System Identification

6.435

SET 1

- –Review of Linear Systems
- –Review of Stochastic Processes
- –Defining a General Framework

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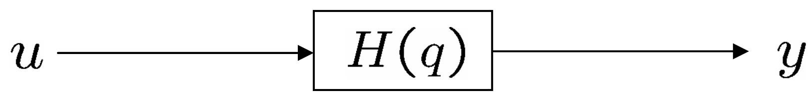
Review

LTI discrete–time systems



$$y(t) = \sum_{k=0}^{\infty} h(t-k)u(k)$$

$H =$ transfer function $= \sum_{t=0}^{\infty} h(t)q^{-t}$

Note ($q \simeq z \simeq \lambda$ different notations)

Stability

$$\sum_{t=0}^{\infty} |h(t)| < \infty$$

$\Leftrightarrow$ (real rational $H(q)$) poles of H are outside the disc

Strict causality

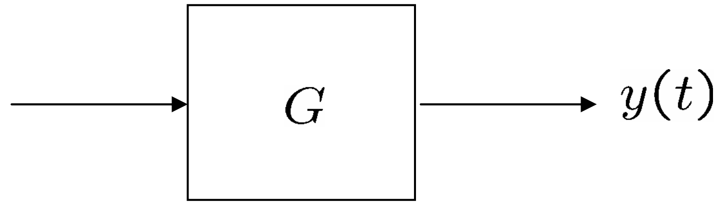
$h(0) = 0$ system has a delay.

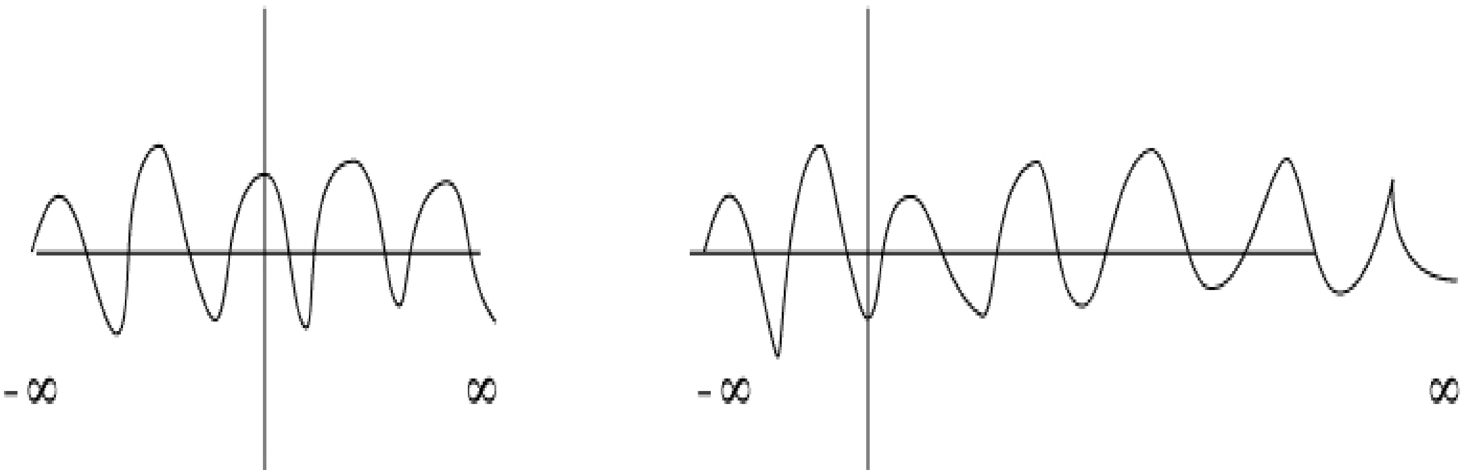
Strict Stability

$$\sum_{t=0}^{\infty} t\,|h(t)| < \infty$$

Frequency Response

$$u(t) = cos(\omega t) = Re\left(e^{i\omega t}\right)$$





$$y(t) = R_e\left[\sum_{k=0}^{\infty} g(k)e^{i\omega(t-k)}\right]$$

$$= R_e\left[e^{i\omega t}\left(\sum_{k=0}^{\infty} g(k)e^{-i\omega k}\right)\right]$$

$$= R_e\left[e^{i\omega t}G\left(e^{i\omega}\right)\right]$$

$$= \left|G\left(e^{i\omega}\right)\right| cos\left(\omega t + avgG\left(e^{i\omega}\right)\right).$$

