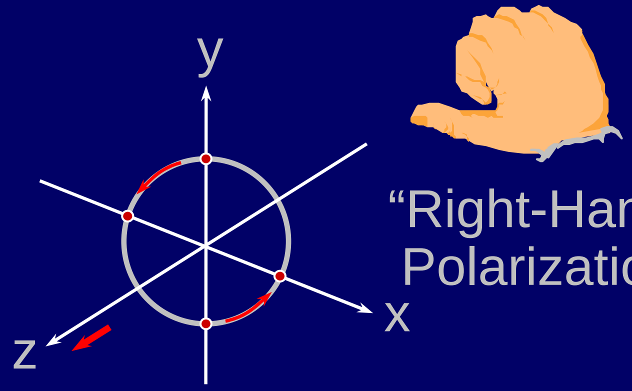
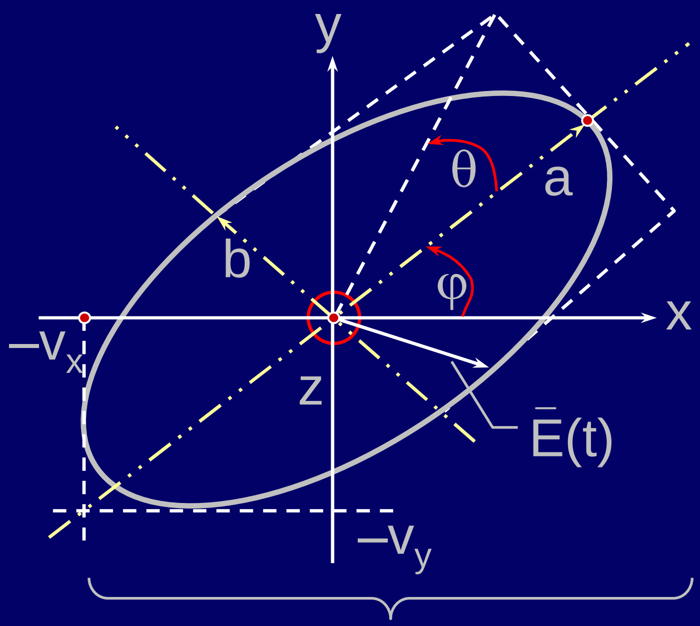
Monochromatic Radiation is Always 100% Polarized



“Right-Hand” Polarization

Propagation

Polarization Ellipse



3 Parameters Specify Ellipse

e.g. a, b, φ
a, φ, θ
v_x, v_y, θ

Also, (need “+” or “–” to ⇒ right or left elliptical)

Polarization of Narrowband Radiation

$$\text{Let } \overline{E}(t) = \hat{x} v_x(t) \cos[\omega t + \phi(t)] + \hat{y} v_y(t) \cos[\omega t + \phi(t) + \delta(t)]$$

$v_x(t)$ and $v_y(t)$ are slowly varying and random;
$\langle v_x \rangle$, $\langle v_y \rangle$, and $\langle \delta \rangle$ may be non-zero

"Stokes Parameters"

$$I \equiv S_o \equiv \frac{\left[\left\langle v_x^2(t) \right\rangle + \left\langle v_y^2(t) \right\rangle\right]}{2\eta_o}$$

[W m^{-2}] total power

$$Q \equiv S_o \equiv \frac{\left[\left\langle v_x^2(t) \right\rangle - \left\langle v_y^2(t) \right\rangle\right]}{2\eta_o}$$

"x-ness"

$$U \equiv S_2 \equiv \frac{2\left\langle v_x(t) \bullet v_y(t) \cos \delta(t) \right\rangle}{2\eta_o}$$

"45°-ness"

$$V \equiv S_3 \equiv \frac{2\left\langle v_x(t) \bullet v_y(t) \sin \delta(t) \right\rangle}{2\eta_o}$$

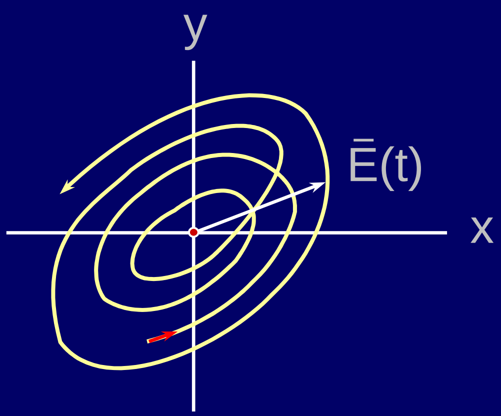
"circularity"

100% Polarized Narrowband Waves

Let $\bar{E}(t) = \hat{x} v_x(t) \cos[\omega t + \phi(t)] + \hat{y} v_y(t) \cos[\omega t + \phi(t) + \delta(t)]$

$v_x(t)$ and $v_y(t)$ are slowly varying and random;
$\langle v_x \rangle$, $\langle v_y \rangle$, and $\langle \delta \rangle$ may be non-zero

$\delta(t) = \delta_o$ and $v_x / v_y(t) = \text{constant} \Rightarrow$ fixed ellipse, variable size



