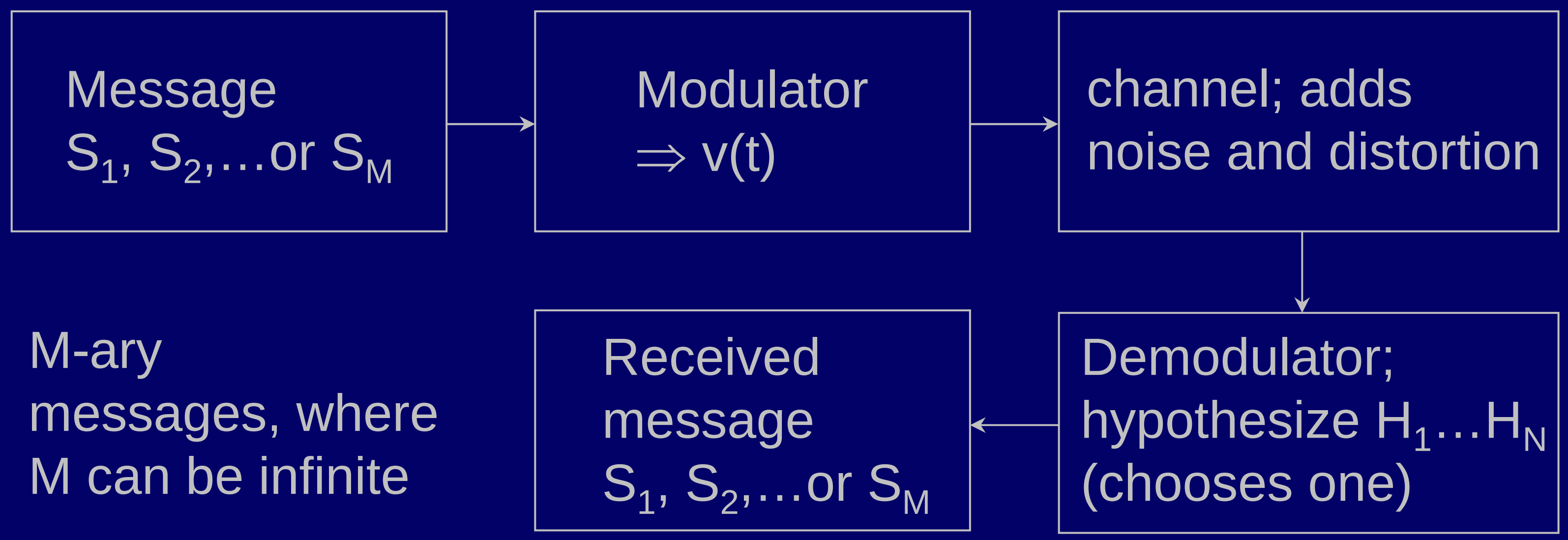
Types of Communication

Analog: continuous variables with noise $\Rightarrow$
P{error = 0} = 0 (imperfect)

Digital: decisions, discrete choices, quantized, noise $\Rightarrow$
P{error} $\rightarrow$ 0 (usually perfect)



The channel can be radio, optical, acoustic, a memory device (recorder), or other objects of interest as in radar, sonar, lidar, or other scientific observations.

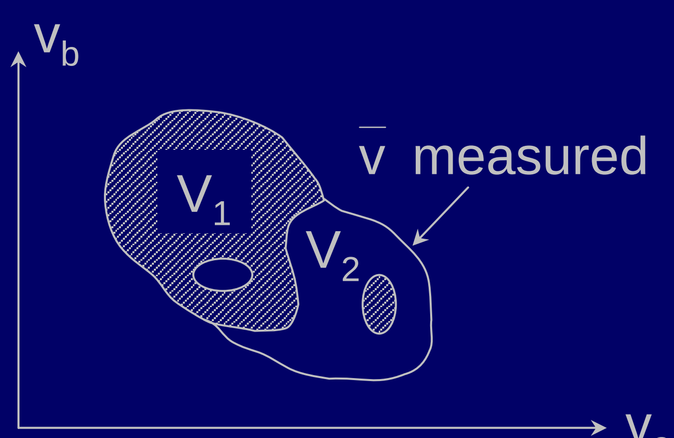
Optimum Demodulator for Binary Messages

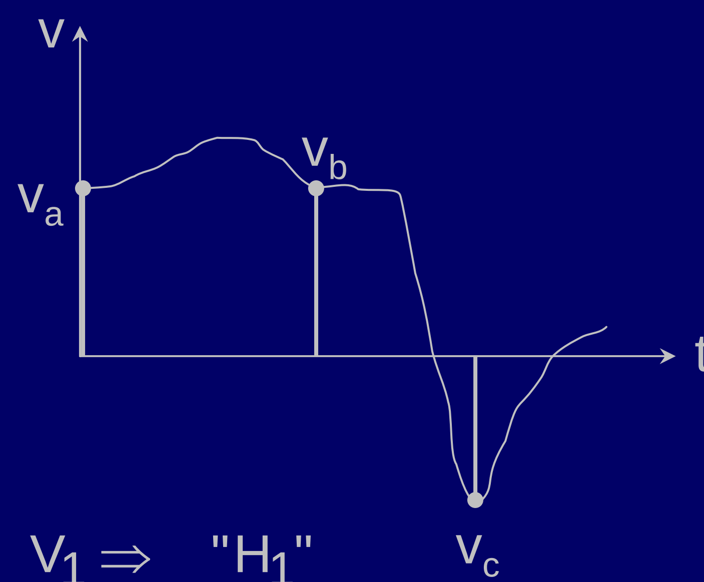
	Hypothesis:	H_1	H_2	Probability a priori
Message:	S_1	OK	ERROR	P_1
	S_2	ERROR	OK	P_2

Demodulator design

E.G.

