Computational Ocean Acoustics

- Ray Tracing
- Wavenumber Integration
- Normal Modes
- Parabolic Equation

Parabolic Equation

- Mathematical Derivation (6.2)
  - Phase Errors and Angular Limitations (6.2.4)
- Starting Fields (6.4)
  - Modal starter
  - PE Self Starter
  - Analytical Starters
- PE Solvers
  - Split-Step Fourier Algorithm (6.5)
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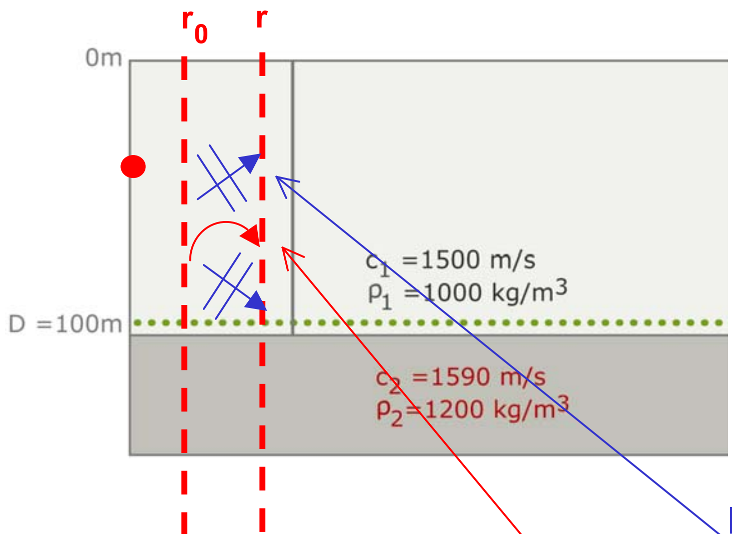
Solutions by FFTs

The Split-Step Fourier Algorithm

Fourier Transform PE - n locally constant



$$2ik_0 \frac{\partial \psi}{\partial r} - k_z^2 \psi + k_0^2 \left(n^2 - 1\right) \psi = 0 \, ,$$

$$\Rightarrow$$

$$\frac{\partial \psi}{\partial r} + \frac{k_0^2 \left(n^2 - 1\right) - k_z^2}{2ik_0} \psi = 0 \, .$$

Solution

Diffraction

$$\psi(r, k_z) = \psi(r_0, k_z)\, e^{-\frac{k_0^2 (n^2-1) - k_z^2}{2ik_0} (r - r_0)} \, .$$

Refraction

Inverse Transform

$$\psi(r, z) = e^{\frac{ik_0}{2}(n^2-1)(r-r_0)} \int_{-\infty}^{\infty} \psi(r_0, k_z)\, e^{-\frac{i(r-r_0)}{2k_0} k_z^2}\, e^{ik_z z} dk_z \, .$$

Field Solution - Split-Step Marching Algorithm

$$\psi(r, z) = \boxed{e^{\frac{ik_0}{2}\left[n^2(r_0,z)-1\right]\Delta r}}\, \mathcal{F}^{-1}\left\{\boxed{e^{-\frac{i\Delta r}{2k_0}k_z^2}\ \mathcal{F}\left\{\psi(r_0, z)\right\}}\right\} ,$$

Generalized Operator Formalism

Standard Parabolic Wave Equation

$$\frac{\partial \psi}{\partial r} = \left[\frac{ik_0}{2}(n^2 - 1) + \frac{i}{2k_0} \frac{\partial^2}{\partial z^2} \right] \psi \,,$$

PE Operators

$$A = A(r,z) = \frac{ik_0}{2} \left[n^2(r,z) - 1 \right]; \qquad B = B(z) = \frac{i}{2k_0} \frac{\partial^2}{\partial z^2} \,,$$

PE Compact Form

$$\frac{\partial \psi}{\partial r} = (A + B)\,\psi = U(r,z)\,\psi \,.$$

Split-Step Algorithm

$$\begin{aligned} \psi(r,z) &= e^{\int_{r_0}^{r_0+\Delta r} U(r,z) dr} \; \psi(r_0, z) \\ &\simeq e^{\tilde{U} \Delta r} \, \psi(r_0, z) \,, \end{aligned}$$

Exponential Operator Splitting

$$\text{(I)}: \qquad e^{(A+B)\Delta r} \simeq e^{A\Delta r}\, e^{B\Delta r}\,,$$

$$\text{(II)}: \qquad e^{(A+B)\Delta r} \simeq e^{B\Delta r}\, e^{A\Delta r}\,,$$

$$\text{(III)}: \qquad e^{(A+B)\Delta r} \simeq e^{\frac{A}{2}\Delta r}\, e^{B\Delta r}\, e^{\frac{A}{2}\Delta r}\,,$$

$$\text{(IV)}: \qquad e^{(A+B)\Delta r} \simeq e^{\frac{B}{2}\Delta r}\, e^{A\Delta r}\, e^{\frac{B}{2}\Delta r}\,.$$

