filter as $\Delta\omega \to 0$. We have that

$$\langle [\Delta i'(t)]^2 \rangle = K_{i'i'}(0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{S}_{i'i'}(\omega) = 2\int_{0}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{S}_{i'i'}(\omega) \tag{20}$$

$$= 2\int_{0}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{S}_{ii}(\omega)|H(\omega)|^2 = 2\int_{\omega_c-\Delta\omega/2}^{\omega_c+\Delta\omega/2} \frac{\mathrm{d}\omega}{2\pi}\, \mathcal{S}_{ii}(\omega) \approx (\Delta\omega/\pi)\mathcal{S}_{ii}(\omega_c). \tag{21}$$

Variances cannot be negative and ω_c was arbitrary, so this calculation shows that $\mathcal{S}_{ii}(\omega) \geq 0$ prevails at all frequencies. However, it is the physical interpretation of Eq. (21) that we are really seeking. For the extremely narrowband passband filter, we have that $i'(t)$ consists of those components of the photocurrent $i(t)$ that lie with a $2\Delta\omega$ bilateral bandwidth[9] of the filter's center frequency, all of which have been passed by the filter *without* change. Thus the variance of $i'(t)$ is the mean-squared fluctuation strength in those frequency components of $i(t)$. Equation (21) then tells us that $\mathcal{S}_{ii}(\omega_c)/2\pi$ is the mean-squared fluctuation strength per unit bilateral bandwidth in the frequency ω_c component of the process $i(t)$.[10] It is therefore appropriate to refer to $\mathcal{S}_{ii}(\omega)$ as the noise spectrum or noise spectral density of the photocurrent $i(t)$.

[8] We have assumed that $H(\omega)$ is dimensionless, giving $h(t)$ the units sec^{-1} and ensuring that $i'(t)$ has the units of current.

[9] Bilateral bandwidth means that we are including both the positive frequency and negative frequency components.

[10] Here we are measuring bandwidth in rad/sec. If we measure bandwidth in Hz, then $\mathcal{S}_{ii}(\omega)$ is the mean-squared fluctuation strength per unit bilateral bandwidth in the frequency ω_c component of the process $i(t)$.

The derivation we have provided in the previous paragraph was couched in terms of the photocurrent-noise spectrum. It should be clear, however, that it applies to the noise spectrum of any stationary random process. In particular, it applies to the noise spectrum $\mathcal{S}_{PP}(\omega)$ of statistically stationary illumination power in semiclassical photodetection. Thus, whereas quantum photodetection only requires the photocurrent-noise spectrum that results from detection of statistically stationary illumination to be non-negative (assuming unity quantum efficiency detection, see below), Fourier transformation of Eq. (8) gives the more restrictive inequality stated last time, viz.,

$$\mathcal{S}_{ii}(\omega) = q\langle i\rangle + \frac{q^2\eta^2\mathcal{S}_{PP}(\omega)}{(\hbar\omega_0)^2} \geq q\langle i\rangle = \frac{q^2\eta\langle P\rangle}{\hbar\omega_0} > 0, \tag{22}$$

where the last inequality assumes $\langle P\rangle > 0$, i.e., that there is non-zero illumination. Equation (22) is the shot-noise limit on the photocurrent-noise spectrum for statistically stationary illumination. Any quantum state that leads to a sub-shot-noise spectrum, $\mathcal{S}_{ii}(\omega) < q\langle i\rangle$, must be a non-classical state.

Balanced Homodyne Detection

Slide 7 shows the continuous-time configuration for balanced homodyne detection. Signal light and local oscillator light are combined on a 50/50 beam splitter whose output beams illuminate a pair of almost-ideal (quantum efficiency η) photodetectors. The difference of the resulting photocurrents is passed through an ideal low-pass filter

$$H_{\rm LP}(\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \Delta\omega/2 \\ 0, & \text{otherwise,} \end{cases} \tag{23}$$

to obtain the homodyne current $i_{\rm hom}(t)$. The positive-frequency, photon-units signal and local oscillator fields (classical) or field operators (quantum) in slide 7 have a common center frequency ω_0. The associated baseband classical fields are $E(t)$ for the signal and $E_{\rm LO} = \sqrt{P_{\rm LO}/\hbar\omega_0}\,e^{j\theta}$ for the local oscillator. The associated baseband field operators are $\hat{E}(t)$ and $\hat{E}_{\rm LO}(t)$, where the latter is taken to be in the coherent state $|\sqrt{P_{\rm LO}/\hbar\omega_0}\,e^{j\theta}\rangle$. For both the classical and quantum fields we will take $P_{\rm LO} \to \infty$, i.e., we will operate our balanced homodyne system in the strong local-oscillator limit.