Also : $S_o^2 = S_1^2 + S_2^2 + S_3^2$

Therefore, any 3 Stokes parameters specify polarization

Partially Polarized Narrowband Radiation

“Stokes Parameters”

$$I \equiv S_o \equiv \frac{[\langle v_x^2(t)\rangle + \langle v_y^2(t)\rangle]}{2\eta_o}$$

[W m^{-2}] total power

$$Q \equiv S_1 \equiv \frac{[\langle v_x^2(t)\rangle - \langle v_y^2(t)\rangle]}{2\eta_o}$$

“x-ness”

$$U \equiv S_2 \equiv \frac{2\langle v_x(t) \bullet v_y(t) \cos \delta(t)\rangle}{2\eta_o}$$

“45°-ness”

$$V \equiv S_3 \equiv \frac{2\langle v_x(t) \bullet v_y(t) \sin \delta(t)\rangle}{2\eta_o}$$

“circularity”

Note: For 2 uncorrelated waves superimposed (A+B), we have $S_{i_{A+B}} = S_{i_A} + S_{i_B}$ where i = 0, 1, 2, 3

For 0% polarization, Stokes: S_o; $S_1 = S_2 = S_3 = 0$

Therefore, for partially polarized wave:

$$\boxed{\begin{array}{l}[S_o, S_1, S_2, S_3] = [S_u, 0, 0, 0] + [S_o - S_u, S_1, S_2, S_3] \\ \text{where } (S_o - S_u)^2 = S_1^2 + S_2^2 + S_3^2\end{array}}$$

Define percentage polarization $= \underbrace{\left(\frac{S_o - S_u}{S_o}\right)}_{\triangleq\ m,\ 0 \le m \le 1} \bullet 100\%$

Coherency Matrix $\underline{\overline{\overline{J}}}$

$$\underline{\overline{\overline{J}}} \overset{\Delta}{=} \frac{1}{\eta_o}\begin{bmatrix}\langle \underline{E}_x \underline{E}_x^* \rangle & \langle \underline{E}_x \underline{E}_y^* \rangle \\ \langle \underline{E}_x^* \underline{E}_y \rangle & \langle \underline{E}_y \underline{E}_y^* \rangle\end{bmatrix}$$

where $E(t) = xR_e\left\{\underline{E}_x(t)e^{j\omega t}\right\} + yR_e\left\{\underline{E}_y(t)e^{j\omega t}\right\}$

where $\underline{E}_x(t), \underline{E}_y(t)$ vary slowly

e.g. X-polarization $$\underline{\overline{\overline{J}}}_x = 2S_o\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}$$

RCP (right-circular) $$\underline{\overline{\overline{J}}}_{RC} = S_o\begin{bmatrix}1 & -j \\ j & 1\end{bmatrix}$$

Unpolarized $$\underline{\overline{\overline{J}}}_u = S_o\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}$$

Finding Orthogonal Polarization $\underline{\overline{\overline{J}}}_{RC}$

e.g. X-polarization $$\underline{\overline{\overline{J}}}_x = 2S_o \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

RCP (right-circular) $$\underline{\overline{\overline{J}}}_{RC} = S_o \begin{bmatrix} 1 & -j \\ j & 1 \end{bmatrix}$$

Unpolarized $$\underline{\overline{\overline{J}}}_u = S_o \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Note : $\underline{\overline{\overline{J}}}_x + \underline{\overline{\overline{J}}}_y = 2\underline{\overline{\overline{J}}}_u$ $\qquad$ $\underline{\overline{\overline{J}}}_{RC} + \underline{\overline{\overline{J}}}_{LC} = 2\underline{\overline{\overline{J}}}_u$

$$\text{If } \left.\begin{array}{l} \underline{\overline{\overline{J}}}_A \perp \underline{\overline{\overline{J}}}_B, \text{and} \\ T_r\underline{\overline{\overline{J}}}_A = T_r\underline{\overline{\overline{J}}}_B \end{array}\right\} \text{then } \underline{\overline{\overline{J}}}_A + \underline{\overline{\overline{J}}}_B = 2\underline{\overline{\overline{J}}}_u$$

Therefore, we can find orthogonal polarization $\underline{\overline{\overline{J}}}_B = 2\underline{\overline{\overline{J}}}_n - \underline{\overline{\overline{J}}}_A$

Polarized Antennas

Far Fields

$$\text{Define}\quad \frac{G_{ij}(\theta,\phi)}{G(\theta,\phi)} = \frac{A_{ij}(\theta,\phi)}{A(\theta,\phi)} = \frac{\underline{E}_i \underline{E}_j}{\underline{E}_x \underline{E}_x^* + \underline{E}_y \underline{E}_y^*}$$

$$\text{e.g.}\ \{i,j\} = \{x,y\},\{r,\ell\},\{a,b\}\,(b \perp a)$$

$$\overline{\overline{\underline{A}}} = \begin{bmatrix} \underline{A}_{xx} & \underline{A}_{xy} \\ \underline{A}_{yx} & \underline{A}_{yy} \end{bmatrix};\ \text{claim}\quad \boxed{P_{rec} = \frac{1}{2} T_r\Big[\underset{\substack{\uparrow\\ [m^2]}}{\overline{\overline{\underline{A}}}} \bullet \underset{\substack{\uparrow\\ [Wm^2]}}{\overline{\overline{\underline{J}}}{}^{t}_{inc}}\Big] [W]}$$

