6.003: Signals and Systems

CT Fourier Transform

April 8, 2010

CT Fourier Transform

Representing signals by their frequency content.

$$X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \qquad \text{(``analysis'' equation)}$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad \text{(``synthesis'' equation)}$$

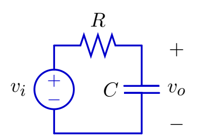
- generalizes Fourier series to represent aperiodic signals.
- equals Laplace transform $X(s)|_{s=j\omega}$ if ROC includes $j\omega$ axis.
  → inherits properties of Laplace transform.
- complex-valued function of **real** domain ω.
- simple "inverse" relation
  → more general than table-lookup method for inverse Laplace.
  → "duality."
- **filtering**.
- **applications in physics**.

Filtering

Notion of a filter.

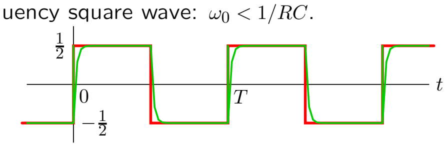
LTI systems
- cannot create new frequencies.
- can only scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

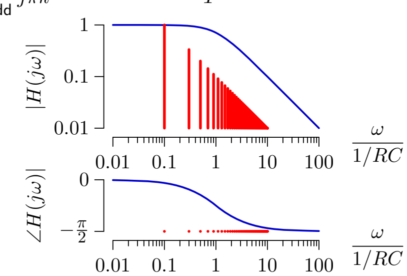


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

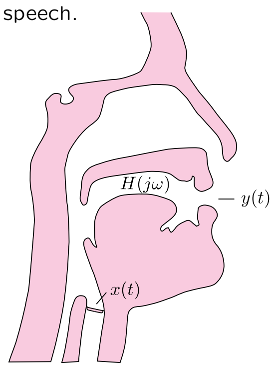


$$x(t)=\sum_{k \text{ odd}} \frac{1}{j\pi k}e^{j\omega_0 kt} \;;\quad \omega_0=\frac{2\pi}{T}$$



Source-Filter Model of Speech Production

Vibrations of the vocal cords are "filtered" by the mouth and nasal cavities to generate speech.



buzz from vocal cords → throat and nasal cavities → speech

Filtering

LTI systems "filter" signals based on their frequency content.

Fourier transforms represent signals as sums of complex exponentials.

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Complex exponentials are eigenfunctions of LTI systems.

$$e^{j\omega t} \to H(j\omega)e^{j\omega t}$$

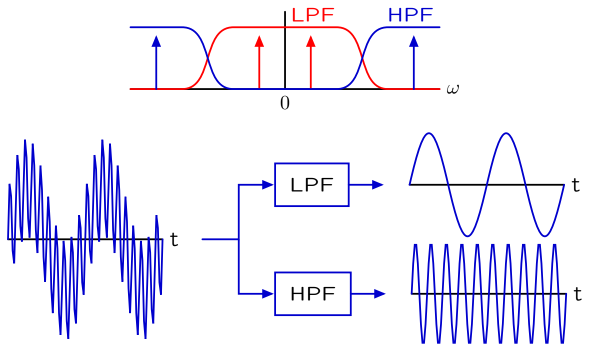
LTI systems "filter" signals by adjusting the amplitudes and phases of each frequency component.

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \;\to\; y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)X(j\omega)e^{j\omega t}d\omega$$

Filtering

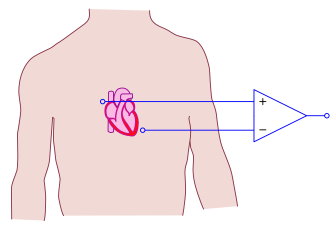
Systems can be designed to selectively pass certain frequency bands.

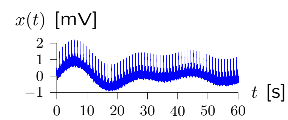
Examples: low-pass filter (LPF) and high-pass filter (HPF).



Filtering Example: Electrocardiogram

An electrocardiogram is a record of electrical potentials that are generated by the heart and measured on the surface of the chest.