From our semiclassical theory for continuous-time direct detection we have that $i_\pm(t)$ are inhomogeneous Poisson impulse trains with rate functions

$$\lambda_\pm(t) = \eta\left|\frac{E(t) \pm E_{\rm LO}(t)}{\sqrt{2}}\right|^2 = \frac{\eta}{2}\left[|E(t)|^2 + \frac{P_{\rm LO}}{\hbar\omega_0} \pm 2\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}{\rm Re}[E(t)e^{-j\theta}]\right] \tag{24}$$

given knowledge of the signal field $E(t)$. We then then find that

$$\langle i_+(t) - i_-(t)\rangle = q[\lambda_+(t) - \lambda_-(t)] = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[E(t)e^{-j\theta}], \tag{25}$$

which, assuming that the baseband classical field's frequency content is limited to $|\omega| \leq \Delta\omega/2$, is also the average homodyne current $\langle i_{\rm hom}(t)\rangle$. Thus our semiclassical treatment shows that the homodyne current contains, in its mean, information about the θ-quadrature of the baseband classical field $E(t)$.[11]

Given knowledge of the signal field, $E(t)$, the homodyne current's covariance function is found to be

$$K_{i_{\rm hom}i_{\rm hom}}(t,u) \equiv \langle \Delta i_{\rm hom}(t)\Delta i_{\rm hom}(u)\rangle = K_{i'_+i'_+}(t,u) + K_{i'_-i'_-}(t,u) \tag{26}$$

where $i'_\pm(t)$ denote the currents obtained by passing $i_\pm(t)$ through the low-pass filter $H_{\rm LP}(\omega)$. Here, the second equality follows from the the photocurrent noises being entirely shot noises when the classical field is known, and the statistical independence of shot noises from different photodetectors. In the strong local-oscillator limit, these shot noises are predominantly due to the local oscillator, and *before* low-pass filtering we get

$$K_{i_\pm i_\pm}(t,u) = \frac{q^2\eta P_{\rm LO}}{2\hbar\omega_0}\,\delta(t-u), \tag{27}$$

i.e., before low-pass filtering the local-oscillator shot noises are statistically stationary with white-noise (constant at all frequencies) spectra

$$\mathcal{S}_{i_\pm i_\pm}(\omega) = \frac{q^2\eta P_{\rm LO}}{2\hbar\omega_0}. \tag{28}$$

The preceding results allow us to establish the following semiclassical decomposition of the homodyne current:

$$i_{\rm hom}(t) = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[E(t)e^{-j\theta}] + i_{\rm LO}(t), \tag{29}$$

where $E(t)$ can now be allowed to be a random process if necessary. Here, $i_{\rm LO}(t)$ is the low-pass filtered local-oscillator shot noise. It is a zero-mean, stationary random process that is statistically independent of $E(t)$ with noise spectrum given by

$$\mathcal{S}_{i_{\rm LO}i_{\rm LO}}(\omega) = \begin{cases} \dfrac{q^2\eta P_{\rm LO}}{\hbar\omega_0}, & \text{for } |\omega| \leq \Delta\omega/2 \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Furthermore, the strong local-oscillator limit ensures that there will be a very large number of local-oscillator induced photodetection events in the $\sim 1/\Delta\omega$ time constant of the low-pass filter. Hence the Central Limit Theorem implies that $i_{\rm LO}(t)$ can be

[11]Equivalently, we can say that $\langle i_{\rm hom}(t)\rangle$ contains, in its mean, information about the positive-frequency optical field, $E(t)e^{-j\omega_0 t}$, after it has been beat down to baseband—by mixing with the frequency-ω_0 local oscillator field and photodetection (an intrinsically square-law process)—and the θ-quadrature has been extracted.

taken to be a stationary Gaussian random process and thus completely characterized by knowledge of its mean function $\langle i_{\rm LO}\rangle = 0$ and its noise spectral density $\mathcal{S}_{i_{\rm LO}i_{\rm LO}}(\omega)$.