Exact Only if A and B Commute

Exponential Function Expansion - Splitting I

$$V(r_0, z) = e^{B\Delta r}\, \psi(r_0, z)\,; \qquad B = \frac{i}{2k_0}\, \frac{\partial^2}{\partial z^2}\,.$$

$$\begin{aligned} V(r_0, z) &= \left[1 + \Delta r\, B + \frac{(\Delta r)^2}{2}\, BB + \cdots\right] \psi(r_0, z) \\ &= \left[1 + \frac{i\Delta r}{2k_0}\, \frac{\partial^2}{\partial z^2} + \frac{1}{2}\left(\frac{i\Delta r}{2k_0}\right)^2 \frac{\partial^4}{\partial z^4} + \cdots\right] \psi(r_0, z)\,, \end{aligned}$$

Fourier Transform

$$\begin{aligned} V(r_0, k_z) &= \left[1 - \frac{i\Delta r}{2k_0}\, k_z^2 + \frac{1}{2}\left(\frac{i\Delta r}{2k_0}\right)^2 k_z^4 - \cdots\right] \psi(r_0, k_z)\,, \\ &= e^{-\frac{i\Delta r}{2k_0} k_z^2}\, \psi(r_0, k_z)\,. \end{aligned}$$

Diffraction Operator

Inverse Transform

$$V(r_0,z) = \mathcal{F}^{-1}\left\{e^{-\frac{i\Delta r}{2k_0}k_z^2}\ \mathcal{F}\left\{\psi(r_0,z)\right\}\right\}.$$

Refraction

Split-Step Algorithms

Diffraction

$$\psi_{\mathrm{I}}(r,z) = e^{\frac{ik_0}{2}\left[n^2(r_0,z)-1\right]\Delta r}\ \mathcal{F}^{-1}\left\{e^{-\frac{i\Delta r}{2k_0}k_z^2}\ \mathcal{F}\left\{\psi(r_0,z)\right\}\right\}+O((\Delta r)^2),$$

$$\psi_{\mathrm{II}}(r,z) = \mathcal{F}^{-1}\left\{e^{-\frac{i\Delta r}{2k_0}k_z^2}\ \mathcal{F}\left\{e^{\frac{ik_0}{2}\left[n^2(r_0,z)-1\right]\Delta r}\psi(r_0,z)\right\}\right\}+O((\Delta r)^2),$$

$$\psi_{\mathrm{III}}(r,z) = e^{\frac{ik_0}{4}\left[n^2(r_0,z)-1\right]\Delta r}\ \mathcal{F}^{-1}\left\{e^{-\frac{i\Delta r}{2k_0}k_z^2}\ \mathcal{F}\left\{e^{\frac{ik_0}{4}\left[n^2(r_0,z)-1\right]\Delta r}\psi(r_0,z)\right\}\right\}+O((\Delta r)^2),$$

$$\psi_{\mathrm{IV}}(r,z) = \mathcal{F}^{-1}\left\{e^{-\frac{i\Delta r}{4k_0}k_z^2}\mathcal{F}\left\{e^{\frac{ik_0}{2}\left[n^2(r_0,z)-1\right]\Delta r}\mathcal{F}^{-1}\left[e^{-\frac{i\Delta r}{4k_0}k_z^2}\mathcal{F}\left[\psi(r_0,z)\right]\right]\right\}\right\}+O((\Delta r)^2).$$

The Split-Step Fourier Algorithm

Error Analysis

Compact Form PE

$$\psi' = U\psi = (A + B)\,\psi\,,$$

Taylor Expansion

$$\psi_{j+1} = \psi_j + \psi'_j\,\Delta r + \psi''_j\,\frac{(\Delta r)^2}{2} + \psi'''_j\,\frac{(\Delta r)^3}{6} + \cdots\,,$$

Power Series Solution

$$\psi_{j+1} \;=\; \left[1 + U\Delta r + (U' + U^2)\,\frac{(\Delta r)^2}{2} + (U'' + 2UU' + U'U + U^3)\,\frac{(\Delta r)^3}{6}\right]_j \psi_j$$
$$+O(\Delta r^4).$$

Exponentiation Error

Split-Step Algorithm

$$\psi(r,z) = e^{\int_{r_0}^{r_0+\Delta r} U(r,z)dr}\ \psi(r_0,z)$$
$$\boxed{\simeq}\ e^{\widetilde{U}\Delta r}\ \psi(r_0,z)\,,$$