for incident plane wave

Polarized Antennas

$$\overline{\underline{A}} = \begin{bmatrix} \underline{A}_{xx} & \underline{A}_{xy} \\ \underline{A}_{yx} & \underline{A}_{yy} \end{bmatrix};$$ claim

$$\boxed{P_{rec} = \frac{1}{2} T_r\left[\underset{\substack{\uparrow \\ [m^2]}}{\overline{\underline{A}}} \bullet \underset{\substack{\uparrow \\ [Wm^2]}}{\overline{\underline{J}}{}^t_{inc}}\right][W]}$$

for incident plane wave

$$\text{So } P_{rec} = \frac{1}{2}\left[A_{11}J_{11} + A_{12}J_{12} + A_{21}J_{21} + A_{22}J_{22}\right]$$

for incident uniform plane wave $\overline{\underline{J}}$ on antenna $\overline{\underline{A}}$

$$\text{For } \Omega_s \neq 0: \; P_{rec} = \frac{1}{2}\int_{4\pi} T_r\left[\overline{\underline{A}}(\theta,\phi)\overline{\underline{J}}^t(\theta,\phi)\right] d\Omega$$

To Measure Polarization

Measure 4 powers; use 4 antennas

e.g.

$$\begin{bmatrix} M_a \\ M_b \\ M_c \\ M_d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \underline{A}11_a & \underline{A}12_a & \underline{A}21_a & \underline{A}22_a \\ \underline{A}11_b & \underline{A}12_b & \bullet & \bullet \\ \underline{A}11_c & \bullet & \bullet & \bullet \\ \underline{A}11_d & \bullet & \bullet & \underline{A}11_d \end{bmatrix}\begin{bmatrix} \underline{J}_{11} \\ \underline{J}_{12} \\ \underline{J}_{21} \\ \underline{J}_{22} \end{bmatrix}$$

$$\overline{M} = \frac{1}{2}\underline{\overline{\overline{A}}}\,\underline{\overline{\overline{J}}}, \quad \text{so} \quad \underline{\overline{\hat{J}}} = 2\underline{\overline{\overline{A}}}^{-1}\overline{M}\left(\underline{\overline{\hat{J}}} \text{ is estimate}\right)$$

Is $\underline{\overline{\overline{A}}}$ singular?

To Measure Polarization

$$\overline{M} = \frac{1}{2}\underline{\overline{\overline{A}}}\,\underline{\overline{J}}, \quad \text{so} \quad \underline{\hat{\overline{J}}} = 2\underline{\overline{\overline{A}}}^{-1}\overline{M}\left(\underline{\hat{\overline{J}}} \text{ is estimate}\right)$$

Is $\underline{\overline{\overline{A}}}$ singular?

For x, y, RC, LC POL:

$$\underline{\overline{\overline{A}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -j & j & 1 \\ 1 & j & -j & 1 \end{bmatrix}$$

↓

$$\det \underline{\overline{\overline{A}}} = 0$$

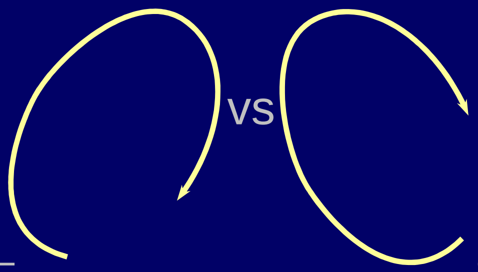
For x, 45°, RC, LC:

$$\underline{\overline{\overline{A}}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -j & j & 1 \\ 1 & j & -j & 1 \end{bmatrix}$$