From our quantum theory of continuous-time direct detection we have that the photocurrents $i_\pm(t)$ have statistics which are equivalent to those of the following photocurrent operators,

$$\begin{aligned}\hat{i}_\pm(t) &\equiv q\left[\frac{\sqrt{\eta}\,[\hat{E}(t)\pm\hat{E}_{\rm LO}(t)]}{\sqrt{2}} + \sqrt{1-\eta}\,\hat{E}_{\eta_\pm}(t)\right]^\dagger \\ &\times \left[\frac{\sqrt{\eta}\,[\hat{E}(t)\pm\hat{E}_{\rm LO}(t)]}{\sqrt{2}} + \sqrt{1-\eta}\,\hat{E}_{\eta_\pm}(t)\right], \qquad (31)\end{aligned}$$

where $\hat{E}_{\eta_\pm}(t)$ are vacuum-state field operators that account for the sub-unity quantum efficiencies of the two detectors. It is now easy to calculate that the photocurrent difference, $i_+(t) - i_-(t)$, has statistics which are equivalent to

$$\hat{i}_+(t) - \hat{i}_-(t) = 2q\eta\,{\rm Re}[\hat{E}(t)\hat{E}^\dagger_{\rm LO}(t)] + 2q\sqrt{\eta(1-\eta)}\,{\rm Re}[\hat{E}_\eta(t)\hat{E}^\dagger_{\rm LO}(t)], \qquad (32)$$

where we have suppressed terms involving $\hat{E}^\dagger_{\eta_\pm}(t)\hat{E}_{\eta_\pm}(t)$, because their measurement will always yield zero as these are photon-flux operators associated with vacuum-state field operators, and we have introduced

$$\hat{E}_\eta(t) \equiv \frac{\hat{E}_{\eta_+}(t) + \hat{E}_{\eta_-}(t)}{\sqrt{2}}, \qquad (33)$$

to account for the sub-unity quantum efficiency noises on the two detectors. This field operator has the proper commutator, $\left[\hat{E}_\eta(t), \hat{E}^\dagger_\eta(u)\right] = \delta(t-u)$, and is in its vacuum state. It can be shown—but we won't do it—that the strong coherent-state local oscillator behaves classically in the preceding characterization of $\hat{i}_+(t) - \hat{i}_-(t)$, i.e., Eq. (32) reduces to

$$\hat{i}_+(t) - \hat{i}_-(t) = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[\hat{E}(t)e^{-j\theta}] + 2q\sqrt{\frac{\eta(1-\eta)P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[\hat{E}_\eta(t)e^{-j\theta}]. \qquad (34)$$

Because $\hat{E}_\eta(t)$ is in its vacuum state, it turns out that the second term on the right in Eq. (34) is equivalent to a *classical* zero-mean, stationary Gaussian random process, $i_\eta(t)$, with the white-noise spectral density[12]

$$\mathcal{S}_{i_\eta i_\eta}(\omega) = \frac{q^2\eta(1-\eta)P_{\rm LO}}{\hbar\omega_0}. \qquad (35)$$

[12]To show that this is so, evaluate $\langle[\hat{E}_\eta(t)e^{-j\theta} + \hat{E}^\dagger_\eta(t)e^{j\theta}][\hat{E}_\eta(u)e^{-j\theta} + \hat{E}^\dagger_\eta(u)e^{j\theta}]\rangle$ by multiplying out, employing the field commutator, and using the fact that the field $\hat{E}_\eta(t)$ has all its modes in their vacuum states. This procedure will lead to a δ-function covariance expression whose Fourier transform is the given noise spectrum.

Thus, incorporating $i_\eta(t)$ into Eq. (34) and applying the impulse response, $h_{\rm LP}(t)$, of the low-pass filter, we see that the homodyne photocurrent is equivalent to the measurement of the following operator,

$$\hat{i}_{\rm hom}(t) = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[\hat{E}(t)e^{-j\theta}]\star h_{\rm LP}(t) + i_\eta(t)\star h_{\rm LP}(t), \tag{36}$$

where $\star$ denotes convolution. In words, we have that the homodyne photocurrent consists of a scaled measurement of the low-pass filtered θ-quadrature of the baseband signal field embedded in a low-pass filtered white Gaussian noise.

