Wed the 25^{th}

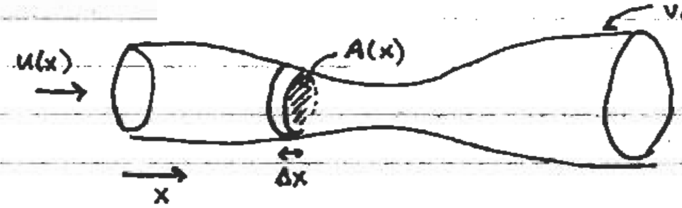
(due Feb. 27)

Compressible Flows (Lecture 2)

Reschedule Lect. 3

Office hours Friday?

<u>Variable area flows</u> (e.g. nozzles and diffusers) [chpt. 6]



variations are slow
→ approx flow as 1D

Suppose <u>steady</u> <u>state</u>

average density on area:

$$\tilde{\rho} = \frac{1}{A}\int_A \rho \, dA$$

Cons. of mass in "slice"

$$\frac{\partial}{\partial t}(\rho A \Delta x) = (\rho u A)\big|_x - (\rho u A)\big|_{x+\Delta x} = 0$$

$$\cancelto{0}{\frac{\partial}{\partial t}(\rho A)} + \frac{\rho u A|_{x+\Delta x} - \rho u A|_x}{\Delta x} = 0$$

$$\boxed{\frac{\partial}{\partial x}(\rho u A) = 0}$$

$$\frac{1}{\rho u A}\left\{ \rho u \frac{\partial A}{\partial x} + \rho A \frac{\partial u}{\partial x} + u A \frac{\partial \rho}{\partial x} = 0 \right\}$$

$$\boxed{\frac{1}{A}\frac{\partial A}{\partial x} + \frac{1}{u}\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial x} = 0} \qquad (1)$$

<u>1D Euler</u>:

$$\boxed{\rho u \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0} \qquad (2)$$

<u>Isentropic</u>:

$$\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial \rho}\right)_s \frac{\partial \rho}{\partial x} + \cancelto{0}{\left(\frac{\partial P}{\partial s}\right)_\rho} \frac{\partial s}{\partial x} = c^2 \frac{\partial \rho}{\partial x}$$

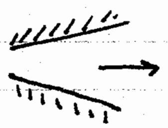
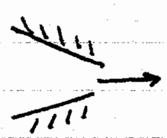
$$\frac{1}{u}\frac{du}{dx} = -\frac{1}{\rho}\frac{d\rho}{dx} - \frac{1}{A}\frac{dA}{dx} = -\frac{1}{\rho c^2}\frac{dP}{dx} - \frac{1}{A}\frac{dA}{dx}$$

↑
partials become totals
since $u = u(x)$

$$\frac{1}{u}\frac{du}{dx} = \frac{u}{c^2}\frac{du}{dx} - \frac{1}{A}\frac{dA}{dx}$$

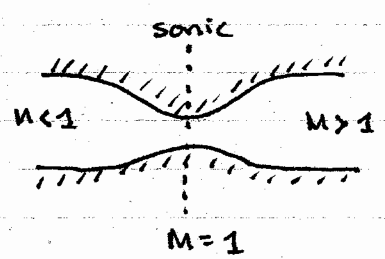
$$\frac{1}{u}\frac{du}{dx}\left(1 - M^2\right) = -\frac{1}{A}\frac{dA}{dx}$$

$$\boxed{\frac{1}{u}\cdot\frac{du}{dx} = \frac{1}{M^2-1}\,\frac{1}{A}\frac{dA}{dx}}$$

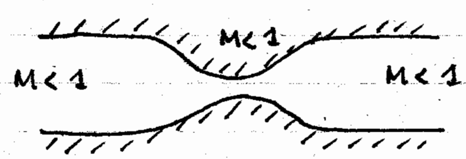
		$M < 1$	$M > 1$
	$dA > 0$	(decel.) $du < 0$, $dP > 0$ subsonic diffuser	(accel) $du > 0$, $dP < 0$ supersonic nozzle
	$dA < 0$	$du > 0$, $dP < 0$ subsonic nozzle	$du < 0$, $dP > 0$ supersonic diffuser

At $M = 1$, if du is finite, $dA = 0$

$\Rightarrow$ $M = 1$ (sonic condition) always occurs @ a throat.