↓

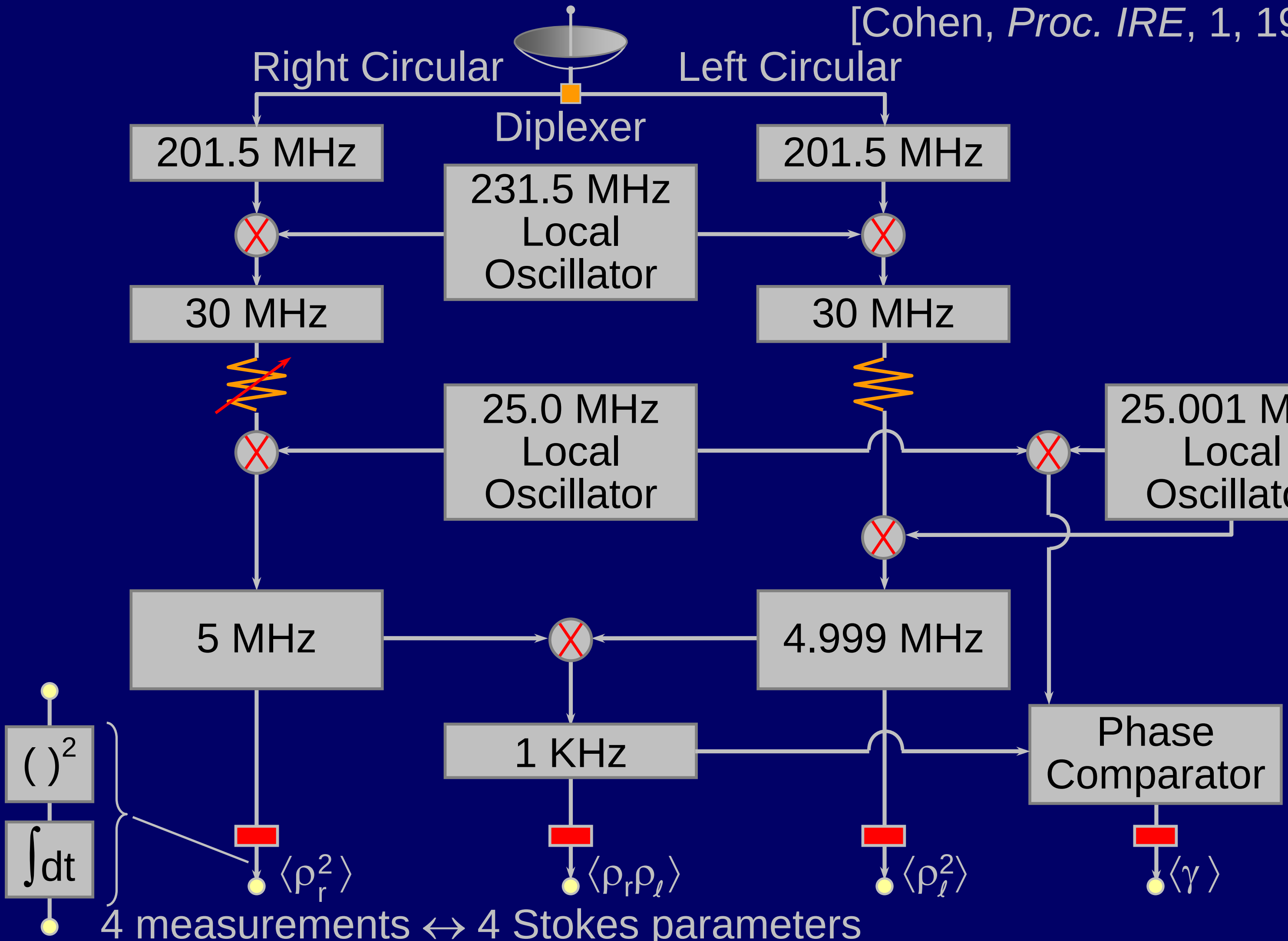
$$\det \underline{\overline{\overline{A}}} \neq 0 \quad \text{"ok"}$$

Can not distinguish



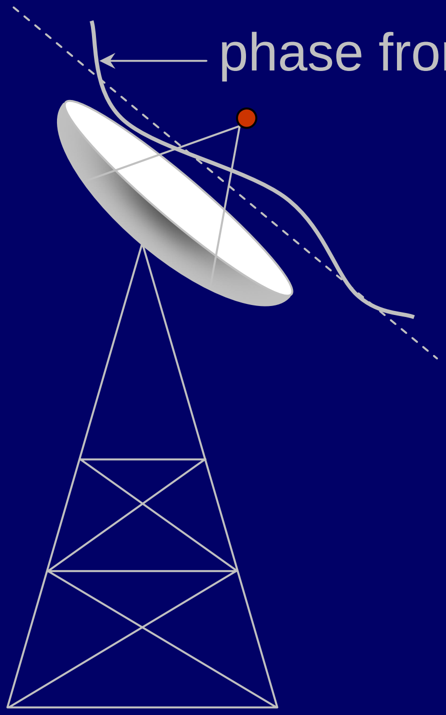
Example of a Polarimeter

[Cohen, *Proc. IRE*, 1, 1958]



4 measurements ↔ 4 Stokes parameters

Antenna Phase Errors



Systematic antenna phase errors:

1) poor design and fabrication
2) gravity, wind, thermal (gravity and thermal limits near 1 arc minute)
3) feed offset

Random antenna phase errors:

1) machine tolerances, surface roughness
2) adjustment errors
3) feed offset

Examples of Antenna Phase Errors

Random antenna phase errors:

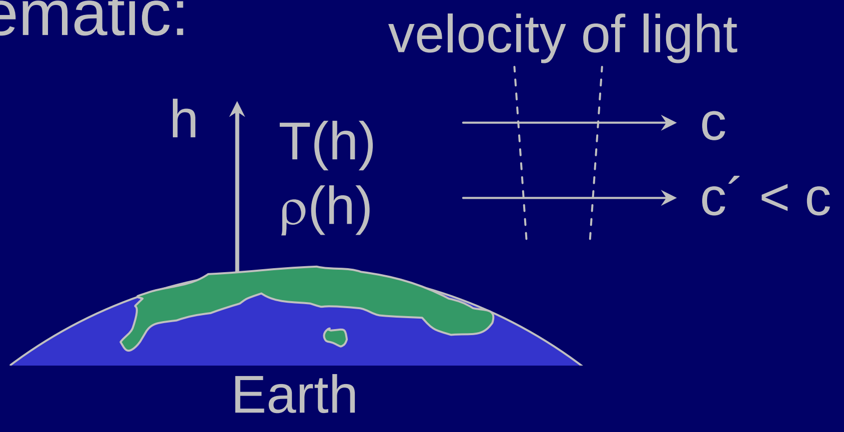
1) matching tolerances, surface roughness
2) adjustment errors
3) feed offset