2-D case





$\bar{v} \in V_1 \Rightarrow$ "H_1"

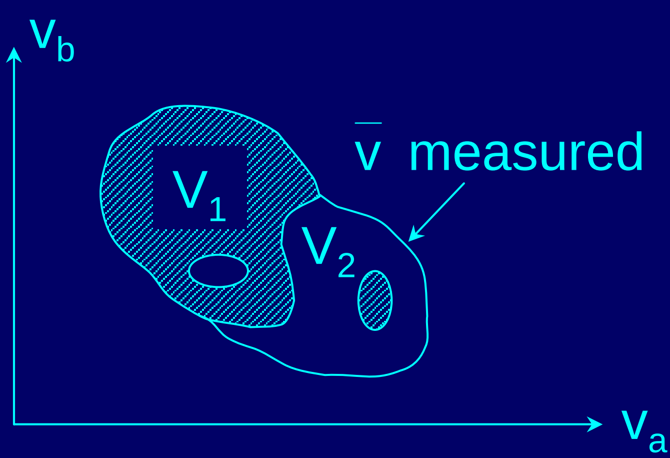
$\bar{v} \in V_2 \Rightarrow$ "H_2"

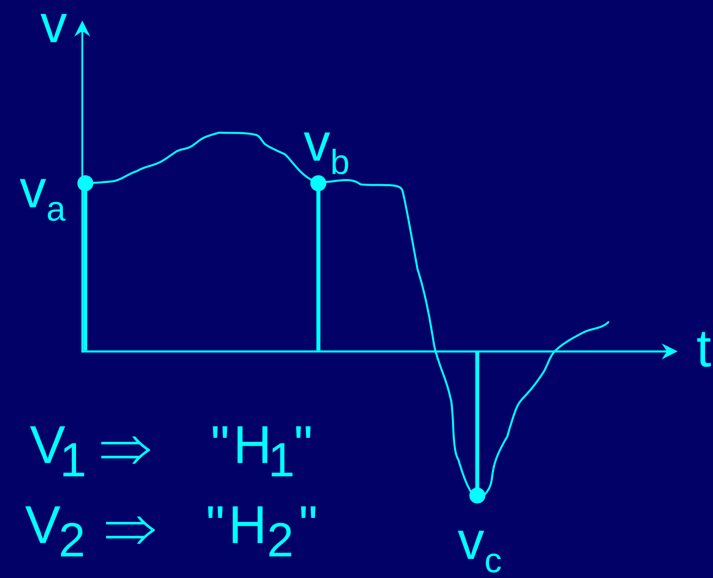
How to define V_1, V_2?

Optimum Demodulator for Binary Messages

E.G.

2-D case





$\bar{v} \in V_1 \Rightarrow$ "H_1"

$\bar{v} \in V_2 \Rightarrow$ "H_2"

How to define V_1, V_2?

Minimize $P_{error} \triangleq P_e = P_1 \underbrace{\int_{V_2} p\{\bar{v} | S_1\}}_{\text{replace with } \int_{V_1}} d\bar{v} + P_2 \int_{V_1} p\{\bar{v} | S_2\}\, d\bar{v}$

$$= P_1 + \int_{V_1} \left[P_2 p\{\bar{v} | S_2\} - P_1 p\{\bar{v} | S_1\} \right] d\bar{v}$$

Note: $\int_{V_1} p\{\bar{v} | S_1\} d\bar{v} + \int_{V_2} p\{\bar{v} | S_1\} d\bar{v} = 1$

Optimum Demodulator for Binary Messages

$$P_e = P_1 + \int_{V_1} \left[P_2 p\{\bar{v}|S_2\} - P_1 p\{\bar{v}|S_1\} \right] \overline{dv}$$

To minimize P_{error} , choose $V_1 \ni P_1 p\{\bar{v}|S_1\} > P_2 p\{\bar{v}|S_2\}$

Very general solution ↑

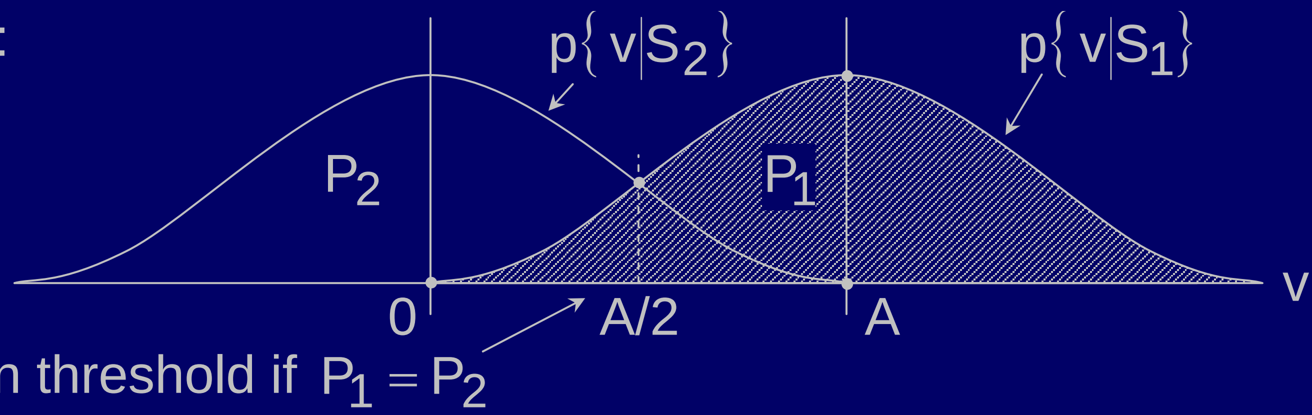
[i.e., choose maximum *a posteriori* P ("MAP" estimate)]

