System Identification

6.435

<u>SET 6</u>

- Parametrized model structures
- One-step predictor
- Identifiability

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Models of LTI Systems

- A complete model

$$y = Gu + He$$

$\boldsymbol{u}$ = input

$\boldsymbol{y}$ = output

$\boldsymbol{e}$ = noise (with $f_e(\cdot)$ PDF).

$$G = \sum_{k=1}^{\infty} g(k) q^{-k} \qquad H = 1 + \sum_{k=1}^{\infty} h(k) q^{-k}$$

- A parametrized model

$$y = G(\theta, q)u + H(\theta, q)e$$

$\boldsymbol{u}$ = input

$\boldsymbol{y}$ = output

$\boldsymbol{e}$ = noise (with $f_e(\cdot, \theta)$ PDF of $\boldsymbol{e}$).

$\boldsymbol{e}$ white noise

$$\theta \in D \subseteq \Re^d$$

One Step Linear Predictor

- e is WN, $\text{Var}(e) = \lambda^2(\theta)I$
- Want to find $\hat{y}(\cdot|\theta)$ that minimizes $E(y-\hat{y})^T(y-\hat{y})$
- $\hat{y}(t,\theta) := H^{-1}(q,\theta)G(q,\theta)u(t) + \left(1 - H^{-1}(q,\theta)\right)y(t)$
- Minimum Prediction Error Paradigm

Data: $[u(t), y(t)|t \leq N]$

$$\hat{\theta}_N = \text{Estimate of } \theta \text{ at time } N$$

$$= \text{argmin} \frac{1}{N}\sum_{t=1}^{N} ||\varepsilon(t|\theta)||_2^2$$

$$\varepsilon(t|\theta) = y - \hat{y}(t|\theta) = H^{-1}(q,\theta)\left[y - G(q,\theta)u\right]$$

One Step Linear Predictor (Derivation)

The predicted output has the form

$$\widehat{y}(\cdot,\theta) = L_1(q,\theta)y + L_2(q,\theta)u$$

Both L_1 & L_2 have a delay. Past inputs and outputs are mapped to give the new predicted output.

$$y - \widehat{y} = Gu + He - L_2 u - L_1 y$$

Notice that:

$$\begin{aligned} & y = Gu + He \\ \Leftrightarrow \quad & H^{-1}y = H^{-1}Gu + e \\ \Leftrightarrow \quad & y = \left(I - H^{-1}\right)y + H^{-1}Gu + e \end{aligned}$$

Now:

$$y - \hat{y} = \underbrace{\left(I - H^{-1} - L_1\right)}_{\text{has at least one delay}} y + \underbrace{\left(H^{-1}G - L_2\right)}_{\text{has at least one delay}} u + e$$

$$= z + e$$

$$E\,(y - \hat{y})^T\,(y - \hat{y}) = E\left(Z^T Z\right) + E\left(e^T e\right)$$

$$\geq \lambda^2(\theta) I$$

The lower bound is achieved if $z = 0$

Equivalently

$$L_1 = I - H^{-1}$$

$$L_2 = H^{-1}G$$

Result

$$\hat{y}(\cdot|\theta) = \left(I - H^{-1}\right) y + H^{-1} G u$$

$$\varepsilon(\cdot|\theta) = y - \hat{y} = H^{-1}(q,\theta)[y - G(q,\theta)u]$$

$$= e$$

Examples of Transfer Function Models

– ARX (Autoregressive with exogenous input)

- Description

$$y(t) + a_1 y(t-1) + \ldots + a_{na} y(t-n_a)$$
$$= b_1 u(t-1) + \ldots + b_{nb} u(t-n_b) + e(t)$$

$$\theta = [a_1 \ldots a_{na} \quad b_1 \ldots b_{nb}] \quad , \quad e \text{ is } WN$$

$$A(q) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-n_a}$$

$$B(q) = b_1 q^{-1} + \ldots + b_{nb} q^{-n_b}$$

- Matched with the model $y = Gu + He$ $\quad G = \frac{B}{A} \quad H = \frac{1}{A}$

Examples … ARX

- One step predictor

$$\widehat{y}(t|\theta) = Bu + (1 - A)y$$

- Linear Regression

$$\phi(t) = [-y(t-1) \ldots -y(t-n_a) \quad u(t-1) \ldots u(t-n_b)]^T$$

(a function of past data)

