6.003: Signals and Systems

Z Transform

February 23, 2010

Mid-term Examination #1

Wednesday, March 3, 7:30-9:30pm.

No recitations on the day of the exam.

Coverage: Representations of CT and DT Systems
Lectures 1–7
Recitations 1–8
Homeworks 1–4

Homework 4 will not collected or graded. Solutions will be posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Z Transform

Z transform is discrete-time analog of Laplace transform.

Furthermore, you already know about Z transforms (we just haven't called them Z transforms)!

Example: Fibonacci system

difference equation	$y[n] = x[n] + y[n-1] + y[n-2]$
operator expression	$Y = X + \mathcal{R}Y + \mathcal{R}^2 Y$
system functional	$\frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$
unit-sample response	$h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Check Yourself

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What is the relation between system functional and $h[n]$?

Check Yourself

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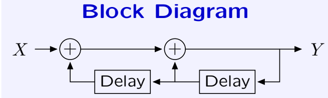
$$\frac{Y}{X} = \sum_n h[n]\mathcal{R}^n$$

What's the relation between $H(z)$ and $h[n]$?

Concept Map: Discrete-Time Systems

Relation between Unit-Sample Response and System Function.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$$

series

$\mathcal{R} \to \frac{1}{z}$

Unit-Sample Response

$h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Z transform

Difference Equation

$y[n] = x[n] + y[n-1] + y[n-2]$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1-z-z^2}$$

Z Transform

Z transform is discrete-time analog of Laplace transform.

Z transform maps a function of discrete time n to a function of z.

$$X(z) = \sum_n x[n]z^{-n}$$

There are two important variants:

Unilateral

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

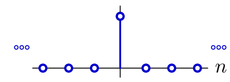
Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Differences are analogous to those for the Laplace transform.

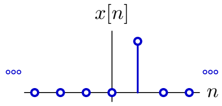
Check Yourself

Find the Z transform of the unit-sample signal.



Check Yourself

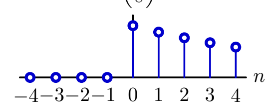
Find the Z transform of a delayed unit-sample signal.



Z Transforms

Example: Find the Z transform of the following signal.

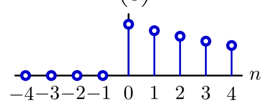
$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



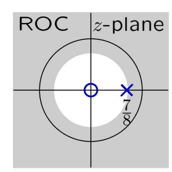
$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n}u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1-\frac{7}{8}z^{-1}} = \frac{z}{z-\frac{7}{8}}$$

provided $\left|\frac{7}{8}z^{-1}\right| < 1$, i.e., $|z| > \frac{7}{8}$.

$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



$$\frac{z}{z-\frac{7}{8}}$$



Shape of ROC

Regions of converge for Z transform are delimited by circles.

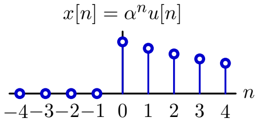
Example: $x[n] = \alpha^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$

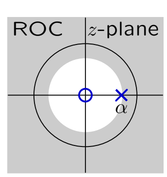
$$= \frac{1}{1-\alpha z^{-1}}\,; \quad \left|\alpha z^{-1}\right| < 1$$

$$= \frac{z}{z-\alpha}\,; \quad |z| > |\alpha|$$

$$x[n] = \alpha^n u[n]$$



$$\frac{z}{z-\alpha}$$



Shape of ROC

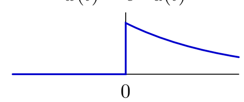
Regions of converge for Laplace transform delimited by vertical lines.

Example: $x(t) = e^{\sigma t}u(t)$

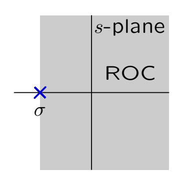
$$X(s) = \int_{-\infty}^{\infty} e^{\sigma t}u(t)e^{-st}dt = \int_0^{\infty} e^{\sigma t}e^{-st}dt$$

$$= \frac{1}{s-\sigma}\,; \quad \mathrm{Re}(s) > \mathrm{Re}(\sigma)$$

$$x(t) = e^{\sigma t}u(t)$$



$$\frac{1}{s-\sigma}$$



Distinguishing Features of Transforms

Most-important feature of Laplace transforms is the derivative rule:

$$x(t) \leftrightarrow X(s)$$
$$\dot{x}(t) \leftrightarrow sX(s)$$

→ allows us to use Laplace transforms to solve differential equations.

Similarly, most-important feature of Z transforms is the delay rule:

$$x[n] \leftrightarrow X(z)$$
$$x[n-1] \leftrightarrow z^{-1}X(z)$$

→ allows us to use Z transforms to solve difference equations.

Distinguishing Features of Transforms

Delay property

$$x[n] \leftrightarrow X(z)$$
$$x[n-1] \leftrightarrow z^{-1}X(z)$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Let $y[n] = x[n-1]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n-1]z^{-n}$$

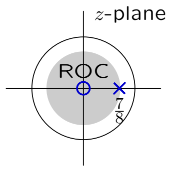
Substitute $m = n-1$

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m]z^{-m-1} = z^{-1}X(z)$$

Check Yourself

What DT signal has the following Z transform?

$$\frac{z}{z-\frac{7}{8}}\,; \quad |z| < \frac{7}{8}$$



Check Yourself

Find the inverse transform of

$$X(z) = \frac{-3z}{2z^2 - 5z + 2}$$

given that the ROC includes the unit circle.

Solving Difference Equations with Z Transforms

Start with difference equation:

$$y[n] - \frac{1}{2}y[n-1] = \delta[n]$$

Take the Z transform of this equation:

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = 1$$

Solve for $Y(z)$:

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Take the inverse Z transform (by recognizing the form of the transform):

$$y[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse Z transform

The inverse Z transform is defined by an integral that is not particularly easy to solve.

Formally,

$$x[n] = \frac{1}{2\pi j} \int_C X(z) z^{n-1} ds$$

were C represents a closed contour that circles the origin by running in a counterclockwise direction through the region of convergence. This integral is not generally easy to compute.

This equation can be useful to prove theorems.

There are better ways (e.g., partial fractions) to compute inverse transforms for the kinds of systems that we frequently encounter.

Properties of Z Transforms

The use of Z Transforms to solve differential equations depends on several important properties.

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n-1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

Check Yourself

Find the inverse transform of $Y(z)=\left(\dfrac{z}{z-1}\right)^2$; $|z|>1$.

Concept Map: Discrete-Time Systems

Relations among representations.