Example: Binary Scalar Signal Case

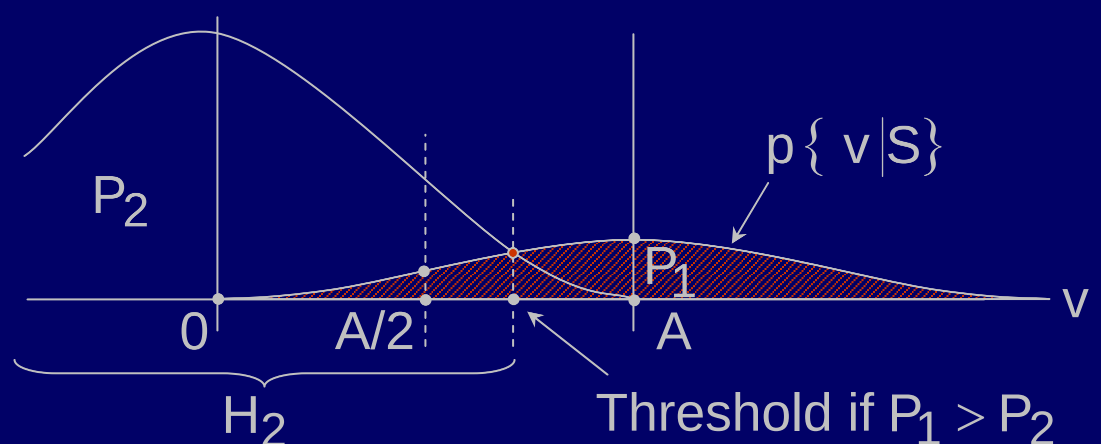
$$S_1 \triangleq A \text{ volts}, \quad S_2 \triangleq O \text{ volts}, \quad \sigma_n^2 \triangleq N, \text{ Gaussian noise}$$

$$\therefore p\{v|S_1\} = \frac{1}{\sqrt{2\pi N}} e^{-(v-A)^2/2N} \qquad p\{v|S_2\} = \frac{1}{\sqrt{2\pi N}} e^{-v/2N}$$

If $P_1 = P_2$:



(bias choise toward H_2 and *a priori* information)



Rule For Defining V_1 : (Binary Scalar Case)

Choose $V_1 \ni P_1 p\{\bar{v}|S_1\} > P_2 p\{\bar{v}|S_2\}$ $\qquad p\{v|S_2\} = \frac{1}{\sqrt{2\pi N}} e^{-v^2/2N}$

(binary case)
"Likelihood ratio" $\qquad \ell \triangleq \frac{p\{\bar{v}|S_1\}}{p\{\bar{v}|S_2\}} > \frac{P_2}{P_1} \Rightarrow "V_1"$

or (equivalently) $\qquad \ell n\ \ell > \ell n(P_2/P_1) \Rightarrow "V_1"$

For additive Gaussian noise,

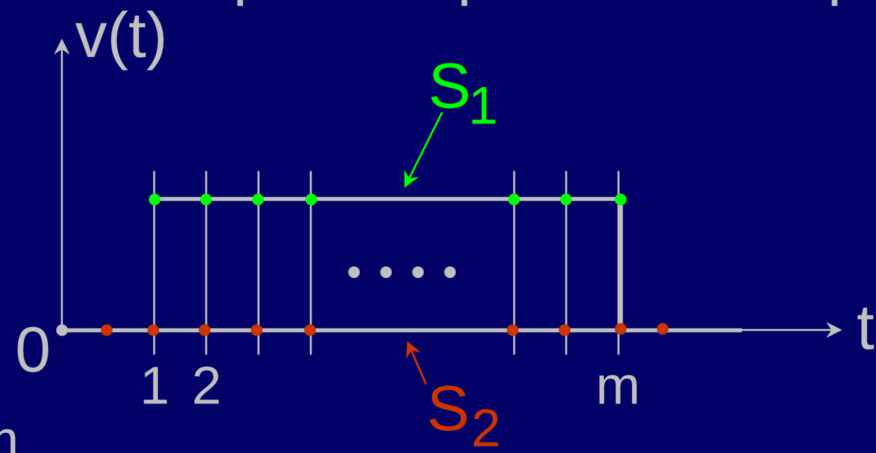
$$\ell n\ \ell = \left[-(v-A)^2/2N\right] + v^2/2N = \left(2vA - A^2\right)/2N \overset{?}{>} \ell n(P_2/P_1)$$

$$\therefore \text{choose } V_1 \text{ if } v > \frac{A^2 + 2N\ell n(P_2/P_1)}{2A}, \text{ or } v > \frac{A}{2} + \underbrace{\frac{N}{A}\ell n(P_2/P_1)}_{\text{bias}}$$

Binary Vector Signal Case

For better performance, use multiple independent samples:

$$\ell \triangleq \frac{p\{\bar{v}|S_1\}}{p\{\bar{v}|S_2\}} \overset{?}{>} \frac{P_2}{P_1}$$



Here $P\{v_1, v_2, \ldots, v_m|S_1\} = \prod_{i=1}^{m} p\{v_i|S_1\}$ (independent noise samples)

Where $p\{v_i|S_i\} = \dfrac{1}{\sqrt{2\pi N}} e^{-\left(v_i - S_{1i}\right)^2 / 2N}$

$$p\{\bar{v}|S_i\} = \frac{1}{\left(\sqrt{2\pi N}\right)^m} e^{-\sum_{i=1}^{m} \left(v_i - S_{1i}\right)^2 / 2N}$$

Binary Vector Signal Case

$$p\{\bar{v}|S_i\} = \frac{1}{\left(\sqrt{2\pi N}\right)^m} e^{-\sum_{i-1}^{m}\left(v_i - S_{1_i}\right)^2 / 2N} \qquad \ell \triangleq \frac{p\{\bar{v}|S_1\}}{p\{\bar{v}|S_2\}} \overset{?}{>} \frac{P_2}{P_1}$$

Thus the test becomes:

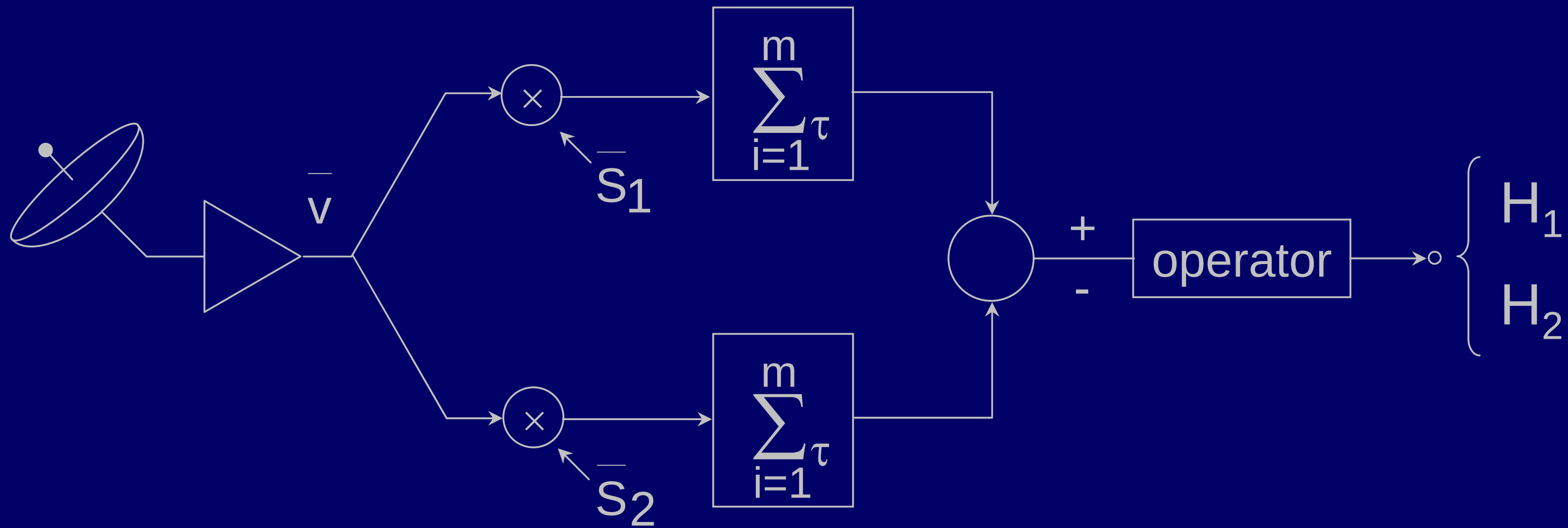
$$\ell n \ \ell = \frac{1}{2N}\Bigg[\underbrace{\sum_{i=1}^{m}\left(v_i - S_{2_i}\right)^2}_{|\bar{v}-\bar{S}_2|^2} - \underbrace{\sum_{i=1}^{m}\left(v_i - S_{1_i}\right)^2}_{|\bar{v}-\bar{S}_1|^2}\Bigg] \overset{?}{>} \ell n \frac{P_2}{P_1}$$

But $|\bar{v} - \bar{S}_2|^2 - |\bar{v} - \bar{S}_1|^2 = -2\bar{v} \bullet \bar{S}_2 + 2\bar{v} \bullet \bar{S}_1 + \bar{S}_2 \bullet \bar{S}_2 - \bar{S}_1 \bullet \bar{S}_1$

Therefore $\bar{v} \in V_1$ iff $\bar{v} \bullet (\bar{S}_1 - \bar{S}_2) > \underbrace{\frac{\bar{S}_1 \bullet \bar{S}_1^* - \bar{S}_2 \bullet \bar{S}_2^*}{2}}_{\text{Bias} = 0 \text{ if energy } E_1 = E_2} + \underbrace{N \ \ell n\left(\frac{P_2}{P_1}\right)}_{\text{Bias} = 0 \text{ if } P_2 = P_1}$

Binary Vector Signal Case

$$V_1 \text{ iff } \bar{v}_1 \bullet (\bar{S}_1 - \bar{S}_2) > \frac{\bar{S}_1 \bullet \bar{S}_1 - \bar{S}_2 \bullet \bar{S}_2}{2} + N\,\ell n\left(\frac{P_2}{P_1}\right)$$



Multiple hypothesis generalization:

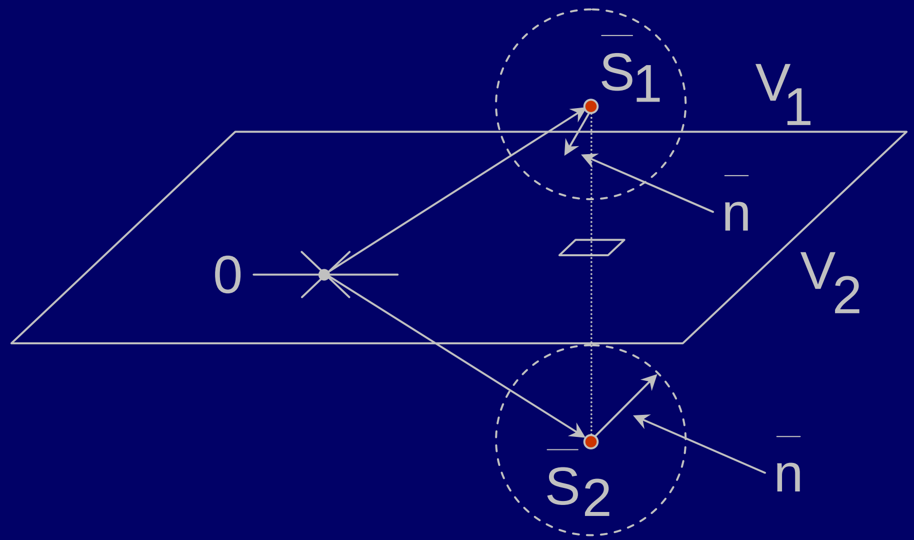
$$\text{Choose } H_i \text{ if } f_i \overset{\Delta}{=} \bar{v} \bullet \bar{S}_i - \frac{\bar{S}_i \bullet \bar{S}_i}{2} + N\,\ell n\, P_i \overset{?}{>} \text{all } f_{j \neq i}$$