- Prediction error

$$\varepsilon(t|\theta) = y(t) - \widehat{y}(t|\theta)$$
$$= y(t) - \phi^T(t)\theta$$

Examples .... ARMAX

– ARMAX (Autoregressive moving average with exogenous input)

- Description

$$Ay(t) = Bu(t) + Ce(t)$$

$$A(q) = 1 + a_1 q^{-1} + \ldots + a_{na} q^{-n_a}$$

$$B(q) = b_1 q^{-1} + \ldots + b_{nb} q^{-n_b}$$

$$C(q) = 1 + c_1 q^{-1} + \ldots + c_{nc} q^{-n_c}$$

$$e : WN \qquad \theta = (a_1 \ldots a_{na} \quad b_1 \ldots b_{nb} \quad c_1 \ldots c_{nc})$$

- Standard model $G(q) = \dfrac{B}{A}$ $H(q) = \dfrac{C}{A}$

- More general, includes ARX model structure.

Examples .... ARMAX

- One step predictor

$$\hat{y}(t|\theta) = \frac{B}{C}u + (1 - \frac{A}{C})y$$

or

$$\hat{y}(t|\theta) = Bu + (C - A)y + \underbrace{(1 - C)\hat{y}}$$

past predictions

- Pseudo-linear Regression

$$\Phi(t,\theta) = [-y(t-1)\ldots -y(t-n_a) \quad u(t-1)\ldots u(t-n_b) \quad \varepsilon(t-1|\theta)\ldots$$
$$\ldots \varepsilon(t - n_c|\theta)]^T$$

where

$$\varepsilon(t|\theta) = y - \hat{y}(t|\theta) = (1-c)\varepsilon(t|\theta) + Bu + (1-A)y$$

or simply

$$\hat{y}(t|\theta) = \Phi^T(t,\theta)\theta$$ Not linear in θ

Examples …. OE

- OE (Output Error)
- Description

$$y(t) = \frac{B}{F}u + e$$

$$F(q) = 1 + f_1 q^{-1} + \ldots + f_{nf} q^{-n_f}$$

$$B(q) = b_1 q^{-1} + \ldots + b_{nb} q^{-n_b}$$

$$\theta = \begin{pmatrix} b_1 \ldots b_{nb} & f_1 \ldots f_{nf} \end{pmatrix}$$

- One step predictor $\hat{y}(t|\theta) = \frac{B}{F}u$
- Standard $G = \frac{B}{F} \quad H = 1$

Examples .... OE

- Nonlinear Regression Vector

$$\hat{y} = \frac{B}{F}u \quad \Rightarrow \quad F\hat{y} = BU$$

$$\Rightarrow \hat{y}(t|\theta) = Bu + (1 - F)\hat{y}$$

Define

$$\Phi(t,\theta) = (u(t-1)\ldots u(t-n_b) \quad -w(t-1|\theta)\ldots -w(t-n_f|\theta))$$

$$\omega(t|\theta) = \frac{B}{F}u \qquad (= \hat{y}(t|\theta))$$

$$\hat{y} = \Phi^T(t,\theta)\theta.$$

Examples …. Box-Jenkins

$$y = \frac{B}{F}u + \frac{C}{D}e$$

$$\hat{y} = \frac{B}{F}\frac{D}{C}u + \frac{C-D}{C}y$$

An even more general model

$$Ay = \frac{B}{F}u + \frac{C}{D}e \qquad \ldots\ldots$$

Examples …. State-Space

– State-Space Models

- Description

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) + \omega(t)$$

$$y(t) = C(\theta)x(t) + \nu(t)$$

$$A(\theta) \in \Re^{n\times n} \quad B(\theta) \in \Re^{n\times m} \quad C(\theta) \in \Re^{p\times n}$$

- Noise: two components
  - disturbance $\omega(t)$
  - Output noise $\nu(t)$

Usual assumptions

$$E\omega\omega^T = R_1(\theta) \qquad Evv^T(t) = R_2(\theta) \qquad Ewv^T(t) = R_{12}(\theta)$$

$\boldsymbol{w}$, $\boldsymbol{v}$ are white.