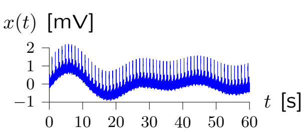
ECG and analysis by T. F. Weiss

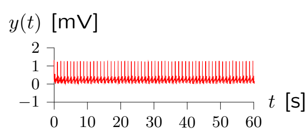
Filtering Example: Electrocardiogram

In addition to picking up electrical responses of the heart, electrodes on the skin also pick up a variety of other electrical signals that we regard as "noise."

We wish to design a filter to eliminate the noise.

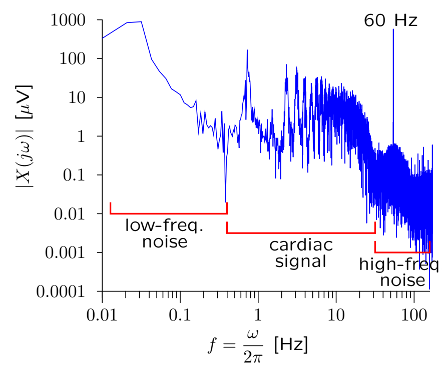
$x(t) \rightarrow$ filter $\rightarrow y(t)$





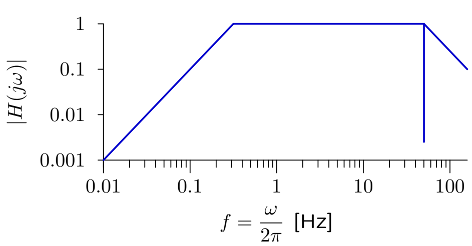
Filtering Example: Electrocardiogram

We can identify the "noise" by breaking the electrocardiogram into frequency components using the Fourier transform.



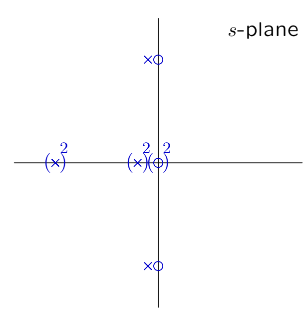
Filtering Example: Electrocardiogram

Filter design: low-pass flter + high-pass filter + notch.



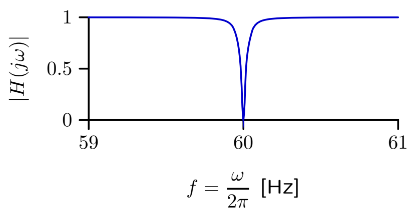
Electrocardiogram: Check Yourself

Which poles and zeros are associated with
- the high-pass filter?
- the low-pass filter?
- the notch filter?



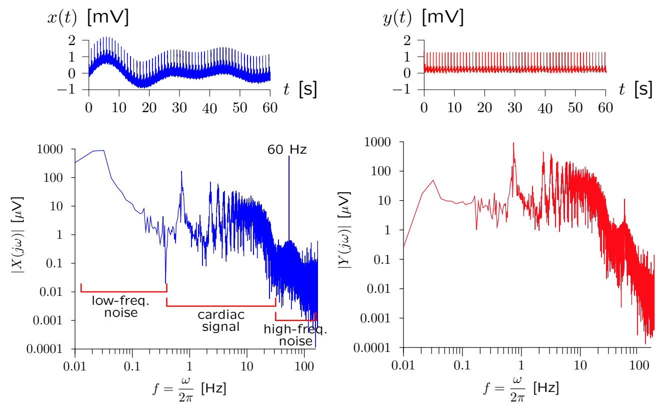
Filtering Example: Electrocardiogram

By placing the poles of the notch filter very close to the zeros, the width of the notch can be made quite small.



Filtering Example: Electrocardiogram

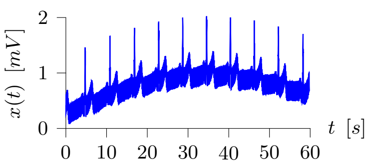
Comparision of filtered and unfiltered electrocardiograms.



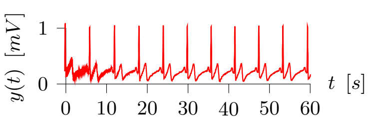
Filtering Example: Electrocardiogram

Reducing the frequency components that are not generated by the heart simplifies the output, making it easier to diagnose cardiac problems.

