Theory of Parallel Hardware April 26, 2004
Massachusetts Institute of Technology 6.896
Charles Leiserson, Michael Bender, Bradley Kuszmaul Handout 17

Solution Set 8

Due: In class on Wednesday, April 14. Starred problems are optional.

Problem 7-1.

(a) Draw an electrical schematic (transistor-level circuit) for a 2-input NOR gate.

Solution: To draw an electrical schematic for a 2-input NOR gate:

If A or B is 1 → X is 0
If A and B are 0 → X is 1

A NOR B		X
0	0	1
0	1	0
1	0	0
1	1	0

Figure 1: Truth table for NOR gate

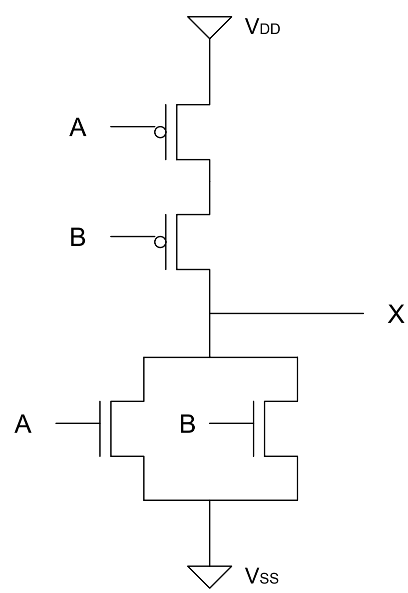

Figure 2: Electric schematic for NOR gate

(b) Draw a CMOS layout of a 2-input NOR gate.

Solution:

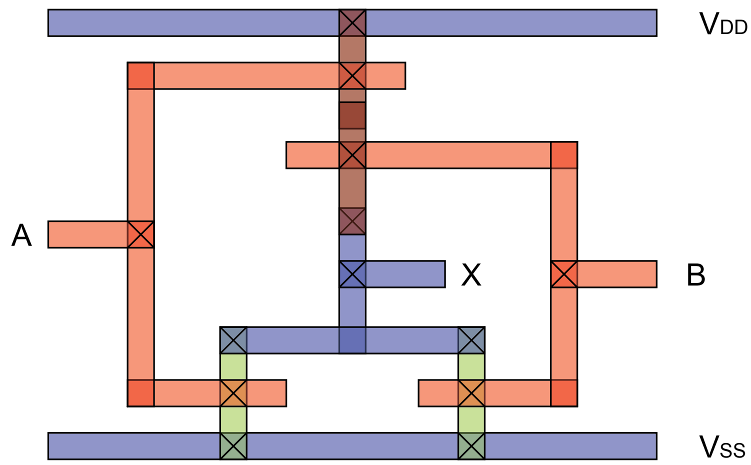

Figure 3: CMOS layout for NOR gate

(c) Draw an electrical schematic for a gate that computes $F = \overline{(A + B + C) \cdot D}$.

Solution: See Figure 4.

Problem 7-2. * Consider a bounded-degree routing network with n inputs and n outputs, where each input contains 1 packet. Show that if each of the n packets chooses an output destination randomly and independently, then the congestion is $\Omega(\lg n / \lg \lg n)$.

Solution: We will show that with very high probability there exists a destination to which $\Omega(\lg n / \lg \lg n)$ packets want to go. Since there are only $O(1)$ edges going into that destination, the congestion on at least one has to be $\Omega(\lg n / \lg \lg n)$. Since this happens whp, the expected congestion is also $\Omega(\lg n / \lg \lg n)$ (there's a lower bound of 1 in all cases).

Let $T = (1/2) \lg n / \lg \lg n$. Let E_i = the event that there are less than T packets that want to go to destination i. We want $\Pr[E_1 \cap \ldots \cap E_n] \leq 2^{-n^{\Omega(1)}}$. We can write:

$$
\begin{aligned}
\Pr\left[\bigcap E_i\right] &= \Pr[E_1 \mid E_2 \cap \ldots \cap E_n] \cdot \Pr[E_2 \cap \ldots \cap E_n] \\
&= \Pr[E_1 \mid E_2 \cap \ldots \cap E_n] \cdot \Pr[E_2 \mid E_3 \cap \ldots \cap E_n] \cdot \Pr[E_3 \cap \ldots \cap E_n] \\
&= \ldots = \prod_{i=1}^{n} \Pr\left[E_i \mid \bigcap_{j>i} E_j\right] \\
&\leq \left(1 - \frac{\Omega(1)}{T^T}\right)^n \qquad \text{(proof below)} \\
&= \left(1 - \Omega(1) \cdot 2^{-T \lg T}\right)^n \leq \left(1 - 2^{-(1/2)\lg n}\right)^n \\
&= \left[\left(1 - \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}\right]^{\sqrt{n}} = \left(\frac{1}{e} + o(1)\right)^{\sqrt{n}} = 2^{-n^{\Omega(1)}}
\end{aligned}
$$

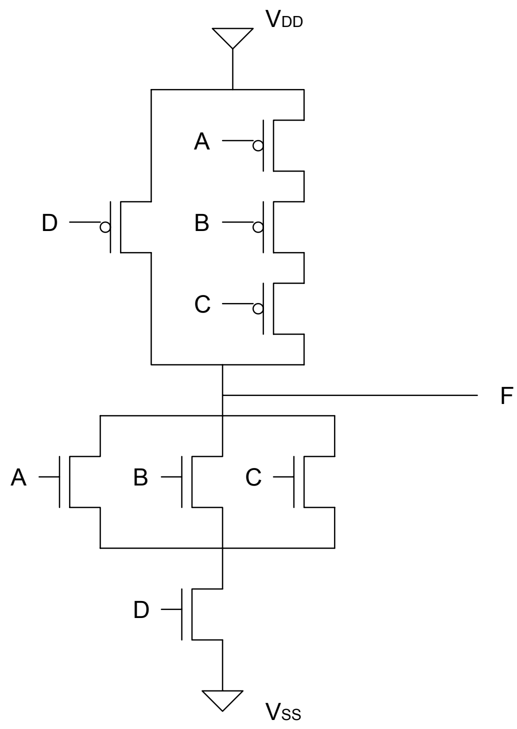

Figure 4: electrical schematic for a gate that computes $F = \overline{(A + B + C) \cdot D}$

It remains to show that $\Pr[E_i \mid \bigcap_{j>i} E_j] \leq 1 - \Omega(T^{-T})$. We analyze $\overline{E_i}$. This is the event that at least T packets want to go to destination i. This is at least the probability that exactly T packets want to go to destination i. There are $\binom{n}{T}$ choices for the T packets. The probability that one such T-subset of the packets decides to go to i, and the others decide to go somewhere else, is at least $\left(\frac{1}{n}\right)^T \left(1 - \frac{1}{n}\right)^{n-T}$. This is precisely the probability if we had no conditioning. Conditioning that not many elements go to other destinations can only increase the probability that more packets go to i. So our probability is:

$$\begin{aligned}
\Pr[\overline{E_i} \mid \bigcap_{j>i} E_j] &\geq \binom{n}{T} \left(\frac{1}{n}\right)^T \left(1 - \frac{1}{n}\right)^{n-T} \\
&\geq \left(\frac{n}{T}\right)^T \frac{1}{n^T} \left(1 - \frac{1}{n}\right)^n \\
&\geq \frac{1}{T^T} \left(\frac{1}{e} + o(1)\right) = \Omega(T^{-T})
\end{aligned}$$