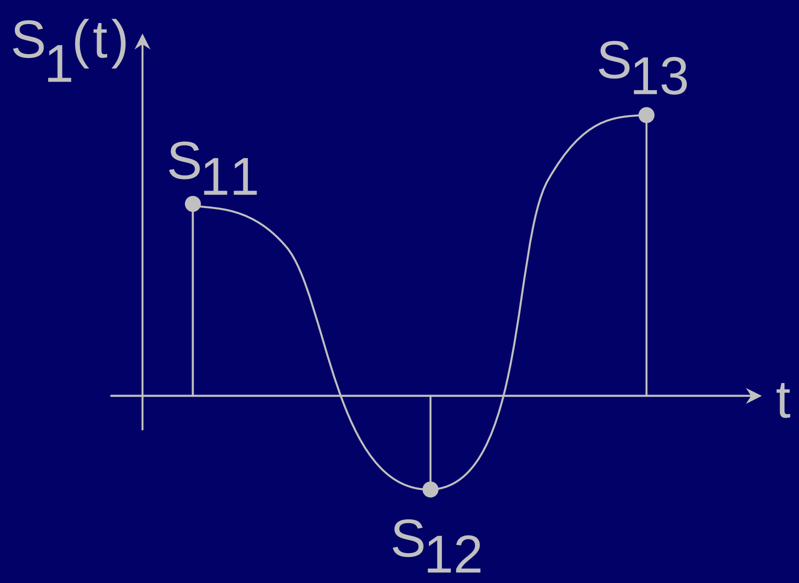
This "matched filter" receiver minimizes P_{error}

Graphical Representation of Received Signals

3-D Case:

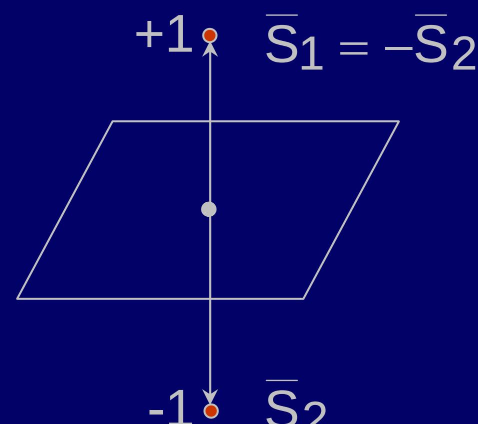
$$\text{Average energy} = \sum_{i=1}^{2} \left|\overline{S}_i\right|^2 P_i$$



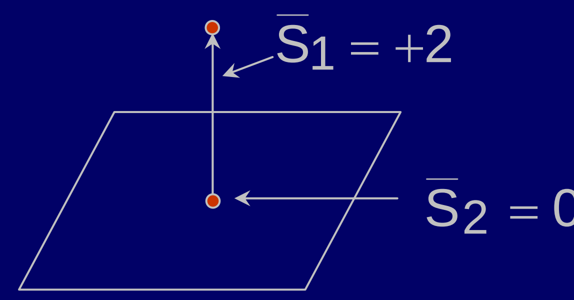


Design of Signals $\bar{S}_i$

E.G. consider



vs.



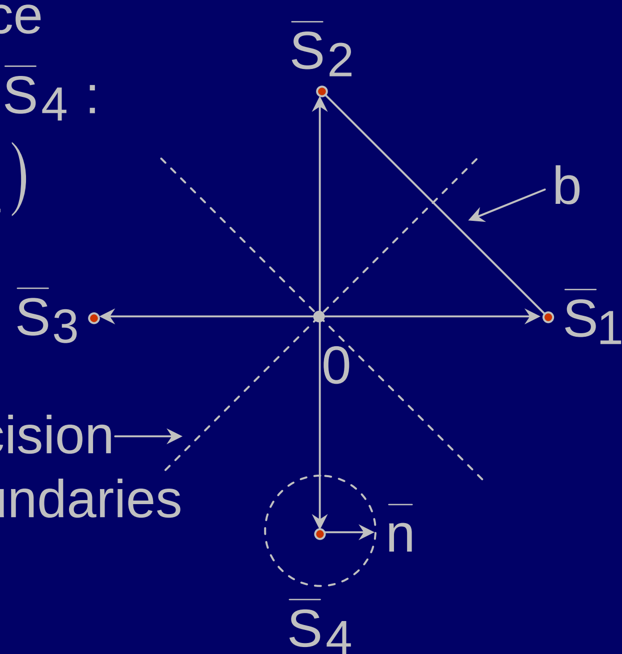
$\langle E \rangle = 2(P_1 = P_2)$

Average energy $\left\langle |S|^2 \right\rangle = \langle E \rangle = 1$

E.G. 2-D space

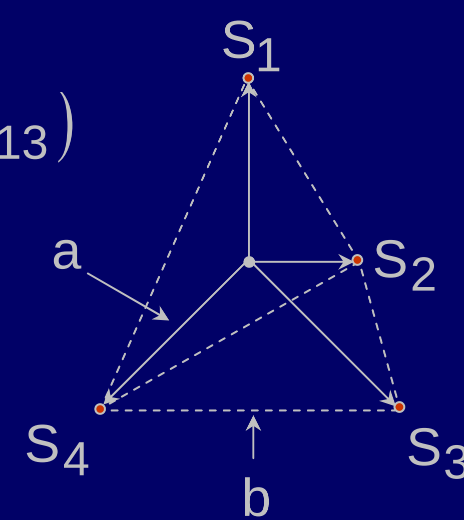
for $\bar{S}_1, \bar{S}_2, \bar{S}_3, \bar{S}_4$:

$(\bar{S}_1 = S_{11}, S_{12})$



3-D space

$(\bar{S}_1 = S_{11}, S_{12}, S_{13})$

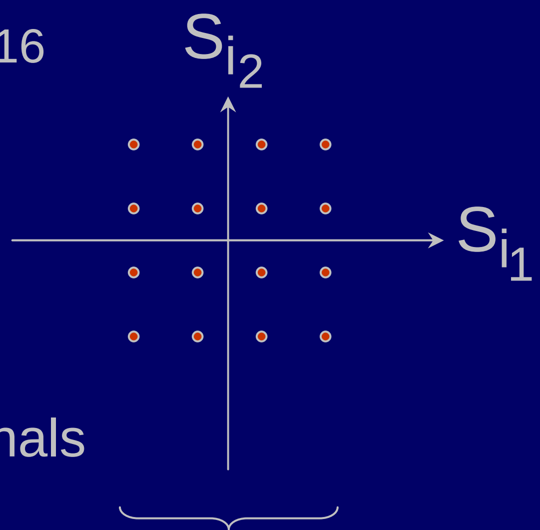


Better $\frac{b}{a}$ ratio

Design of Signals $\bar{S}_i$

2-D space:

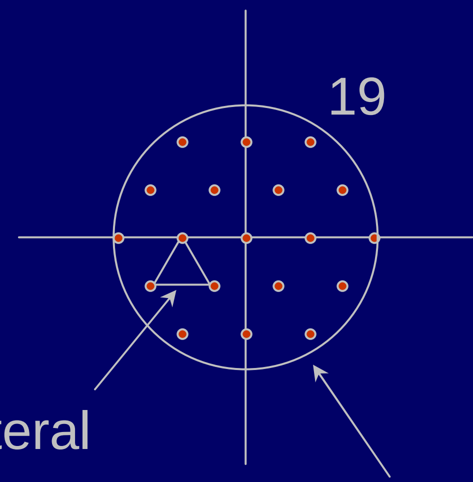
$\bar{S}_1,\ldots,\bar{S}_{16}$



16-ary signals

or magnitude/phase

vs



equilateral triangle

slightly lower average signal energy for same p{error}

n-Dimensional sphere packing optimization unsolved

Calculation of p{error} $\stackrel{\Delta}{=} P_e$:

Binary case:

For additive Gaussian noise, optimum is

$$\text{"}H_1\text{" if } \quad \bar{v} \bullet (\bar{S}_1 - \bar{S}_2) > \frac{|\bar{S}_1|^2 - |\bar{S}_2|^2}{2} + N \,\ell n \frac{P_2}{P_1}$$

Where $\bar{v} = \bar{S} + \bar{n}$

$$N \stackrel{\Delta}{=} \overline{n^2(t)} = N_o B = kT_s B \quad \left(N_o / 2 \left[W \ Hz^{-1}\right] \times 2B, \text{ double sideband}\right)$$

$$P_e\Big|_{S_1} = p\left\{\bar{v} \bullet (\bar{S}_1 - \bar{S}_2) < \frac{|\bar{S}_1|^2 - |\bar{S}_2|^2}{2} + N \,\ell n \frac{P_2}{P_1}\right\}$$

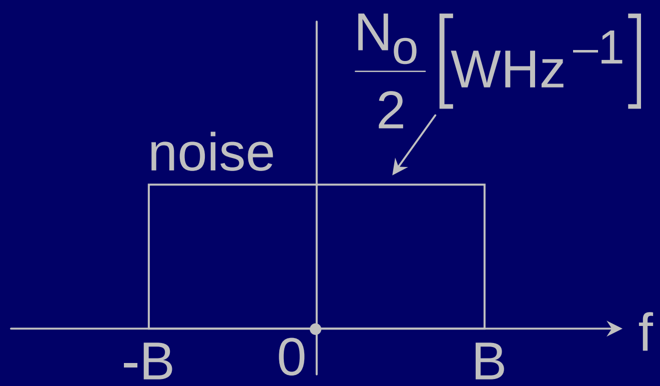
$$= p\Bigg\{\underbrace{\bar{n} \bullet (\bar{S}_1 - \bar{S}_2)}_{y \,\bullet\, 2B[GRVZM]} < \underbrace{\frac{-|\bar{S}_1 - \bar{S}_2|^2}{2} + N \,\ell n \frac{P_2}{P_1}}_{-b \,\bullet\, 2B}\Bigg\} = p\{y < -b\}$$

Duality of Continuous and Sampled Signals

$$P_e\Big|_{S_1} = p\left\{ \underbrace{\bar{n} \bullet \left(\bar{S}_1 - \bar{S}_2\right)}_{y \bullet 2B[\text{GRVZM}]} < \underbrace{\frac{-\left|\bar{S}_1 - \bar{S}_2\right|^2}{2} + N\,\ell n\frac{P_2}{P_1}}_{-b \bullet 2B} \right\} = p\{y < -b\}$$

Conversion to continuous signals assuming nyquist sampling is helpful here, $S_1(t)[0 < t < T] \leftrightarrow \bar{S}_1$ (2BT samples, sampling theorem)

$$y \overset{\Delta}{=} \int_o^T n(t) \bullet [S_1(t) - S_2(t)]\,dt$$



$$b \overset{\Delta}{=} \frac{1}{2}\int_o^T [S_1(t) - S_2(t)]^2\,dt - \frac{N_o}{2}\,\ell n\left(P_2/P_1\right)$$

$$\sigma_y^2 \overset{\Delta}{=} E\left[y^2\right] = E\left\{\left[\frac{1}{2B}\sum_{j=1}^{2BT} n_j\left(S_{1j} - S_{2j}\right)\right]^2\right\}$$

Calculation of P_e, continued

$$\sigma_y^2 \stackrel{\Delta}{=} E\left[y^2\right] = E\left\{\left[\frac{1}{2B}\sum_{j=1}^{2BT} n_j\left(S_{1j} - S_{2j}\right)\right]^2\right\}$$

$$= \left(\frac{1}{2B}\right)^2 E\left\{\sum_{i=1}^{2BT}\sum_{j=1}^{2BT} n_i n_j \left(S_{1i} - S_{2_i}\right)\left(S_{1j} - S_{2_j}\right)\right\}$$

where $E\left[n_i n_j\right] = N\delta_{ij}$

$$\sigma_y^2 = \left(\frac{1}{2B}\right)^2 N\left|\bar{S}_1 - \bar{S}_2\right|^2 = \frac{N_o}{2}\int_o^T \left[S_1(t) - S_2(t)\right]^2 dt$$

where $N = N_oB$ and $\left|\bar{S}_1 - \bar{S}_2\right|^2 = 2B\int_o^T \left[S_1(t) - S_2(t)\right]^2 dt$