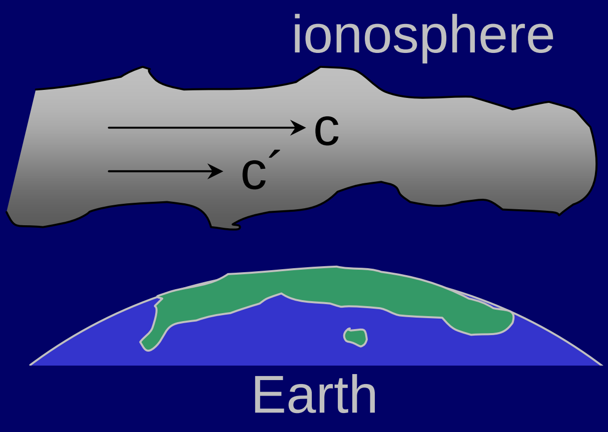
300-ft parabolic reflector antenna at NRAO, Greenbank, West Virginia

1) systematic sag ⇒ fix backup; footprints on mesh
2) steamrolled mesh ⇒ long waves
3) new panels: $\theta_B \, \tilde{>} \, 0.5 - 1$ arc minute

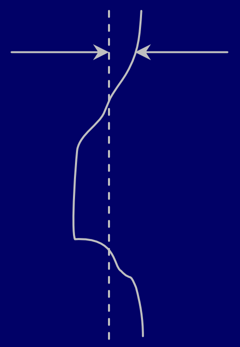
Types of optical and radio propagation phase errors

Systematic:



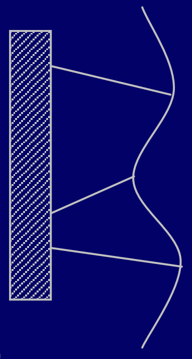


Random phase:
+ amplitude?



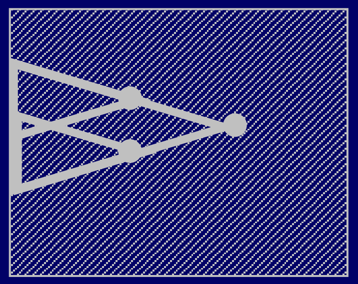
RMS $\overset{\sim}{<} \lambda/2\pi$, weak fluctuations

RMS $>> \lambda/2\pi$, strong fluctuations



Thin screen
(constant amplitude)

vs

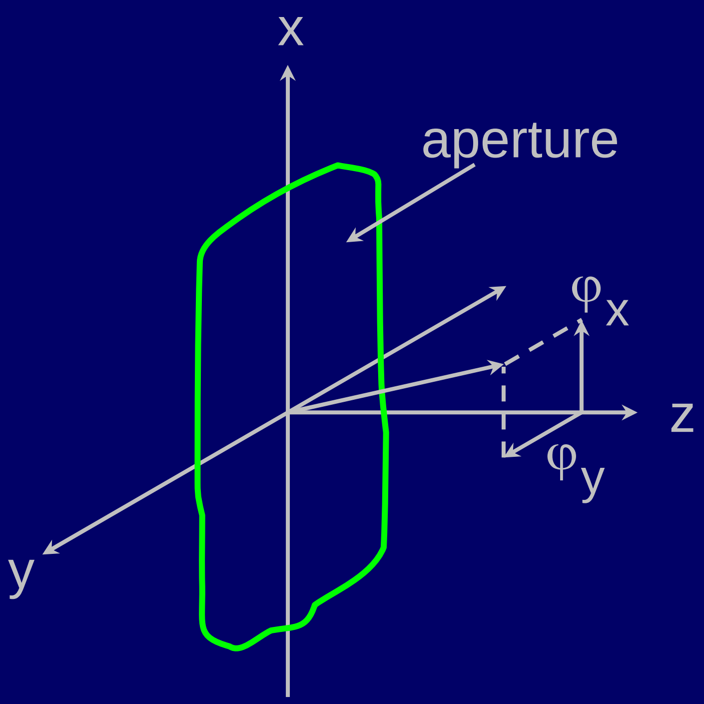


Thick screen

Δ pathlength = $\Delta L \geq n\lambda$

$\Rightarrow$ interference and nulls

Effect of Phase Variation on Directivity



For x-polarization:

$$\begin{tikzcd}
\underline{E}_x(x,y) \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \underline{E}_x(\varphi_x, \varphi_y) \arrow[d] \\
R_{\underline{E}_x}(\bar{\tau}) \arrow[r, leftrightarrow, "\sim"] & \left|\underline{E}_x(\bar{\varphi})\right|^2 \propto D(\bar{\varphi}), G(\bar{\varphi})
\end{tikzcd}$$

$$D(f,\theta,\phi) = \left[\pi(1+\cos\theta)^2 / \lambda^2\right] \bullet \frac{\int_A R_{\underline{E}_x}(\bar{\tau}) e^{-j\frac{2\pi}{\lambda}(\bar{\varphi}\bullet\bar{\tau})} d\tau_x d\tau_y}{\int_A \left|\underline{E}_x(x,y)\right|^2 dxdy}$$