Unfiltered ECG



Filtered ECG



Continuous-Time Fourier Transform: Summary

Fourier transforms represent signals by their frequency content.

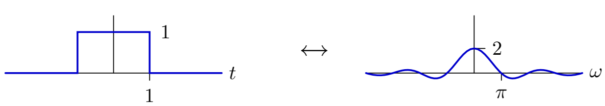
→ useful for many signals, e.g., electrocardiogram.

→ motivates representing a system as a filter.

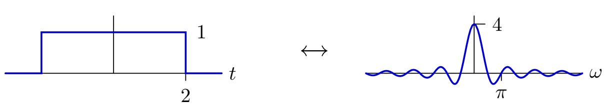
→ useful for many systems.

Visualizing the Fourier Transform

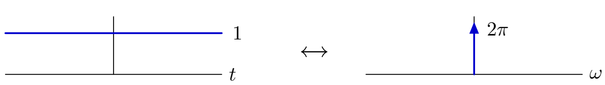
Fourier transforms provide alternate **views** of signals.



Pulses contain all frequencies except harmonics of 2π/width.



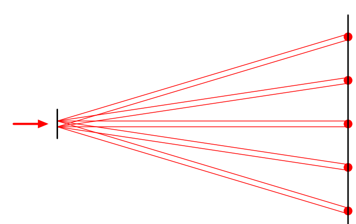
Wider pulses contain more low frequencies than narrow pulses.



Constants (in time) contain only frequencies at $\omega = 0$.

Fourier Transforms in Physics: Diffraction

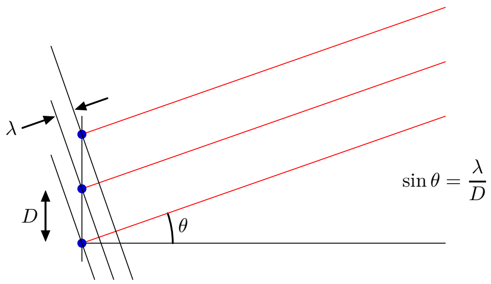
A diffraction grating breaks a laser beam input into multiple beams.



Demonstration.

Fourier Transforms in Physics: Diffraction

The grating has a periodic structure (period $= D$).



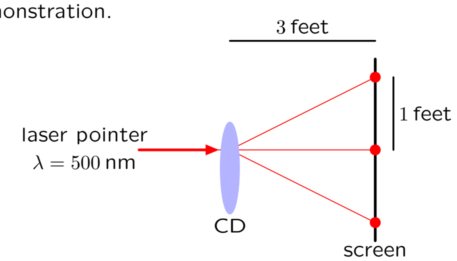
The "far field" image is formed by interference of scattered light.

Viewed from angle θ, the scatterers are separated by $D\sin\theta$.

If this distance is an integer number of wavelengths $\lambda \rightarrow$ constructive interference.

Check Yourself

CD demonstration.

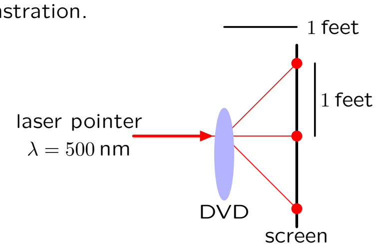


What is the spacing of the tracks on the CD?

1. $160\,\text{nm}$ 2. $1600\,\text{nm}$ 3. 16μm 4. 160μm

Check Yourself

DVD demonstration.

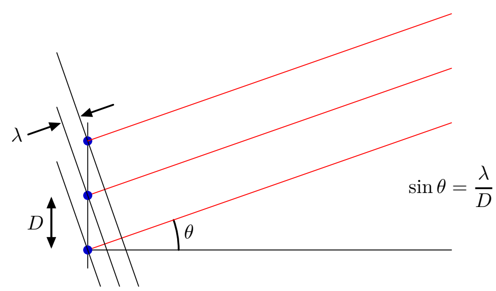


What is track spacing on DVD divided by that for CD?