It is instructive to conclude this section by examining the quantum statistics for balanced homodyne detection when the signal field $\hat{E}(t)$ is in the coherent state, $|E(t)\rangle$, whose eigenfunction only contains frequencies satisfying $|\omega| \le \Delta\omega/2$, so that $E(t)$ is unaffected by the low-pass filter. Equation (36) can then be used to show that $i_{\rm hom}(t) = \langle i_{\rm hom}(t)\rangle + \Delta i_{\rm hom}(t)$, where

$$\langle i_{\rm hom}(t)\rangle = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[E(t)e^{-j\theta}], \tag{37}$$

and $\Delta i_{\rm hom}(t)$ is a zero-mean, stationary Gaussian noise process, whose spectral density is

$$\mathcal{S}_{\Delta i_{\rm hom}\Delta i_{\rm hom}}(\omega) = \underbrace{\frac{q^2\eta^2P_{\rm LO}}{\hbar\omega_0}}_{\text{signal quantum noise}} + \underbrace{\frac{q^2\eta(1-\eta)P_{\rm LO}}{\hbar\omega_0}}_{\eta<1\text{ quantum noise}} = \frac{q^2\eta P_{\rm LO}}{\hbar\omega_0}, \quad \text{for } |\omega| \le \Delta\omega/2. \tag{38}$$

Because the covariance function of $i_{\rm hom}(t)$ equals the covariance function of $\Delta i_{\rm hom}(t)$—addition of a mean does not change covariances or noise spectral densities—the quantum theory of continuous-time homodyne detection with a coherent-state signal field yields measurement statistics that are identical to those of the semiclassical theory when the latter employs deterministic illumination with the same baseband classical field $E(t)$. Both theories tell us that the mean photocurrent contains a scaled version of the θ-quadrature of the baseband classical field $E(t)$. Both theories tell us that this mean is embedded in a zero-mean, bandlimited Gaussian noise with spectrum $q^2\eta P_{\rm LO}/\hbar\omega_0$. However, the physical interpretation of this noise is very different in the two theories. In semiclassical theory it is local-oscillator shot noise, but in quantum theory it is the due to the signal light quantum noise plus the quantum noise contributed by having sub-unity quantum efficiency detectors.

In general, the semiclassical theory must have *at least* local-oscillator shot noise in its homodyne noise spectrum, i.e., when $E_\theta(t) \equiv {\rm Re}[E(t)e^{-j\theta}]$ is a stationary classical random process semiclassical theory teaches that

$$\mathcal{S}_{i_{\rm hom}i_{\rm hom}}(\omega) = \underbrace{\frac{q^2\eta P_{\rm LO}}{\hbar\omega_0}}_{\text{LO shot noise}} + \underbrace{\frac{4q^2\eta^2P_{\rm LO}\mathcal{S}_{E_\theta E_\theta}(\omega)}{\hbar\omega_0}}_{\text{signal-beam excess noise}}, \quad \text{for } |\omega| \le \Delta\omega/2, \tag{39}$$

where *both* terms on the right are non-negative. On the other hand, the photocurrent-noise spectrum from quantum photodetection theory, when the illumination is in a statistically stationary field state, satisfies the weaker bound

$$\mathcal{S}_{i_{\rm hom}i_{\rm hom}}(\omega) \geq \mathcal{S}_{i_\eta i_\eta}(\omega) = \frac{q^2\eta(1-\eta)P_{\rm LO}}{\hbar\omega_0}, \quad \text{for } |\omega| \leq \Delta\omega/2. \tag{40}$$

Thus, whenever balanced homodyne detection yields a noise-spectrum measurement obeying $\mathcal{S}_{i_{\rm hom}i_{\rm hom}}(\omega) < q^2\eta P_{\rm LO}/\hbar\omega_0$, at some frequency within the loss-pass filter's passband, then we know that the signal beam was in a *non-classical* state.