- One Step Predictor = Kalman Filter

$$\hat{x}(t+1|\theta) = A(\theta)\hat{x}(t|\theta) + B(\theta)u + K(\theta)\left[y - C(\theta)\hat{x}(t|\theta)\right]$$

$$\hat{y}(t|\theta) = C(\theta)\hat{x}(t|\theta)$$

$$K(\theta) = \left[A(\theta)\bar{P}(\theta)C^T(\theta) + R_{12}(\theta)\right] \cdot \left[C(\theta)\bar{P}(\theta)C^T(\theta) + R_2(\theta)\right]^{-1}$$

$\bar{P}(\theta)$ is a positive semi-definite solution of the steady-state Riccati equation:

$$\bar{P}(\theta) = A(\theta)\bar{P}(\theta)A^T(\theta) + R_1(\theta) - \left(A(\theta)\bar{P}(\theta)C^T(\theta) + R_{12}(\theta)\right)\left[C(\theta)\bar{P}(\theta)C^T(\theta) + R_2(\theta)\right]^{-1} \cdot \left(A(\theta)\bar{P}(\theta)C^T(\theta) + R_{12}(\theta)\right)^T$$

$$\bar{P}(\theta) = E(x - \hat{x})(x - \hat{x})^T \qquad \text{(error covariance)}$$

Innovation Representation

$$\hat{x}(t+1|\theta) = A(\theta)\hat{x}(t|\theta) + B(\theta)u + K(\theta)e$$

$$y(t) = C(\theta)\hat{x}(t|\theta) + e$$

$$E\left(ee^T\right) = C(\theta)\bar{P}(\theta)C^T(\theta) + R_2(\theta)$$

In general

$$\hat{y}(t,\theta) = C(\theta)\left(qI - A(\theta) + K(\theta)C(\theta)\right)^{-1} B(\theta)u +$$

$$C(\theta)\left(qI - A(\theta) + K(\theta)C(\theta)\right)^{-1} K(\theta)y$$

General Notation

Define

$$T(q) = \begin{bmatrix} G(q) & H(q) \end{bmatrix}, \quad X(t) = \begin{bmatrix} u(t) \\ e(t) \end{bmatrix}$$

It follows

$y(t) = T(q)X(t)$; Describes the model structure

Predictor:

$$\hat{y}(t) = W(q)Z(t) \qquad Z(t) = \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}$$

$$= \begin{bmatrix} W_u(q) & W_y(q) \end{bmatrix} Z(t)$$

For a given $T(q)$,

$$W(q,\theta) = \left[\begin{array}{cc} H^{-1}(q,\theta)G(q,\theta) & I - H^{-1}(q,\theta) \end{array} \right]$$

- Notice: $W(q,\theta)$ can be stable even though $T(q,\theta)$ is not!

- $T(q,\theta) \simeq W(q,\theta)$

Predictor Models

Def: A predictor model is a linear time-invariant stable filter $W(q)$ that defines a predictor

$$\hat{y}(t) = W(q) \begin{bmatrix} u \\ y \end{bmatrix}.$$

Def: A complete probabilistic model of a linear time-invariant system is a pair $(W(q), f_e(x))$ of a predictor model $W(q)$ and the PDF f_e associated with the prediction error (noise).

In most situations, $f_e(x)$ is not complete known. We may work with means & variances.

Stability Requirements

<u>Example</u> ARX

$$A(q)y = B(q)u + e$$

$$\Leftrightarrow \quad y = \frac{B}{A}u + \frac{1}{A}e$$

If $A(q)$ has zeros outside the disc, then the map from $u \to y$ is unstable.

$$\hat{y}(t) = (I - A)y + Bu = W(q)Z$$

is always stable.

Model Sets

Def: A model set is a collection of models

$$m^* = \{W_\alpha(q) | \alpha \in \varphi\} \qquad \varphi: \text{ Index set}$$

Examples

$m^* =$ all linear models

$m^* =$ all models where $W(q)$ is FIR of fixed order

$m^* =$ a finite set of models

$m^* =$ nonlinear fading memory models

Comment: These are “big” sets that are not necessarily parametrized in a nice way.

Model Structures

Model structures are parametrizations of model sets. We require this parametrization to be smooth.