1. $4\times$ 2. $2\times$ 3. $\frac{1}{2}\times$ 4. $\frac{1}{4}\times$

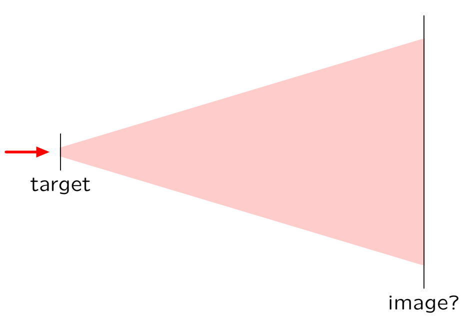
Fourier Transforms in Physics: Diffraction

Macroscopic information in the far field provides microscopic (invisible) information about the grating.



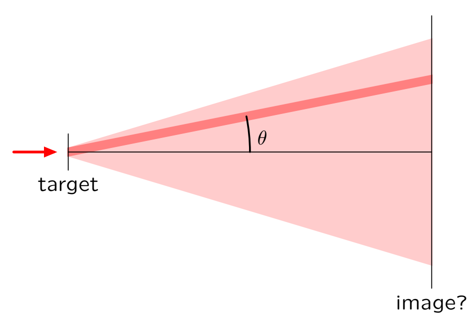
Fourier Transforms in Physics: Crystallography

What if the target is more complicated than a grating?



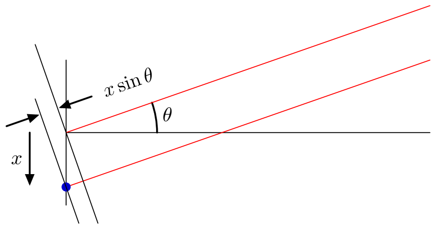
Fourier Transforms in Physics: Crystallography

Part of image at angle θ has contributions for all parts of the target.



Fourier Transforms in Physics: Crystallography

The phase of light scattered from different parts of the target undergo different amounts of phase delay.



Phase at a point x is delayed (i.e., negative) relative to that at 0:

$$\phi = -2\pi\frac{x\sin\theta}{\lambda}$$

Fourier Transforms in Physics: Crystallography

Total light $F(\theta)$ at angle θ is the integral of amount scattered from each part of the target ($f(x)$) appropriately shifted in phase.

$$F(\theta) = \int f(x)e^{-j2\pi\frac{x\sin\theta}{\lambda}}dx$$

Assume small angles so $\sin\theta \approx \theta$.

Let $\omega = 2\pi\frac{\theta}{\lambda}$.

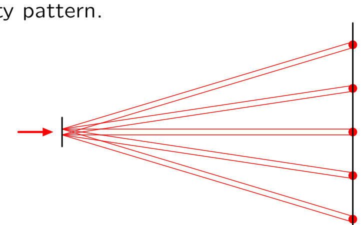
Then the pattern of light at the detector is

$$F(\omega) = \int f(x)e^{-j\omega x}dx$$

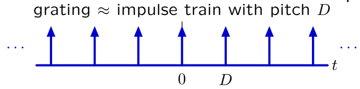
which is the Fourier transform of $f(x)$!

Fourier Transforms in Physics: Diffraction

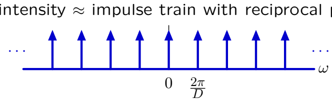
There is a Fourier transform relation between this structure and the far-field intensity pattern.



grating $\approx$ impulse train with pitch D



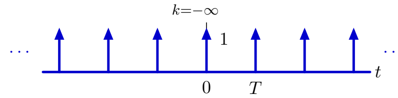
far-field intensity $\approx$ impulse train with reciprocal pitch $\propto \frac{\lambda}{D}$



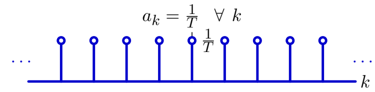
Impulse Train

The Fourier transform of an impulse train is an impulse train.