Suppose $u(t) = 0 \quad \forall \quad t < 0$, then

$$y(t) = \left|G\left(e^{i\omega}\right)\right| cos\left(\omega t + avgG\left(e^{i\omega}\right)\right) - R_e\left[e^{i\omega t}\sum_{k=t}^{\infty} g(k)e^{-i\omega k)}\right]$$

$$\underbrace{\phantom{e^{i\omega t}\sum_{k=t}^{\infty} g(k)e^{-i\omega k)}}}$$

$$\leq \sum_{k=t}^{\infty} |g(k)| \to 0$$

for stable systems.

Periodograms

Given: $\{u(t), t = 1, 2, ..., N\}$, the Fourier transform is given by

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} u(t) e^{-i\omega t}$$

Discrete – Fourier Transform (DFT)

$$\{U_N\left(\frac{2\pi}{N}k\right), k = 1, ..., N\}.$$

Inverse DFT

$$u(t) = \frac{1}{\sqrt{N}} \sum_{k=1-\frac{N}{2}}^{\frac{N}{2}} U_N\left(\frac{2\pi}{N}k\right) e^{i\left(\frac{2\pi}{N}t\right)-k}$$

Periodogram

$$= |U_N(\omega)|^2$$

Nice Property: Parsaval's equality

$$\sum_{k=1}^{N} \left|U_N\left(\frac{2\pi}{N}k\right)\right|^2 = \sum_{t=1}^{N} |u(t)|^2$$

$\left|U_N\left(\frac{2\pi}{N}k\right)\right|^2$ = energy contained in each frequency component.

Properties:

-periodicity $U_N(\omega + 2\pi) = U_N(\omega)$

-If $u(t)$ is real $U_N(-\omega) = \overline{U_N(\omega)}$

Example: $u(t) = Acos(\omega_o t)$ $\omega_o = \frac{2\pi}{N_o}$

(consider $N = sN_o$)

$$U_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} Acos\left(\frac{2\pi}{N_o}t\right) e^{-i\omega t}$$

$$= \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \frac{A}{2} \left[e^{i\omega_o t} + e^{-i\omega_o t}\right] e^{-i\omega t}$$

$$= \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \frac{A}{2} \left[e^{i(\omega_o - \omega)t} + e^{-i(\omega_o + \omega)t}\right]$$

$$= \begin{cases} \frac{\sqrt{N}A}{2} & \text{if } \omega = \pm\omega_o = \frac{2\pi}{N_o} \\ 0 & \text{if } \omega = \frac{2\pi}{N}k, k \neq s \\ ? & \omega \text{ - other} \end{cases}$$

$$|U_N(\omega)|^2 = \begin{cases} \frac{A^2}{4}N & \text{if } \omega = \pm\omega_o = \frac{2\pi}{N_o} = \frac{2\pi s}{N} \\ 0 & \text{if } \omega = \frac{2\pi}{N}k = \frac{2\pi}{sN_o}k, k \neq s \\ ? & \text{other.} \end{cases}$$