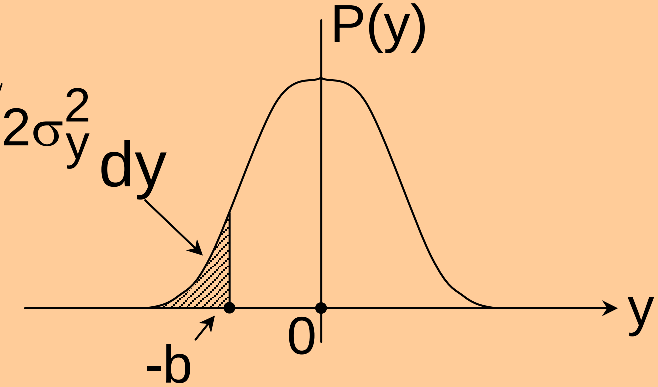
Calculation of P_e, continued

$$\sigma_y^2 = \left(\frac{1}{2B}\right)^2 \underset{\uparrow\, N_oB}{N} \underbrace{\left|\bar{S}_1 - \bar{S}_2\right|^2}_{2B\int_o^T [S_1(t) - S_2(t)]^2 dt} = \frac{N_o}{2}\int_o^T [S_1(t) - S_2(t)]^2 dt$$

$$p(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-y^2 / 2\sigma_y^2} \text{ (GRVZM)}$$

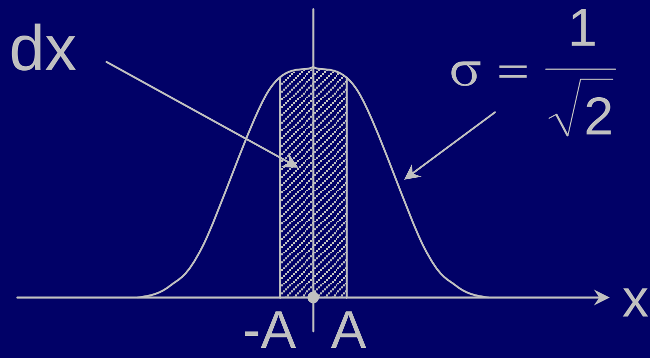
Therefore:

$$P_e\Big|_{S_1} = \int_{-\infty}^{-b} \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-y^2/2\sigma_y^2} dy$$



Definition of ERFC(A)

"Error function" $\text{ERF}(A) \triangleq \frac{1}{\sqrt{\pi}} \int_{-A}^{A} e^{-x^2} dx$



"Complementary error function"

$\text{ERFC}(A) \triangleq 1 - \text{ERF}(A)$

Then $P_e\big|_{S_1} = \frac{1}{2}\text{ERFC}(A)$, where A must be found

If we let $x^2 = y^2 \big/ 2\sigma_y^2$ then

$$\text{ERF}(A) = \frac{1}{\sqrt{\pi}} \int_{-A}^{A} e^{-x^2} dx = \frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{-A\sigma_y\sqrt{2}}^{A\sigma_y\sqrt{2}} e^{-y^2 / 2\sigma_y^2} dy$$

where the new limits $A\sigma\sqrt{2}$ and factor $1\big/\sqrt{2\pi\sigma_y^2}$ arise as follows:

Definition of ERFC(A)

$$\mathrm{ERF}(A) = \frac{1}{\sqrt{\pi}} \int_{-A}^{A} e^{-x^2} dx = \frac{1}{\sqrt{2\pi\sigma_y^2}} \int_{-A\sigma_y\sqrt{2}}^{A\sigma_y\sqrt{2}} e^{-y/2\sigma_y^2} dy$$

where the new limits $A\sigma\sqrt{2}$ and factor $1/\sqrt{2\pi\sigma_y^2}$ arise as follows:

Since $x = y/\sigma_y\sqrt{2}$, the limit $x = A = y/\sigma_y\sqrt{2}$

becomes a limit where $y = A\sigma_y\sqrt{2}$

Also, $dx = dy/\sigma_y\sqrt{2}$ so $1/\sqrt{\pi}$ becomes $1/\sqrt{2\pi\sigma_y^2}$

Solution for P_e for Binary Signals

$$P_e\big|_{S_1} = \frac{1}{2}\text{ERFC}(A) = \frac{1}{2}\text{ERFC}\left(b\big/\sigma_y\sqrt{2}\right) \quad \text{and} \quad P_e = P_1 P_e\big|_{S_1} + P_2 P_e\big|_{S_2}$$

$$\left(\text{where the limit } b = A\sigma_y\sqrt{2}\text{, so } A = b\big/\sigma_y\sqrt{2}\right)$$

If $P_1 = P_2 = \frac{1}{2}$, and since $P_e\big|_{S_1} = P_e\big|_{S_2}$, then

$$P_e = \frac{1}{2}\text{ERFC}\left(b\big/\sigma_y\sqrt{2}\right)$$

$$= \frac{1}{2}\text{ERFC}\left[\frac{\frac{1}{2}\int_0^T [S_1(t) - S_2(t)]^2\,dt}{\sqrt{2}\sqrt{(N_o/2)\int_0^T [S_1(t) - S_2(t)]^2\,dt}}\right]$$

Solution for P_e for Binary Signals

$$P_e = \frac{1}{2}\text{ERFC}\left[\frac{\frac{1}{2}\int_0^T [S_1(t) - S_2(t)]^2\,dt}{\sqrt{2}\sqrt{(N_o/2)\int_0^T [S_1(t) - S_2(t)]^2\,dt}}\right]$$

$$P_e = \frac{1}{2}\text{ERFC}\left[\frac{1}{2}\sqrt{\int_0^T [S_1(t) - S_2(t)]^2\,dt / N_o}\right]$$