Balanced Heterodyne Detection

Slide 9 shows the continuous-time configuration for balanced heterodyne detection. It differs from the balanced homodyne arrangement that we have just considered in two ways. First, the local oscillator is offset, in frequency, from the signal field we're trying to detect. Second, as a consequence of the first, we apply a bandpass filter to the difference of the photocurrents from the two detectors. For the semiclassical treatment, the positive-frequency, photon-units signal field will be taken to be $E_S(t)e^{-j(\omega_0+\omega_{\rm IF})t}$ and the positive-frequency, photon-units local oscillator field will be taken to be $E_{\rm LO}e^{-j\omega_0 t} = \sqrt{P_{\rm LO}/\hbar\omega_0}\, e^{-j\omega_0 t}$. Here, $E_S(t)$ is a baseband field and $\omega_{\rm IF}$ is the intermediate frequency, with the former allowed, in general, to be a random process. For convenience, we shall assume that the bandpass filter has frequency response

$$H_{\rm BP}(\omega) = \begin{cases} 1, & \text{for } |\omega \pm \omega_{\rm IF}| \leq \Delta\omega/2 \\ 0, & \text{otherwise,} \end{cases} \tag{41}$$

where $\omega_{\rm IF} > \Delta\omega/2$. We shall also assume that ${\rm Re}[E_S(t)e^{-j\omega_{\rm IF}t}]$ is unaffected by passage through this filter, i.e., its frequency content lies entirely within the filter's passband. For the quantum approach, the positive-frequency, photon-units field operator that enters the signal port in the slide 9 setup will be taken to be $\hat{E}(t)e^{-j\omega_0 t} = \hat{E}_S(t)e^{-j(\omega_0+\omega_{\rm IF})t} + \hat{E}_I(t)e^{-j(\omega_0-\omega_{\rm IF})t}$, and the positive-frequency, photon-units local oscillator field operator will be taken to be $\hat{E}_{\rm LO}(t)e^{-j\omega_0 t}$. Here, in anticipation of the role played by the passband filter, $\hat{E}_S(t)$ and $\hat{E}_I(t)$ are baseband field operators whose modes only span the frequency range $|\omega| \leq \Delta\omega/2$. In keeping with what we learned about single-mode heterodyne detection, we will allow the baseband signal-field operator, $\hat{E}_S(t)$, to be in an excited state and take the baseband image-band field operator, $\hat{E}_I(t)$, to be in its vacuum state. The local oscillator will be assumed to be in the coherent state $|\sqrt{P_{\rm LO}/\hbar\omega_0}\, e^{-j\omega_0 t}\rangle$, with $P_{\rm LO} \to \infty$. It is a measurement of both quadratures of $\hat{E}_S(t)$ that we are trying to accomplish with the slide 9 setup; the noise contributed by the image-band operator $\hat{E}_I(t)$ must be included in order to accomplish this task without violating the Heisenberg uncertainty principle. The astute reader will notice that $\hat{E}(t)e^{-j\omega_0 t} = \hat{E}_S(t)e^{-j(\omega_0+\omega_{\rm IF})t} + \hat{E}_I(t)e^{-j(\omega_0-\omega_{\rm IF})t}$ does *not* have the proper δ-function commutator, i.e., there are "other modes" that we have neglected. Because these other modes will *not* contribute to the bandpass filter's output, in the strong local-oscillator limit, we have suppressed them at the outset in order to simplify our analysis.

The semiclassical treatment of balanced heterodyne detection closely parallels what we did for balanced homodyne detection. Suppose, for now, that $E_S(t)$ is a deterministic baseband field. Then, the mean functions of the photocurrents $i_\pm(t)$ are as follows:

$$\langle i_\pm(t)\rangle = q\lambda_\pm(t) = \frac{q\eta}{2}\left|\frac{E_S(t)e^{-j\omega_{\rm IF}t} \pm E_{\rm LO}(t)}{\sqrt{2}}\right|^2. \tag{42}$$

From this result we immediately find that

$$\langle i_+(t) - i_-(t)\rangle = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[E_S(t)e^{-j\omega_{\rm IF}t}], \tag{43}$$

and, because this signal is unaffected by the bandpass filter, we get

$$\langle i_{\rm het}(t)\rangle = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[E_S(t)e^{-j\omega_{\rm IF}t}], \tag{44}$$

for the average heterodyne photocurrent when the signal light is deterministic. In the strong local-oscillator limit the noise in $i_+(t)-i_-(t)$, when the signal field is known, is entirely local-oscillator shot noise, i.e., a zero-mean, white Gaussian noise with noise spectral density $q^2\eta P_{\rm LO}/\hbar\omega_0$. It follows that we can express $i_{\rm het}(t)$ in a form similar to what we did in Eq. (29) for the homodyne current, viz.,

$$i_{\rm het}(t) = 2q\eta\sqrt{\frac{P_{\rm LO}}{\hbar\omega_0}}\,{\rm Re}[E_S(t)e^{-j\omega_{\rm IF}t}] + i_{\rm LO}(t), \tag{45}$$