Effect of Phase Variation on Directivity

$$D(f,\theta,\phi)=\left[\pi(1+\cos\theta)^2/\lambda^2\right]\bullet\frac{\int_A \underline{R}_{E_x}(\overline{\tau})e^{-j\frac{2\pi}{\lambda}(\overline{\varphi}\bullet\overline{\tau})}d\tau_x d\tau_y}{\int_A\left|\underline{E}_x(x,y)\right|^2 dxdy}$$

$$E\{D(f,\theta,\phi)\}=\frac{\pi(1+\cos\theta)^2}{\lambda^2\int_A\left|\underline{E}_x(x,y)\right|^2 dxdy}\bullet\int_A E\Big\{\underbrace{R_{\underline{E}_x}(\overline{\tau})}_{\int_A \underbrace{\underline{E}_x(\overline{r})}_{E_o(\overline{r})e^{j\gamma(\overline{r})}}\underline{E}_x^*(\overline{r}-\overline{\tau})d\overline{r}}\Big\}e^{-j\frac{2\pi}{\lambda}(\overline{\varphi}\bullet\overline{\tau})}d\tau_x d\tau_y$$

Therefore $E\left\{R_{\underline{E}_x}(\overline{\tau})\right\}=R_{\underline{E}_o}(\overline{\tau})E\left\{e^{j\gamma(\overline{r})-j\gamma(\overline{r}-\overline{\tau})}\right\}$

Spatial stationarity : $E\left\{e^{j\gamma(\overline{r})-j\gamma(\overline{r}-\overline{\tau})}\right\}=E\left\{e^{j\gamma(o)-j\gamma(\overline{r})}\right\}$

Definition of “Characteristic Function”

It is the Fourier transform of probability distribution p(x) (also called the moment-generating function)

$$E\left[e^{j\omega x}\right] \equiv \int_{-\infty}^{\infty} p(x) e^{j\omega x} dx = F.T.[p(x)]$$

$$\overset{\Delta}{=} \Gamma(\omega\,;x) = \text{“characteristic function of p(x)”}$$

One use of the Fourier transform of p(x) is when we seek $p(x_1 + x_2 + \ldots + x_n) =$

$$p(x_1) * p(x_2) * \ldots * p(x_n) = F.T.\left\{ \underset{i=1}{\overset{n}{\pi}} F.T.[p(x_i)] \right\}$$

Computation of $E\left\{R_{\underline{\underline{E}}_x}\left(\bar{\tau}\right)\right\}$

Thus $\Gamma\left(\omega_1,\omega_2;\gamma(0),\gamma\left(\bar{\tau}\right)\right)=E\left\{e^{j\omega_1\gamma(0)+j\omega_2\gamma\left(\bar{\tau}\right)}\right\}$

Recall: If $\underline{\gamma}_1,\underline{\gamma}_2$ are JGRV, then

$$\Gamma\left(\omega_1,\omega_2,\underline{\gamma}_1,\underline{\gamma}_2\right)=e^{-\frac{1}{2}[\omega_1\omega_2]\begin{bmatrix}\underline{\gamma}_1\underline{\gamma}_1^* & \underline{\gamma}_1\underline{\gamma}_2^* \\ \underline{\gamma}_2\underline{\gamma}_1^* & \underline{\gamma}_2\underline{\gamma}_2^*\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}}$$

Here, $\gamma_1 \overset{\Delta}{=} \gamma(0)$, $\gamma_2 \overset{\Delta}{=} \gamma\left(\bar{\tau}\right)$

Therefore: $E\left\{e^{j\gamma(0)-j\gamma\left(\bar{\tau}\right)}\right\}=\Gamma\left(\omega_1=1,\omega_2=-1;\gamma(0),\gamma\left(\bar{\tau}\right)\right)$

Computation of $E\left\{R_{\underline{\underline{E}}_x}(\bar{\tau})\right\}$

$$\Gamma\left(\omega_1,\omega_2,\underline{\gamma}_1,\underline{\gamma}_2\right)=e^{-\frac{1}{2}[\omega_1\omega_2]\begin{bmatrix}\underline{\gamma}_1\underline{\gamma}_1^* & \underline{\gamma}_1\underline{\gamma}_2^* \\ \underline{\gamma}_2\underline{\gamma}_1^* & \underline{\gamma}_2\underline{\gamma}_2^*\end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2\end{bmatrix}}$$

Here, $\gamma_1 \overset{\Delta}{=} \gamma(0)\,,\ \gamma_2 \overset{\Delta}{=} \gamma(\bar{\tau})$

Therefore : $E\left\{e^{j\gamma(0)-j\gamma(\bar{\tau})}\right\}=\Gamma\left(\omega_1=1,\omega_2=-1;\gamma(0),\gamma(\bar{\tau})\right)$