If $\int_0^T S_1^2(t)dt + \int_0^T S_2^2(t)dt$ is fixed for $P_1 = P_2$ then

To minimize P_e, let $S_2(t) = -S_1(t)$ $\left(\text{maximizes } \int_0^T [S_1(t) - S_2(t)]^2\,dt\right)$

Examples of Binary Communications Systems

$$P_e = \frac{1}{2} \text{ERFC}\left[\frac{1}{2}\sqrt{\int_0^T [S_1(t) - S_2(t)]^2 \, dt / N_o}\right]$$

Assume $P_1 = P_2 = \frac{1}{2}$ and define $\int_0^T s_1^2(t)dt \triangleq E$

Modulation type	$s_1(t)$	$s_2(t)$	P_e
"OOK" (on-off keying)	$A\cos\omega_o t$	0	$\frac{1}{2}\text{ERFC}\sqrt{E/4N_o} = \frac{1}{2}\text{ERFC}\sqrt{E_{avg}/2N_o}$
"FSK" (frequency-shift keying)	$A\cos\omega_1 t$	$A\cos\omega_2 t$	$\frac{1}{2}\text{ERFC}\sqrt{E_{avg}/2N_o}$
"BPSK" binary phase-shift keying)	$A\cos\omega t$	$-A\cos\omega t$	$\frac{1}{2}\text{ERFC}\sqrt{E_{avg}/N_o}$

Examples of Binary Communications Systems

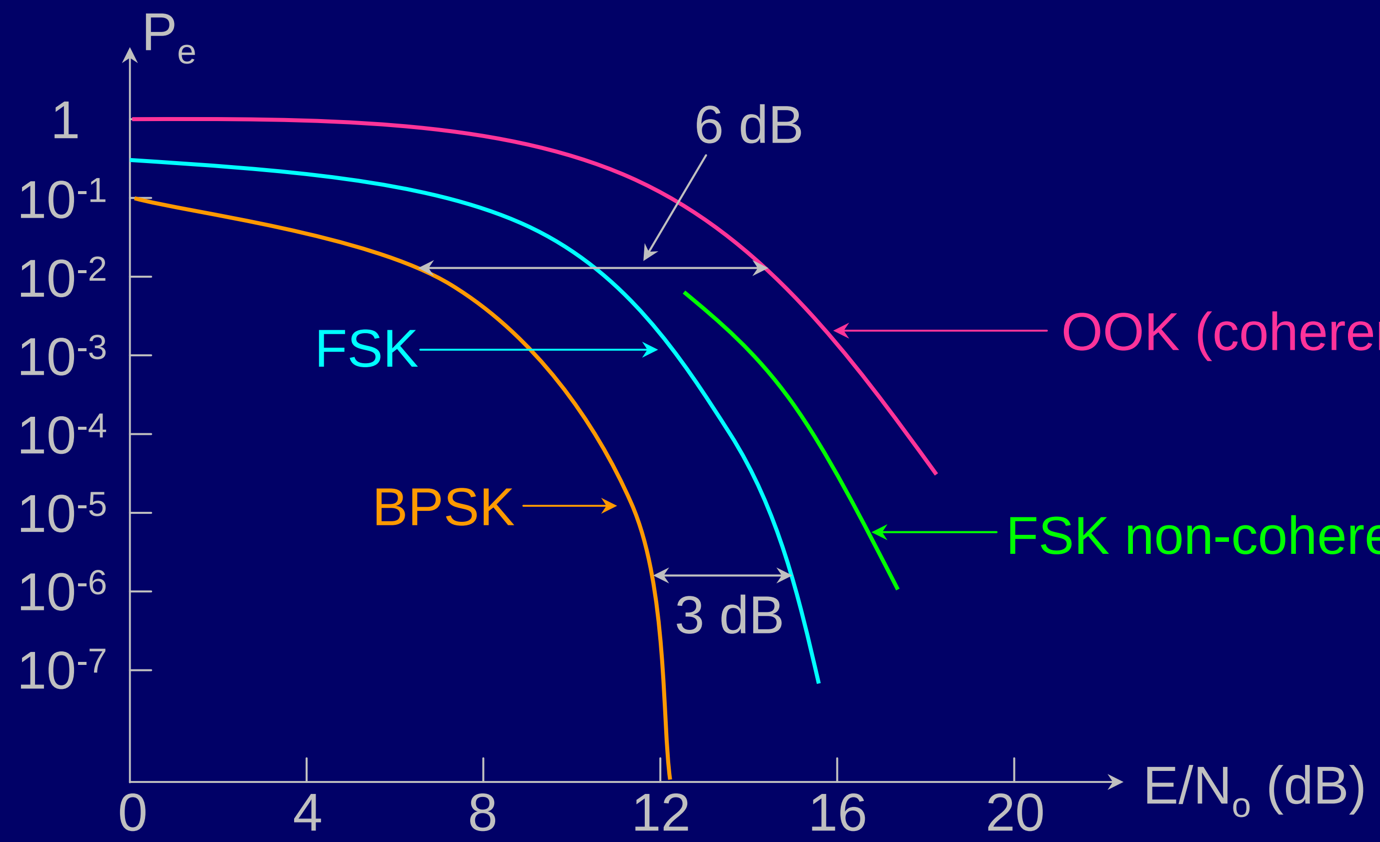
Note:

$$P_e = f\left(E_{AVG}/N_o\right)$$

E_{AVG}: $[J]$ $\qquad$ N_o: $\left[W\,Hz^{-1} = J\right]$

Cost of communications $\propto$ cost of energy, Joules per bit
(e.g. very low bit rates imply very low
power transmitters, small antennas)

Probability of Baud Error



Non-coherent FSK: carrier is unsynchronized so that both sine and cosine terms admitted, increasing noise. Such "envelope detectors have a different form of $P_e(E/N_o)$.

Note how rapidly P_e declines for $E/N_o \gtrsim 12 - 16$ dB