where $E_S(t)$ can now be allowed to be a random process. Here, $i_{\rm LO}(t)$ is the passband-filtered local-oscillator shot noise. It is a zero-mean, stationary Gaussian random process that is statistically independent of $E_S(t)$ with noise spectrum given by,

$$\mathcal{S}_{i_{\rm LO}i_{\rm LO}}(\omega) = \begin{cases} \dfrac{q^2\eta P_{\rm LO}}{\hbar\omega_0}, & \text{for } |\omega \pm \omega_{\rm IF}| \le \Delta\omega/2 \\ 0, & \text{otherwise.}\end{cases} \tag{46}$$

We see, from Eq. (45), that the semiclassical theory gives a heterodyne photocurrent which contains a frequency-downtranslated (from $\omega_0 + \omega_{\rm IF}$ center frequency to $\omega_{\rm IF}$ center frequency) replica of the signal field embedded in a passband-filtered white Gaussian noise arising from local-oscillator shot noise.

For the quantum theory of balanced heterodyne detection, we rely on what we did for the quantum theory of balanced homodyne detection. Specifically, we can say that $i_+(t)$ and $i_-(t)$ have statistics that are equivalent to those of the quantum operators $\hat{i}_+(t)$ and $\hat{i}_-(t)$, respectively, where

$$\hat{i}_+(t) - \hat{i}_-(t) = 2q\eta\,\mathrm{Re}[\hat{E}(t)E^*_{\mathrm{LO}}(t)] + i_\eta(t), \tag{47}$$

where we have used the strong local-oscillator condition to justify replacing the local-oscillator field operator with its coherent-state eigenfunction $E_{\mathrm{LO}}(t) \equiv P_{\mathrm{LO}}/\hbar\omega_0$, and $i_\eta(t)$ is a classical, zero-mean, white Gaussian noise process with spectral density

$$\mathcal{S}_{i_\eta i_\eta}(\omega) = \frac{q^2\eta(1-\eta)P_{\mathrm{LO}}}{\hbar\omega_0}, \tag{48}$$

representing the quantum noise contributed by sub-unity quantum quantum efficiency photodetectors. After the passband filter, we then get the following operator equivalent for the heterodyne photocurrent,

$$\hat{i}_{\mathrm{het}}(t) = 2q\eta\sqrt{\frac{P_{\mathrm{LO}}}{\hbar\omega_0}}\,\mathrm{Re}[\hat{E}_S(t)e^{-j\omega_{\mathrm{IF}}t}] + 2q\eta\sqrt{\frac{P_{\mathrm{LO}}}{\hbar\omega_0}}\,\mathrm{Re}[\hat{E}^\dagger_I(t)e^{-j\omega_{\mathrm{IF}}t}] + i_\eta(t)\star h_{\mathrm{BP}}(t), \tag{49}$$

where $h_{\mathrm{BP}}(t)$ is the bandpass filter's impulse response and $\star$ denotes convolution. Now, because $\hat{E}_I(t)$ is in its vacuum state, its contribution to $\hat{i}_{\mathrm{het}}(t)$ is equivalent to a classical, zero-mean, stationary Gaussian noise process, $i_I(t)$, whose noise spectrum is

$$\mathcal{S}_{i_I i_I}(\omega) = \begin{cases} \dfrac{q^2\eta^2 P_{\mathrm{LO}}}{2\hbar\omega_0}, & \text{for } |\omega \pm \omega_{\mathrm{IF}}| \le \Delta\omega/2 \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

Once again it is instructive to examine what happens in our quantum treatment when the signal field $\hat{E}_S(t)$ is in the coherent state $|E_S(t)\rangle$.[13] Equation (49) can then be used to show that $i_{\mathrm{het}}(t) = \langle i_{\mathrm{het}}(t)\rangle + \Delta i_{\mathrm{het}}(t)$, where

$$\langle i_{\mathrm{het}}(t)\rangle = 2q\eta\sqrt{\frac{P_{\mathrm{LO}}}{\hbar\omega_0}}\,\mathrm{Re}[E_S(t)e^{-j\omega_{\mathrm{IF}}t}], \tag{51}$$

and $\Delta i_{\mathrm{het}}(t)$ is a zero-mean, stationary Gaussian noise process, whose spectral density is,