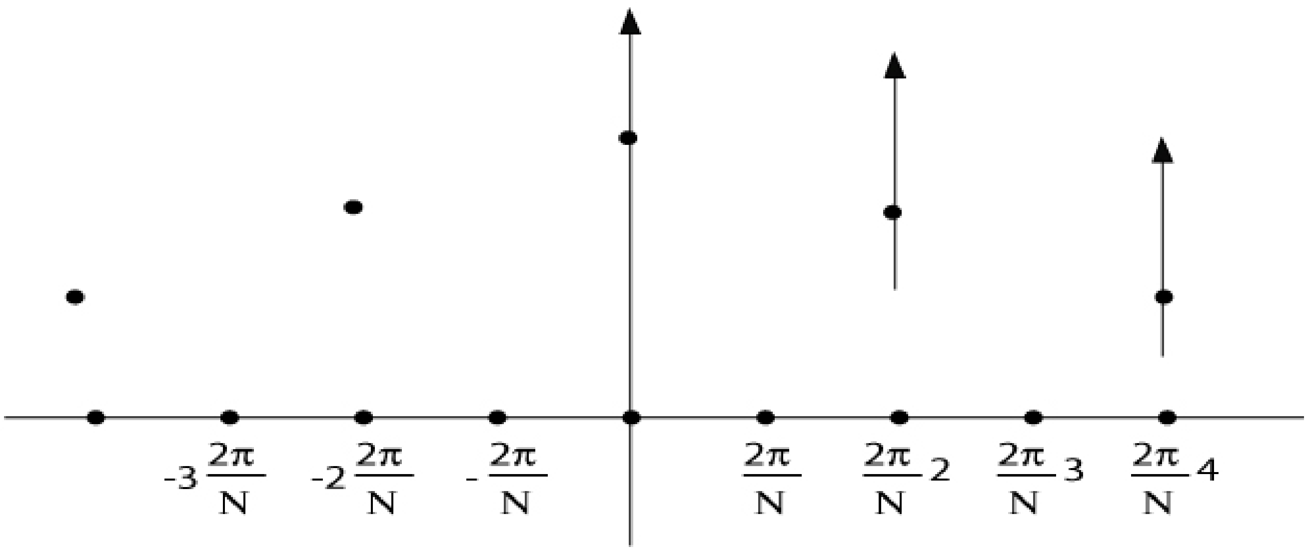
Example: $u(t) = (t + N_o)$ $\qquad N = sN_o$

$$u(t) = \frac{1}{\sqrt{N_o}} \sum_{r=-\frac{N_o}{2}+1}^{\frac{N_o}{2}} A_r \left(e^{i\frac{2\pi}{N_o}t}\right)r \qquad 0 \leq t \leq N_o$$

(Why?) $$A_r = \frac{1}{\sqrt{N_o}} \sum_{t=1}^{N_o} u(t) \left(e^{-\frac{2\pi i}{N_o}r}\right)t$$

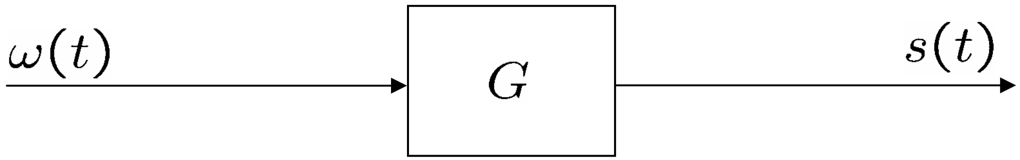
This representation is valid over the whole interval [1,N] since u is periodic over N.

$$|U_N(\omega)|^2 = \begin{cases} s^2\,|A_r|^2 & \text{if } \omega = \frac{2\pi}{N_o} r, r \neq \pm 0, \pm 1, \ldots \pm \frac{N_o}{2} \\ 0 & \text{if } \omega = \frac{2\pi}{N} k, k \neq r \cdot s \\ ? & \omega \text{ other.} \end{cases}$$



$N = 2N_0$

Passing through a filter



$$|\omega(t)| \leq C_W \quad \forall \quad t \in R \qquad\qquad C_G = \sum_{k=1}^{N} k\,|g(k)| < \infty$$