Let a model be index by a parameter $\theta, W(q,\theta)$. We require $W(q,\theta)$ to be differentiable with respect to θ, for $|q| \geq 1$.

$$\Psi(q,\theta) = \frac{d}{d\theta}W(q,\theta)$$

$$= \left[\begin{array}{cc} \frac{d}{d\theta}W_u(q,\theta) & \frac{d}{d\theta}W_y(q,\theta) \end{array} \right]$$

$$\triangleq \left[\begin{array}{cc} \Psi_u(q,\theta) & \Psi_y(q,\theta) \end{array} \right]$$

Notice that

$$\frac{d}{d\theta}\hat{y}(t,\theta) = \frac{d}{d\theta}\left[\ W(q,\theta)Z\ \right]$$

$$= \Psi(q,\theta)Z$$

<u>Def</u>: A model structure m is a differentiable map from a connected open subset $D_m \subseteq \Re^c$ to a model set m^* such that $\Psi(q,\theta)$ the gradient, is defined and stable.

$$m : D_m \longrightarrow m^*$$

$$\theta \longrightarrow m(\theta) = W(q,\theta) \in m^*$$

m : is the map

$m(\theta)$: is one particular model

Example: ARX model structure

$$y(t) + ay(t-1) = b_1 u(t-1) + b_2 u(t-2) + e(t)$$

$$W(q,\theta) = \left(b_1 q^{-1} + b_2 q^{-2}, -aq^{-1}\right)$$

$$\theta = \begin{pmatrix} a & b_1 & b_2 \end{pmatrix}^T$$

$$\Psi(q,\theta) = \begin{bmatrix} 0 & -q^{-1} \\ -q^{-1} & 0 \\ -q^{-2} & 0 \end{bmatrix}$$

stable

General Structure

$$y(t) = G(q,\theta)u + H(q,\theta)e$$

$$\Psi(q,\theta) = \begin{bmatrix} H^{-1}G & 1 - H^{-1} \end{bmatrix}$$

$$\frac{d}{d\theta}W_u(q,\theta) = \frac{d}{d\theta}H^{-1}G = -\frac{1}{H^2}H' + \frac{G'}{H}$$

$$= \frac{1}{H^2}\left(G'H - H'G\right)$$

$$\frac{d}{d\theta}W_y(q,\theta) = \frac{d}{d\theta}\left(1 - H^{-1}\right) = -\frac{1}{H^2}H'$$

$$\Psi(q,\theta) = \frac{1}{H^2}\begin{bmatrix} \frac{d}{d\theta}G & \frac{d}{d\theta}H \end{bmatrix}\begin{bmatrix} H & 0 \\ -G & 1 \end{bmatrix}$$

You need

G, H to be differentiable

H^{-1} is stable

(not sufficient).

Model Structures

Proposition: The parametrization

$$\widehat{y}(t,\theta) = \frac{D}{C}\frac{B}{F}u + \left(1 - \frac{DA}{C}\right)y$$

for θ (= set of parameters of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{F}$)

restricted to the set

$D_m = \{\theta | F(q)C(q)$ has no zeros outside the unit disc

is a model structure

Proof: Follows from the previous general derivative. Notice that $\boldsymbol{H}$ may be unstable!

Proposition: The Kalman Filter parametrization is a model structure if

$$\theta \in \{\theta | A(\theta) - K(\theta)C(\theta) \text{ is stable}\} \hat{=} D_\mu$$

(The stability property is equivalent to $(A(\theta), C(\theta))$ being detectable).

Independent Parametrization

A model structure m is independently parametrized if

$$\theta = \begin{bmatrix} \rho \\ \eta \end{bmatrix}, D_m = D_\rho \times D_\eta, \text{ and}$$

$$T(q,\theta) = \begin{bmatrix} W_u(q,\rho) & W_y(q,\eta) \end{bmatrix} \quad \rho \in D_\rho, \quad \eta \in D_\eta$$

Useful for giving a frequency domain interpretation of the estimate.