Laval nozzle



Venturi nozzle

If $M \neq 1$ at throat then $du = 0$ $\Rightarrow$ no accel so no super/sub sonic transition.

Plots of $c_p + c_v$

In particular for a <u>perfect gas</u> ...

$$Pv = RT \Rightarrow \text{ideal gas}$$

$$Pv = RT \;\; \underline{\text{AND}} \;\; \underset{\substack{\uparrow \\ \text{specific heats} \\ \text{are const.}}}{\gamma = \text{const}} \Rightarrow \text{perfect gas}$$

$$c_p(T) = \left(\frac{\partial h}{\partial T}\right)_p \Rightarrow h = \int c_p(T)\,dT + c_0 \overset{\substack{\text{perfect gas} \\ \downarrow}}{=} c_p T + c_0$$

$$e = \int c_v(T)\,dT + c_1 = c_v T + c_1$$

Recall:

$$dh = T\,ds + v\,dP$$

$$\frac{c_p\,dT}{T} = \frac{dh}{T} = ds + \frac{dP}{\rho T} \Rightarrow ds = \frac{c_p\,dT}{T} - \frac{dP}{P} R$$

Integrate from reference state s_0, P_0, ρ_0 ...

$$s - s_0 = c_p \ln\left(\frac{T}{T_0}\right) - R \ln\left(\frac{P}{P_0}\right)$$

Note: $\gamma = c_p / c_v \qquad c_p - c_v = R$ (ideal gas)

$$\Rightarrow \gamma = R/c_v + 1 \Rightarrow c_v = \frac{R}{\gamma - 1}, \quad c_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{s - s_0}{c_v} = \gamma \ln\left(\frac{T}{T_0}\right) - (\gamma - 1) \ln\left(\frac{P}{P_0}\right)$$

$$\exp\left(\frac{s - s_0}{c_v}\right) = \left[\left(\frac{T}{T_0}\right)^{\gamma} \left(\frac{P}{P_0}\right)^{1-\gamma}\right] = \left[\left(\frac{P}{P_0}\right)\left(\frac{\rho}{\rho_0}\right)^{-\gamma}\right]$$

<u>Isentropic</u> <u>perfect gas</u>:

$$\boxed{\frac{P}{P_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}} \qquad (*)$$

Cons. of energy: $h + u^2/2 + \cancelto{0}{P.E.} = \text{const.}$

$$\Rightarrow c_p T + \cancel{\phi_0} + u^2/2 = c_p T_0 + \cancel{\phi_0}$$

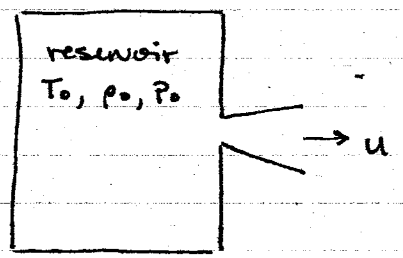
stagnation temp. $(u_0 = 0)$

Recall $c^2 = \gamma R T$

$$\frac{\gamma R T}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma R T_0}{\gamma - 1}$$

$$c^2 + (\gamma - 1)\, u^2/2 = c_0^2$$

stagnation speed of sound



Can use sonic condition as reference state instead of stagnation.

Denote sonic condition w. $*$: $u = c = c_*$

$$c_*^2\left[1 + \frac{\gamma}{2} - \frac{1}{2}\right] = c_0^2 = c_*^2\left(\frac{\gamma+1}{2}\right)$$

$$\Rightarrow c^2 + \frac{\gamma-1}{2} u^2 = c_*^2\left(\frac{\gamma+1}{2}\right)$$

$$1 + \frac{\gamma-1}{2} M^2 = \frac{\gamma+1}{2}\frac{c_*^2}{c^2} = \left(\frac{c_0}{c}\right)^2 = \frac{T_0}{T}$$

$$\boxed{\frac{T_0}{T} = 1 + \left(\frac{\gamma-1}{2}\right) M^2} \qquad (**)$$

From $(*)$ $\frac{p}{p_0} = \left[1 + \left(\frac{\gamma-1}{2}\right) M^2\right]^{-\frac{\gamma}{\gamma-1}}$

For a perfect gas: M, $\frac{T}{T_0}$, $\frac{P}{P_0}$, $\frac{\rho}{\rho_0}$ are all interrelated.