<u>Claim</u>:

$$S_N(\omega) = G\left(e^{i\omega}\right) W_N(\omega) + R_N(\omega)$$

$$|R_N(\omega)| \leq 2\frac{C_W C_G}{\sqrt{N}}$$

Proof:

$$S_N(w) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} s(t) e^{-i\omega t} = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \sum_{k=1}^{\infty} g(k)\omega(t-k) e^{-i\omega t}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=1}^{\infty} g(k) \sum_{t=1}^{N} \omega(t-k) e^{-i\omega t}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=1}^{\infty} g(k) e^{-i\omega k} \sum_{\tau=1-k}^{\tau=N-k} \omega(\tau) e^{-i\omega\tau}$$

Note that:

$$\left| W_N(\omega) - \frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{N-k} \omega(\tau) e^{-i\omega\tau} \right|$$

$$\leq \left| \frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{0} \omega(\tau) e^{-i\omega\tau} \right| + \left| \frac{1}{\sqrt{N}} \sum_{\tau=N-k+1}^{N} \omega(\tau) e^{-i\omega\tau} \right|$$

$$\leq 2kC_W \frac{1}{\sqrt{N}}$$

Hence:

$$\left| S_N(\omega) - G\left(e^{i\omega}\right) W_N(\omega) \right|$$

$$= \left| \sum_{k=1}^{\infty} g(k) e^{-i\omega k} \left[\frac{1}{\sqrt{N}} \sum_{\tau=1-k}^{N-k} \omega(\tau) e^{-i\omega\tau} - W_N(\omega) \right] \right|$$

$$\leq \frac{2C_W}{\sqrt{N}} \sum_{k=1}^{\infty} |g(k)| \cdot k = 2 \frac{C_W C_G}{\sqrt{N}}$$

Stochastic Processes

Definition:

A stochastic process is a sequence of random variables with a joint pdf.

Definition:

$m_x(t) = Ex(t)$ = mean

$R_x(t) = Ex(t_1)x^T(t_2)$ = Correlation / Covariance

$C_x(t_1, t_2) = E\left(x(t_1) - m_x(t_1)\right)\left(x(t_2) - m_x(t_2)\right)^T$

= Covariance

$Var\left(x(t)\right) = R_x(t,t) - E^2\left(x(t)\right)$

Traditional definitions:

$x(t)$ is Wide-sense stationary (WSS) if

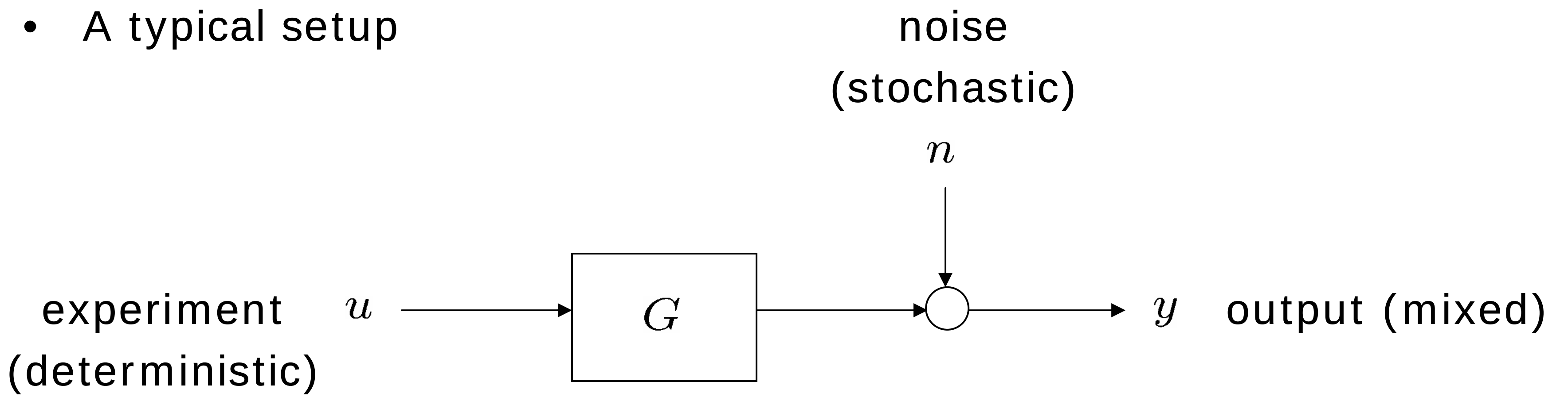
$$m_x(t) = m_x = \text{constant} \quad \forall \quad t$$