Since: $\underline{\gamma}(0)\underline{\gamma}^*(0)\overset{\Delta}{=}\phi(0)$ $\qquad$ $\underline{\gamma}(0)\gamma^*(\bar{\tau})\overset{\Delta}{=}\phi(\bar{\tau})$

$\underline{\gamma}(\bar{\tau})\underline{\gamma}^*(\bar{\tau})=\phi(0)\ \leftarrow$ by stationarity

$$\left(\omega_1=1,\omega_2=-1;\gamma(0),\gamma(\bar{\tau})\right)=e^{-\frac{1}{2}[1-1]\begin{bmatrix}\phi(0) & \phi(\tau) \\ \phi^*(\tau) & \phi(0)\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix}}$$

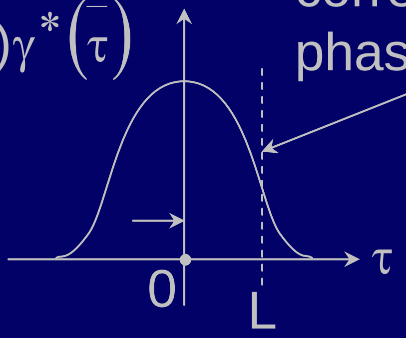
$$=e^{\phi(\bar{\tau})-\phi(0)}$$

Therefore $E\left\{R_{\underline{\underline{E}}_x}(\bar{\tau})\right\}=R_{\underline{\underline{E}}_o}(\bar{\tau})\bullet e^{\phi(\bar{\tau})-\phi(0)}$

Computation of Expected Directivity

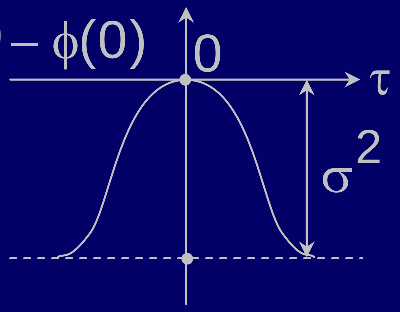
$$E\{D(f,\theta,\phi)\} = \left[\frac{\pi(1+\cos\theta)^2}{\lambda^2 \int_A \left|\underline{E}(\bar{r})\right|^2 da}\right] \bullet \int_A \left[e^{\phi(\bar{\tau})-\phi(0)}\, R_{\underline{E}_o}(\bar{\tau})\right] e^{-j\frac{2\pi}{\lambda}\bar{\varphi}\bullet\bar{\tau}}\, d\tau_x d\tau_y$$

$$\phi(\bar{\tau}) = \overline{\gamma(0)\gamma^*(\bar{\tau})}$$



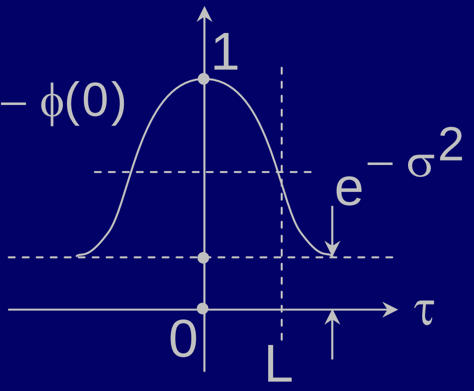
correlation length L of phase irregularities