$$\mathcal{S}_{\Delta i_{\mathrm{het}}\Delta i_{\mathrm{het}}}(\omega) = \underbrace{\frac{q^2\eta^2 P_{\mathrm{LO}}}{2\hbar\omega_0}}_{\text{signal quantum noise}} + \underbrace{\frac{q^2\eta^2 P_{\mathrm{LO}}}{2\hbar\omega_0}}_{\text{image-band quantum noise}} + \underbrace{\frac{q^2\eta(1-\eta)P_{\mathrm{LO}}}{\hbar\omega_0}}_{\eta<1\text{ quantum noise}} \tag{52}$$

$$= \frac{q^2\eta P_{\mathrm{LO}}}{\hbar\omega_0}, \quad \text{for } |\omega \pm \omega_{\mathrm{IF}}| \le \Delta\omega/2. \tag{53}$$

[13] The eigenfunction $\{\, E_S(t) : -\infty < t < \infty \,\}$ must be bandlimited to $|\omega| \le \Delta\omega/2$, for consistency with our earlier assumption about the modes contained in $\hat{E}_S(t)$.

Because the covariance function of $i_{\text{het}}(t)$ equals the covariance function of $\Delta i_{\text{het}}(t)$, the quantum theory of continuous-time heterodyne detection with a coherent-state signal field (and a vacuum-state image-band field) yields measurement statistics that are identical to those of the semiclassical theory when the latter employs deterministic illumination with the same classical baseband signal field $E_S(t)$. Both theories tells us that the mean photocurrent contains a frequency-downtranslated (from $\omega_0 + \omega_{\text{IF}}$ center frequency to ω_{IF} center frequency) replica of the signal field embedded in a passband-filtered white Gaussian noise. However, as we saw for homodyne detection, the physical interpretation of the heterodyne noise is very different in the two photodetection theories. In semiclassical theory it arises from local-oscillator shot noise, but in quantum theory it is the sum of the signal light quantum noise, the image band quantum noise, and the quantum noise contributed by having sub-unity quantum efficiency detectors.

In general, the semiclassical theory for heterodyne detectcion must have *at least* local-oscillator shot noise in its heterodyne noise spectrum, as we found for the homodyne case. Specifically, if $E_{\text{IF}}(t) \equiv \text{Re}[E_S(t)e^{-j\omega_{\text{IF}}t}]$ is a stationary classical random process we have that

$$\mathcal{S}_{I_{\text{het}}i_{\text{het}}}(\omega) = \underbrace{\frac{q^2\eta P_{\text{LO}}}{\hbar\omega_0}}_{\text{LO shot noise}} + \underbrace{\frac{4q^2\eta^2 P_{\text{LO}}\mathcal{S}_{E_{\text{IF}}E_{\text{IF}}}(\omega)}{\hbar\omega_0}}_{\text{signal-beam excess noise}}, \quad \text{for } |\omega \pm \omega_{\text{IF}}| \le \Delta\omega/2, \tag{54}$$

where *both* terms on the right are non-negative. On the other hand, the photocurrent-noise spectrum for quantum photodetection theory, when the illumination is in a statistically stationary field state, satisfies the weaker bound,

$$\mathcal{S}_{i_{\text{het}}i_{\text{het}}}(\omega) \ge \mathcal{S}_{i_I i_I}(\omega) + \mathcal{S}_{i_\eta i_\eta}(\omega) = \frac{q^2\eta(1-\eta/2)P_{\text{LO}}}{\hbar\omega_0}, \quad \text{for } |\omega \pm \omega_{\text{IF}}| \le \Delta\omega/2. \tag{55}$$

Thus, whenever a heterodyne measurement yields a noise spectrum obeying $\mathcal{S}_{i_{\text{het}}i_{\text{het}}}(\omega) > q^2\eta(1-\eta/2)P_{\text{LO}}/\hbar\omega_0$, at some frequency with the bandpass filter's passband, then we know that the signal beam was in a *non-classical* state.

The Road Ahead

In the next lecture we shall turn to our final major topic for the semester, i.e., how we can generate non-classical light through nonlinear optics. The specific nonlinear optical system we'll consider is continuous-wave pumping of a second order ($\chi^{(2)}$) nonlinearity. Ultimately, we will see that such a system can produce quadrature noise squeezing, photon-twins behavior, and polarization entanglement.