$$R_x(t_1, t_2) = R_x(t_1 - t_2)$$

“This may be a limiting definition.” We will discuss shortly.

Common Framework for Deterministic and Stochastic Signals

- A typical setup



$$E_y = \underbrace{G_u}$$

not $\neq$ constant

(loose stationarity).

- Consider signals with the following assumptions:

1) $Es(t) = m_s(t) \quad |m_s(t)| \leq C \quad \forall \quad t$

2) $Es(t)s(r) = R_s(t,r) \quad |R_s(t,r)| \leq C$

and $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} R_s(t, t-\tau) = R_s(\tau) \quad \forall \quad \tau$

$s(t)$ is called quasi-stationary

- If $s(t)$ is a stationary process, then it satisfies 1, 2 trivially.

- If $s(t)$ is a deterministic signal, then

1) $|s(t)| \leq C$

2) $\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} s(t)s(t-\tau) = R_s(\tau)$

- <u>Example</u>: Suppose $s(t)$ has finite energy

$$\left| \frac{1}{N} \sum_{t=1}^{N} u(t)u(t-\tau) \right| \leq \frac{1}{N} ||u||_2^2 \to 0 \quad N \to \infty$$

- In general: $s(t) = \underbrace{x(t)}_{\text{stochastic}} + \underbrace{u(t)}_{\text{deterministic}}$

- Notation: $\bar{E}(\cdot) = \lim \frac{1}{N} \sum_{t=1}^{N} (\cdot)$

quasi-stationary $\cong \bar{E}s(t) = m_s = constant$

$$\bar{E}s(t)s(\tau) = R_s(t-\tau)$$

- $s(t) = x(t) + u(t); \quad \bar{E}s(t) = m_x + m_u$

$$\bar{E}s(t)s(t-\tau) = R_x(\tau) + R_u(\tau) + 2m_x m_u$$

Power Spectrum

Let $x(t)$ be a quasi-stationary process. The power spectrum $\Phi_x\left(e^{i\omega}\right)$ is defined as

$$\Phi_x\left(e^{i\omega}\right) = \sum_{\tau=-\infty}^{\infty} R_x(\tau)e^{-i\omega\tau}$$

= Fourier transform of R_x

- Example: for $x(t) = Acos(\omega_o t)$

$$R_x(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} A^2 cos\omega_o t \cdot cos\omega_o(t-\tau)$$

$$= \frac{A^2}{2} cos(\omega_o \tau)$$

$$\Phi_x\left(e^{i\omega}\right) = \frac{A^2}{4}\left(\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right)$$

Recall

$$|X_N(\omega)|^2 = \begin{cases} \frac{A^2}{4}N & \text{if } \omega = \pm\omega_o \\ (0,?) & \text{other} \end{cases}$$

$$\lim_{N\to\infty} |X_N(\omega)|^2 \xrightarrow{\quad ? \quad} \Phi_x\left(e^{i\omega}\right)$$

- Result ; for any $s(t)$, quasi-stationary

$$E\int_{-\pi}^{\pi} |S_N(\omega)|^2\,\Psi(\omega)d\omega \xrightarrow[N\to\infty]{} \int_{-\pi}^{\pi} \Phi_s\left(e^{i\omega}\right)\Psi(\omega)d\omega$$