Numerical values @ $\gamma = 1.4$ are summarized in table D.1 pg. 585. (calculated from **, etc...)

At the sonic condition:

$$\boxed{\frac{T_0}{T_*} = \frac{\gamma+1}{2} \qquad \frac{P_*}{P_0} = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}}$$

E.g. for air ($\gamma = 1.4$)
$\frac{P_*}{P_0} = 0.5283$

These are also related to area:

$$A\rho u = A_* u_* \rho_*$$

$$\frac{A}{A_*} = \frac{u_*\rho_*}{\rho u} = \underbrace{\frac{\rho_*}{\rho_0}}_{\left(\frac{2}{\gamma+1}\right)^{1/\gamma-1}} \frac{\rho_0}{\rho} \underbrace{\frac{c}{u}}_{M^{-1}} \frac{c_0}{c} \overbrace{\frac{c_*}{c_0}}^{\left(\frac{2}{\gamma+1}\right)^{1/2}}$$

$$= \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}}\left(\frac{2}{\gamma+1}\right)^{1/2}\right] \overbrace{\frac{\rho_0}{\rho}}^{\frac{P_0}{P}\frac{c}{c_0}} \frac{c_0}{c}$$

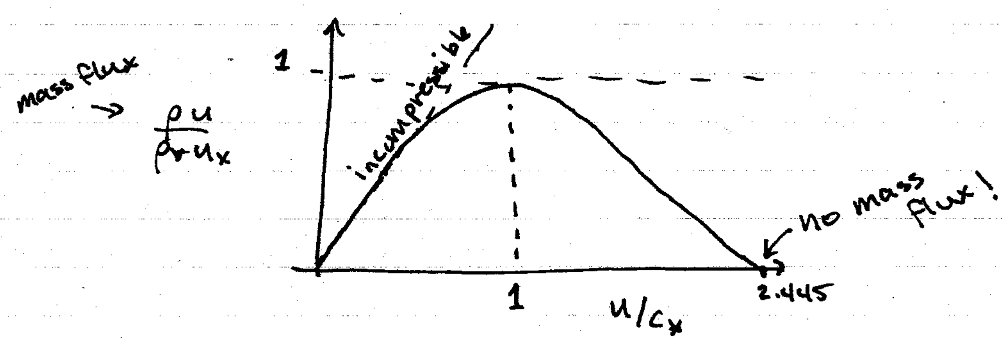
$$\left[1 + (\gamma-1)u^2/2\right]^{1/2}$$

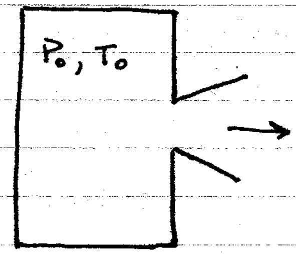
$$= \frac{1}{M}\left[\left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}+\frac{1}{2}}\right]\left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{\frac{\gamma}{\gamma-1}}\left[1 + \left(\frac{\gamma-1}{2}\right)M^2\right]^{-1/2}$$

$$\boxed{\frac{A}{A_*} = \frac{1}{M}\left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

(show fig 6.5 ... also in table D.1.)

Show fig. 5.18. mass flux $= u\rho \propto A$





$P \downarrow$ as fluid flows through nozzle
$\Rightarrow M \uparrow$

As $P \to 0$, $M \to \infty$ but the flow speed remains finite ($c \to 0$)

This limit $M \to \infty$ defines a maximum flow speed. (Maximum speed attainable in inviscid steady state flow.

$$c^2 + \frac{\gamma-1}{2} u^2 = c_0^2 \Rightarrow \frac{c^2}{u^2}^{\nearrow 0} + \frac{\gamma-1}{2} = \frac{c_0^2}{u^2}$$

$$\Rightarrow \boxed{u_{max} = c_0 \sqrt{\frac{2}{\gamma-1}}}$$

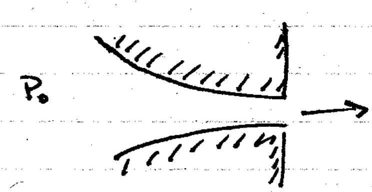
on Hw

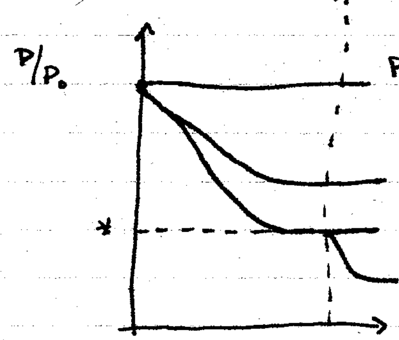
All energy converted to kinetic energy:

$$\frac{1}{2} u_{max} = h_0 \Rightarrow \boxed{u_{max} = \sqrt{2h_0}} = \sqrt{2c_pT_0} = \sqrt{\frac{2\gamma R}{\gamma-1} T_0}$$

$$= c_0 \sqrt{\frac{2}{\gamma-1}}$$

Converging nozzle





$P_0 = P_a \Rightarrow$ no flow

$P_0 > P_a > P_*$ $\Rightarrow$ subsonic

sonic conditions

$P_a < P_*$ "choked"

no signals can propagate upstream (cannot influence upstream flow)

Bernoulli's for compressible flow

$$\rho \frac{D\bar{u}}{Dt} = -\nabla P + \underbrace{\rho\bar{g}}_{\text{more general conservative force}}$$

$-\rho\nabla\psi$ (for $\rho\bar{g}$, $\psi = gz$)

$$\rho\left(\cancelto{\text{st. st.}}{\frac{\partial \bar{u}}{\partial t}} + \bar{u}\cdot\nabla\bar{u}\right) = -\nabla P - \rho\nabla\psi$$

Note: $\bar{u}\cdot\nabla\bar{u} = \nabla\left(\frac{u^2}{2}\right) + \underbrace{(\nabla\times\bar{u})}_{\text{vorticity} \,\equiv\, \bar{\omega} \,=\, \nabla\times\bar{u}}\times\bar{u}$ (vector identity)

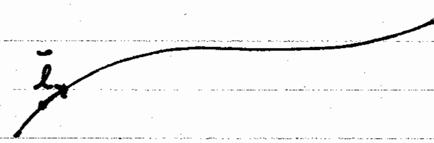
$$(*) \qquad \nabla\left(\frac{u^2}{2}\right) + \frac{1}{\rho}\nabla P + \nabla\psi = \underbrace{-\bar{\omega}\times\bar{u}}$$

Recall: streamlines $\parallel$ to $\bar{u}$

$\bar{\omega}\times\bar{u}$ $\perp$ to both $\bar{u}$ and $\bar{\omega}$

$\therefore$ $\perp$ to streamlines

Recall also directional derivative:



how does a function f vary along the line?

$$\frac{\partial f}{\partial l} = \bar{l}\cdot\nabla f$$

↑ project driv onto l

$\therefore$ take dot product of $(*)$ with $\bar{l}$ along a streamline

$$\frac{d}{dl}\left(\frac{u^2}{2}\right) + \frac{1}{\rho}\frac{dP}{dl} + \frac{d\psi}{dl} = 0$$

$= 0$

$$\Rightarrow \quad d\left(\tfrac{1}{2}u^2\right) + \frac{1}{\rho}dP + \cancelto{\text{small for gas}}{d\psi} = 0$$

$$\Rightarrow \quad \boxed{u\,du + \frac{1}{\rho}dP = 0}$$

along a streamline (inviscid)

In terms of enthalpy: $dh = T\,ds + \frac{1}{\rho}\,dP$

$$\nabla\left(\frac{u^2}{2}\right) + \nabla(\psi) + \nabla h - T\nabla s = -\vec{\omega}\times\vec{u}$$

On a streamline

$$\underbrace{d\left(\tfrac{1}{2}u^2 + h + \psi\right)}_{\text{total energy}} - \underbrace{T\,ds}_{\text{heat}} = 0$$

For isentropic

$$d\left(\tfrac{1}{2}u^2 + h + \psi\right) = 0$$

$$\boxed{\tfrac{1}{2}u^2 + h + \psi = \text{const.}}$$

st. st., isentropic, inviscid along a streamline

$$dh = \frac{1}{\rho}\,dP \qquad \text{(isentropic)}$$

$$\frac{1}{\rho}\,dP + d\left(\tfrac{1}{2}u^2 + \psi\right) = 0$$

$$\boxed{\int_{P_0}^{P} \frac{1}{\rho}\,dP + \tfrac{1}{2}u^2 + \psi = \text{const.}}$$

Compressible Bernoulli's isentropic, inviscid along a streamline