- We can define a <u>model set</u> as the range of a model structure:

$$R(m) = \{m(\theta), \theta \in D_m\}$$

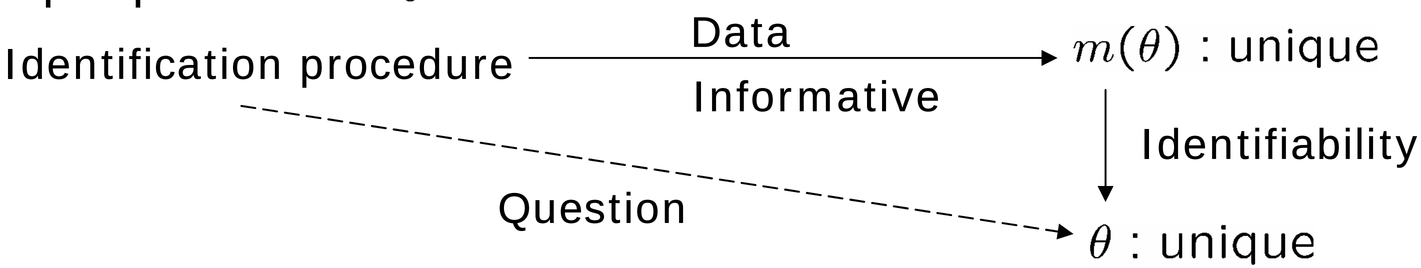
- We can define unions of different model structures:

$$m^* = \bigcup_{k} R(m_k)$$

Useful for model structure determination!

Identifiability

Central question I: Does the identification procedure yield a unique parameter θ?



Def: Model structure is identifiable (globally!) at θ^* if $m(\theta) = m(\theta^*), \theta \in D_m \Rightarrow \theta = \theta^*$. It is locally identifiable at θ^* if there exists an $\varepsilon > 0$ such that $m(\theta) = m(\theta^*), \theta \in B(\theta^*, \varepsilon)$ implies $\theta = \theta^*$.

Def: Model structure is strictly identifiable (local or global) if it is identifiable (local or global) for all $\theta^* \in D_m$.

Central question II: Is the identified parameter equal to the "true parameters" ?

Parametrized structure: $y(t) = G(q,\theta)u + H(q,\theta)e = m$

true system ζ : $y(t) = G_o u + H_o e$

Case I: $\zeta \notin m$

Case II: $\zeta = m(\theta)$ for some θ.

Define:

$$D_T(\zeta, m) = \{\theta \in D_m | G_o = G(q,\theta), H_o = H(q,\theta) \text{ almost everywhere } (q)\}$$

Let $\zeta = m(\theta_o)$ for some θ_o. If $m(\theta)$ is identifiable at θ_o , then $D_T(\zeta, m) = \{\theta_o\}$.

Identifiability of Model Structures

General: $T(q,\theta) = \left[\begin{array}{cc} G(q,\theta) & H(q,\theta) \end{array}\right]$

Identifiable at θ_o $\Leftrightarrow$ $G(q,\theta) \overset{\text{a.e.}}{=} G(q,\theta_o)$, $H(q,\theta) = H(q,\theta_o)$ $\Rightarrow$ $\theta = \theta_o$

ARX: $G = \dfrac{B}{A}$ $\quad H = \dfrac{1}{A}$

If $G(q,\theta) = G(q,\theta_o)$ & $H(q,\theta) = H(q,\theta_o)$

$\Rightarrow \quad A(q,\theta) = A(q,\theta_o)$ & $B(q,\theta) = B(q,\theta_o)$

$\Rightarrow \quad \theta = \theta_o$

ARX is strictly identifiable

OE: $y = \frac{B}{F}u + v$

Suppose $\frac{B}{F} = \frac{\tilde{B}}{\tilde{F}}$. Then

$B = \tilde{B}$ & $F = \tilde{F}$ iff ($\boldsymbol{B}$, $\boldsymbol{F}$) are coprime.

(To do this cleanly, need to consider $q^{n_B}B, q^{n_F}F$ where n_B & n_F are the delay powers in both $\boldsymbol{B}$ & $\boldsymbol{F}$)

Identifiability

<u>Theorem</u>:

$Ay = \frac{B}{F}u + \frac{C}{D}e$ is identifiable at $\theta = \theta^*$ iff

$(C^*(q) = C(q, \theta^*) \ldots\ldots)$

1) There are no common factors of $q^{n_a}A^*, q^{n_b}B^*, q^{n_c}C^*$
2) " $q^{n_b}B^*, q^{n_f}F^*$
3) " $q^{n_c}C^*, q^{n_d}D^*$