Convergence as a distribution

Note: $|S_N(\omega)|^2$ an erratic function

$\Phi_s\left(e^{i\omega}\right)$ well behaved function

Cross Spectrum

$$\Phi_{xy}\left(e^{i\omega}\right) = \sum_{\tau=-\infty}^{\infty} R_{xy}(\tau)e^{-i\omega\tau}$$

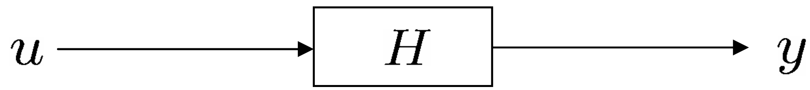
Spectrum of mixed signals

$$s(t) = x(t) + u(t)$$ {zero means}

$$R_s = R_x + R_u$$

$$\Phi_s = \Phi_x + \Phi_u$$

Spectrum of Filtered Signals



$$\Phi_y\left(e^{i\omega}\right) = H\left(e^{i\omega}\right)\Phi_u\left(e^{i\omega}\right)H^*\left(e^{i\omega}\right)$$

$$\overset{\text{SISO}}{=} \Phi_u\left(e^{i\omega}\right)\left|H\left(e^{i\omega}\right)\right|^2$$

If $u = WN$ signal $\phi_u = \lambda^2 I$

$$\Phi_y\left(e^{i\omega}\right) = \lambda^2 H H^*\left(e^{i\omega}\right)$$

$$\overset{\text{SISO}}{=} \lambda^2\left|H\left(e^{i\omega}\right)\right|^2$$

Generation of a process with a given Covariance:

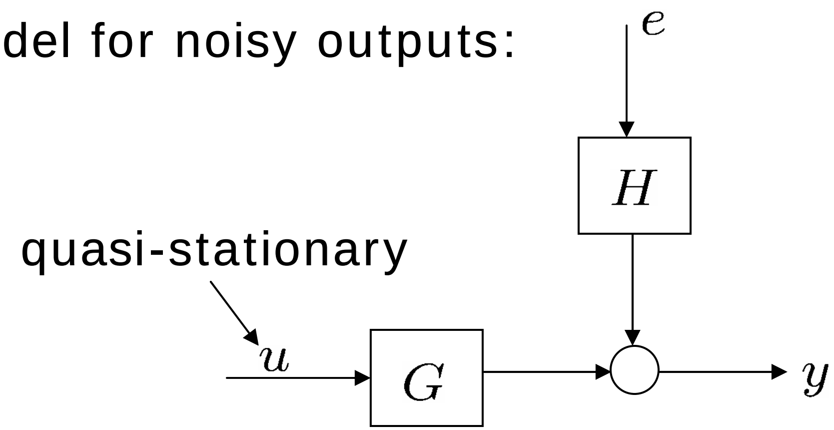
given $R_x(\tau)$, then x is the output of a filter with a WN signal as an input. The Filter is the spectral factor of $\Phi_x\left(e^{i\omega}\right)$;

$$\Phi_x\left(e^{i\omega}\right) = \underbrace{H(e^{i\omega})}_{} \cdot H(e^{i\omega})^*$$

stable minimum phase.

Important relations

A model for noisy outputs:



$$y = Gu + He$$

$$E\left(e(t)e^T(t-\tau)\right) = \underbrace{\Lambda^2}_{\text{constant.}}\delta(\tau)$$

$$\Phi_y\left(e^{i\omega}\right) = G\Phi_u\left(e^{i\omega}\right)G^T\left(e^{i\omega}\right)H\Lambda^2H^*\left(e^{i\omega}\right)$$

$$\Phi_{yu}\left(e^{i\omega}\right) = G\left(e^{i\omega}\right)\Phi_u(\omega)$$

- Very Important relations in system ID.
- Correlation methods are central in identifying an unknown plant.
- Proofs: Messy; Straight forward.