$$\phi(\bar{\tau}) - \phi(0)$$



$$\sigma^2 \equiv \phi(0)$$

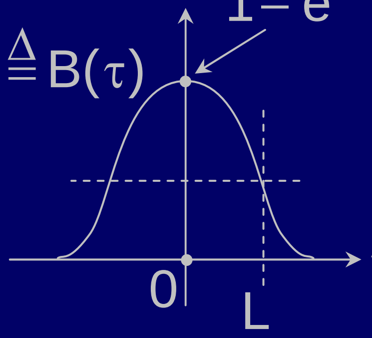
$$e^{\phi(\tau)-\phi(0)}$$



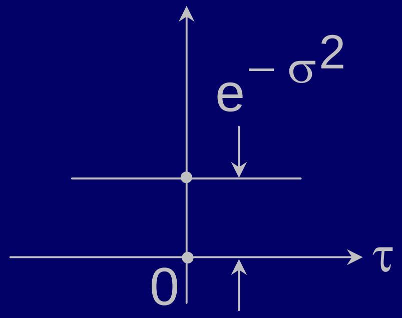
$$=$$

$$\stackrel{\Delta}{=} B(\tau)$$

$$1 - e^{-\sigma^2}$$

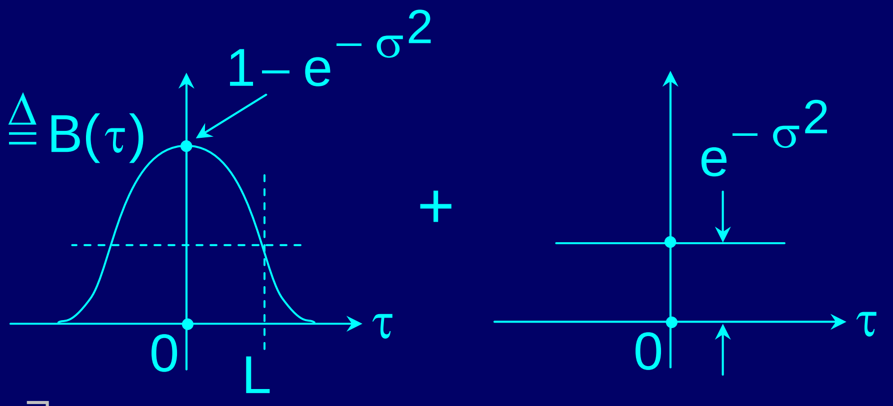


$$+$$



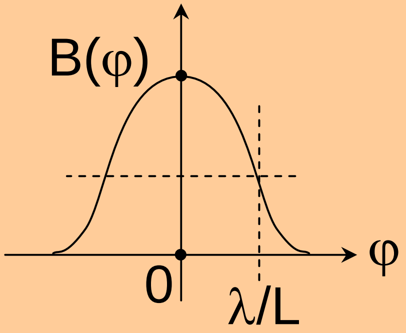
Solution to Expected Directivity

$$e^{\phi(\tau)-\phi(0)} =$$



$$E\{D(f,\theta,\phi)\} = \left[\frac{\pi(1+\cos\theta)^2}{\lambda^2 \int_A \left|E(\bar{r})\right|^2 da}\right] \bullet \int_A \left[e^{-\sigma^2} + B(\bar{\tau})\right] \bullet R_{\underline{E}_o(\bar{\tau})} \bullet e^{-j\frac{2\pi}{\lambda}\bar{\varphi}\bullet\bar{\tau}} d\tau_x d\tau_y$$

$$E\{D(f,\theta,\phi)\} = \underbrace{e^{-\sigma^2} D_o(f,\theta,\phi)}_{\text{gain degradation}} + \underbrace{B(\bar{\varphi}) * D_o(f,\theta,\phi)}_{\text{sidelobe increase}}$$



Examples of Random Antenna Surface

Let b = RMS surface tolerance of reflector antenna

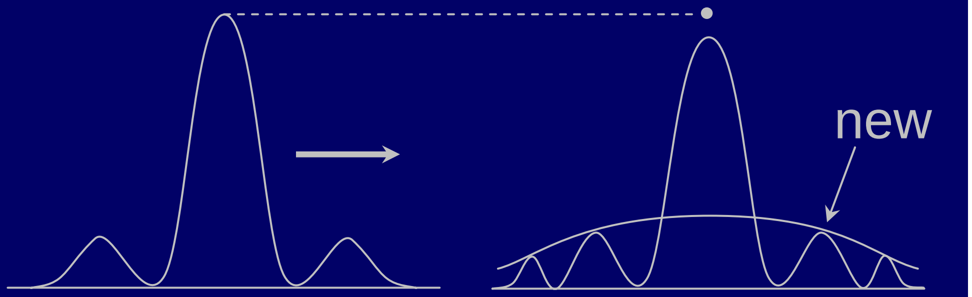
On-axis gain of random antenna

$$G'_o = G_o e^{-\sigma^2} = G_o e^{-(2b\bullet 2\pi/\lambda)^2} = G_o e^{-(b4\pi/\lambda)^2}$$

$$\text{If} \quad b = \lambda/4\pi \Rightarrow G_o \bullet e^{-1}$$

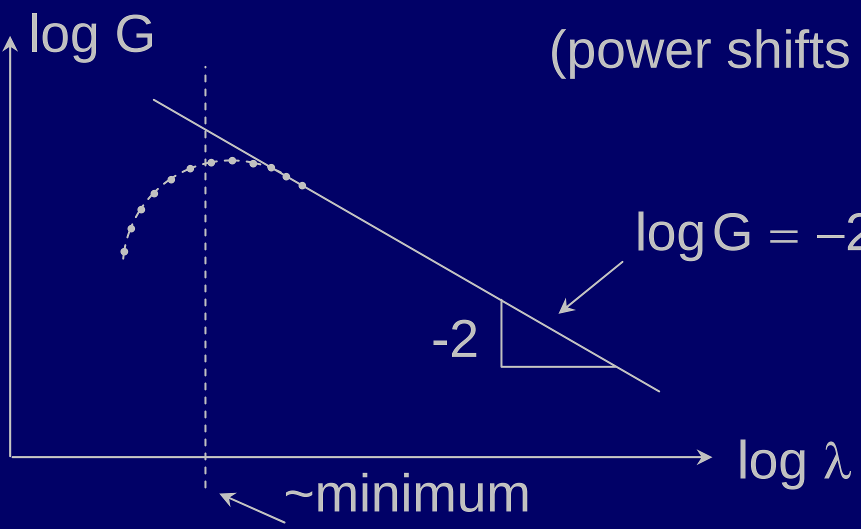
$$b = \lambda/16 \Rightarrow G_o \bullet 0.54$$

$$b = \lambda/32 \Rightarrow G_o \bullet 0.9$$



(power shifts to sidelobes)

Any aperture antenna, fixed illumination



$$\log G = -2\log\lambda + \log 4\pi A_e$$