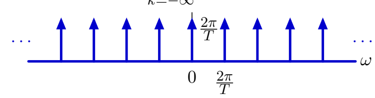
$$x(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT)$$



$$a_k = \frac{1}{T} \quad \forall\ k$$



$$X(j\omega) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{T}\delta(\omega - k\frac{2\pi}{T})$$

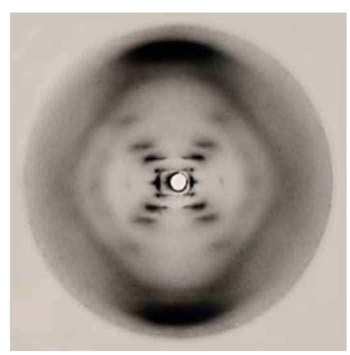


Two Dimensions

Demonstration: 2D grating.

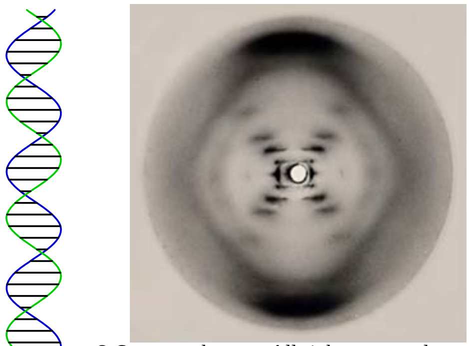
An Historic Fourier Transform

Taken by Rosalind Franklin, this image sparked Watson and Crick's insight into the double helix.



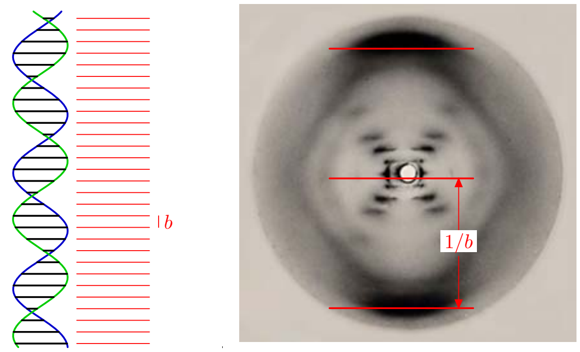
An Historic Fourier Transform

This is an x-ray crystallographic image of DNA, and it shows the Fourier transform of the structure of DNA.



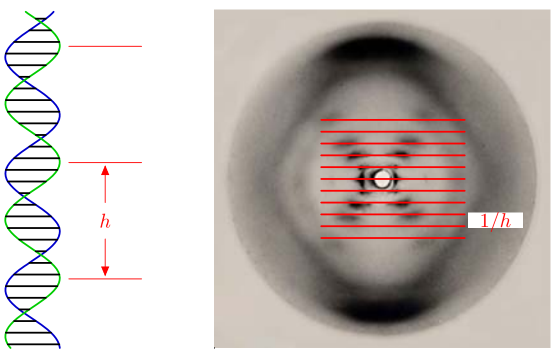
An Historic Fourier Transform

High-frequency bands indicate repreating structure of base pairs.



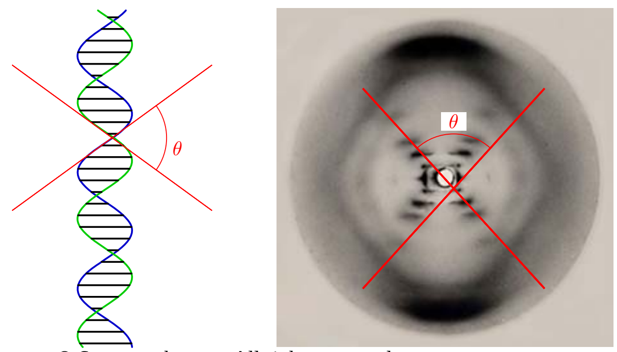
An Historic Fourier Transform

Low-frequency bands indicate a lower frequency repeating structure.



An Historic Fourier Transform

Tilt of low-frequency bands indicates tilt of low-frequency repeating structure: the double helix!



Simulation

Easy to calculate relation between structure and Fourier transform.

Images removed due to copyright restrictions.
Left: double helix drawing. Right: x-ray diffraction image.

Fourier Transform Summary

Represent signals by their frequency content.

Key to “filtering,” and to signal-processing in general.

Important in many physical phenomenon: x-ray crystallography.