Ergodicity

- $x(t)$ is a stochastic process

Sample function or a realization

- Sample mean $= \bar{E}\tilde{x}(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \tilde{x}(t)$

- Sample Covariance $= \bar{E}\tilde{x}(t)\tilde{x}(t-\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \tilde{x}(t)\tilde{x}(t-\tau)$

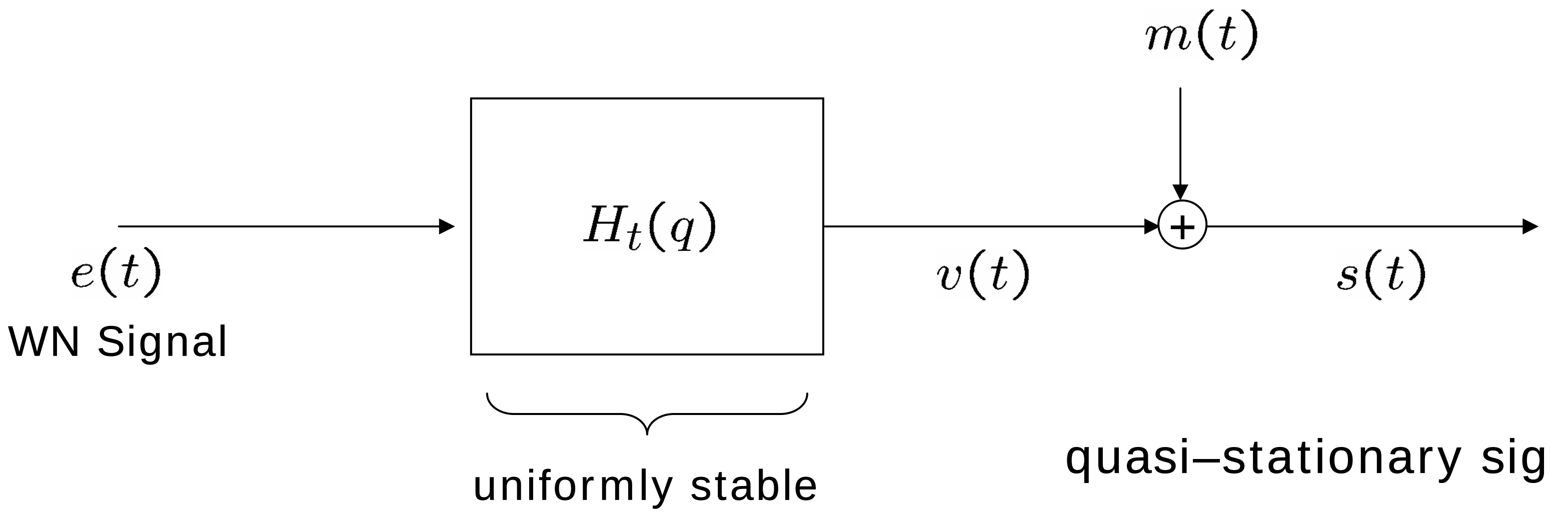
- A process is 2nd-order ergodic if

  mean $\hat{=}\ Ex(t) =$ the sample mean of any realization.

  covariance $\hat{=}\ Ex(t)x(t-\tau) =$ the sample covariance of any realization.

- Sample averages $\simeq$ Ensemble averages

A general ergodic process



quasi–stationary signal

$$Es(t) = m(t)$$

$$\bar{E}s(t)s(t-\tau) = R_s(\tau) \qquad \text{w.p.1}$$

$$\frac{1}{N}\sum_{t=1}^{N}[s(t)m(t-\tau) - Es(t)m(t-\tau)] \to 0 \qquad \text{w.p.1}$$

$$\frac{1}{N}\sum_{t=1}^{N}[s(t)v(t-\tau) - Es(t)v(t-\tau)] \to 0 \qquad \text{w.p.1}$$

Remark:

Most of our computations will depend on a given realization of a quasi-stationary process. Ergodicity will allow us to make statements about repeated experiments.