Constant U

$$\widetilde{U} = U_j$$

$$\psi_{j+1} = \left[1 + U\Delta r + U^2\frac{(\Delta r)^2}{2} + U^3\frac{(\Delta r)^3}{6}\right]_j \psi_j\,.$$

Error

$$E_1 = \frac{\boxed{(\Delta r)^2}}{2} U_j'\,\psi_j + O((\Delta r)^3)\,\psi_j\,,$$

$$U_j' = A_j' = \frac{ik_0}{2}\boxed{\frac{\partial n^2}{\partial r}}.$$

Linear U

$$\widetilde{U} = U_j + U_j'\frac{\Delta r}{2}\,,$$

$$\widetilde{E_1} = \frac{\boxed{(\Delta r)^3}}{12}(2U'' + UU' - U'U)_j\,\psi_j\,.$$

Commutator Error

Splitting I

$$e^{(A+B)\Delta r} \simeq e^{A\Delta r}\, e^{B\Delta r}\,.$$

Expansion of Exponentials $O(\Delta r^3)$

$$\begin{aligned} E_2 &= -\frac{(\Delta r)^2}{2}(AB - BA)\psi_j\,, \\ &= -\frac{\boxed{(\Delta r)^2}}{2}[A,B]\psi_j\,, \end{aligned}$$

CommutatorError

$$[A,B]\psi = \frac{1}{4}\left(\boxed{\frac{\partial^2 n^2}{\partial z^2}}\psi + \boxed{2\frac{\partial n^2}{\partial z}}\frac{\partial \psi}{\partial z}\right).$$

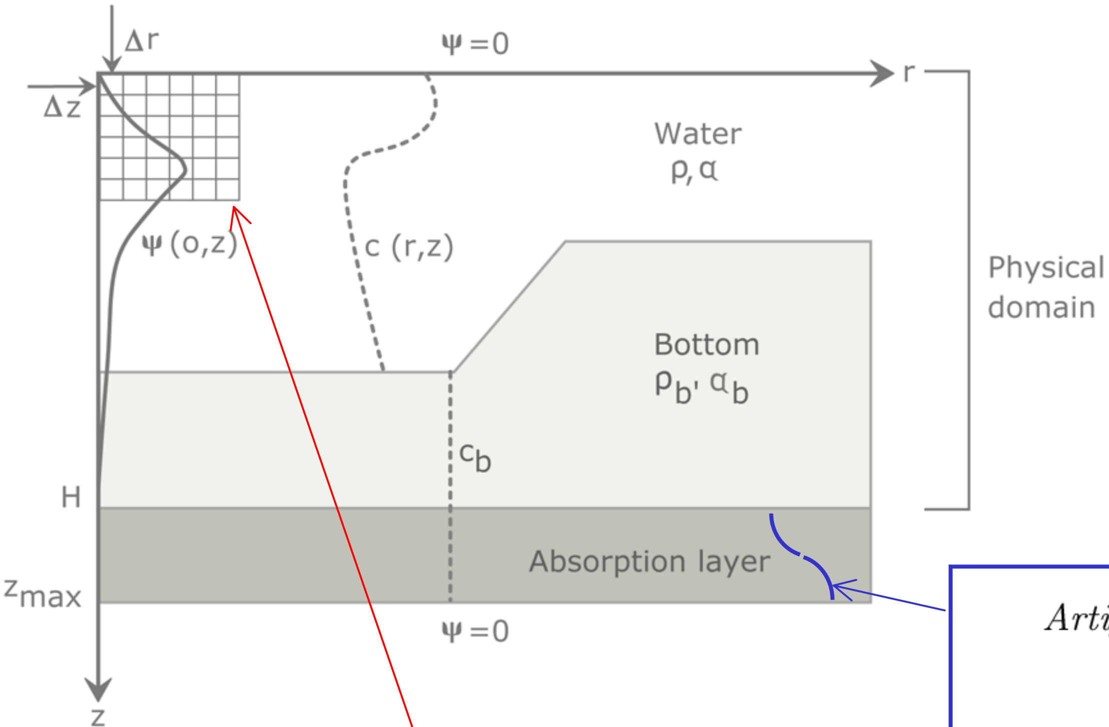
More serious
Depth Dependence Stronger

Higher-Order Splittings

$$e^{(A+B)\Delta r} = e^{\frac{A}{2}\Delta r}\, e^{B\Delta r}\, e^{\frac{A}{2}\Delta r}\,,$$

$$\begin{aligned} E_3 &= \frac{(\Delta r)^3}{6}BAB - \frac{(\Delta r)^3}{12}\left(AB^2 + B^2A + ABA\right) + \frac{(\Delta r)^3}{24}\left(A^2B + BA^2\right) \\ &= \frac{(\Delta r)^3}{24}\left(2B[A,B] - [A,B]2B + A[A,B] - [A,B]A\right). \end{aligned}$$

Numerical Implementation



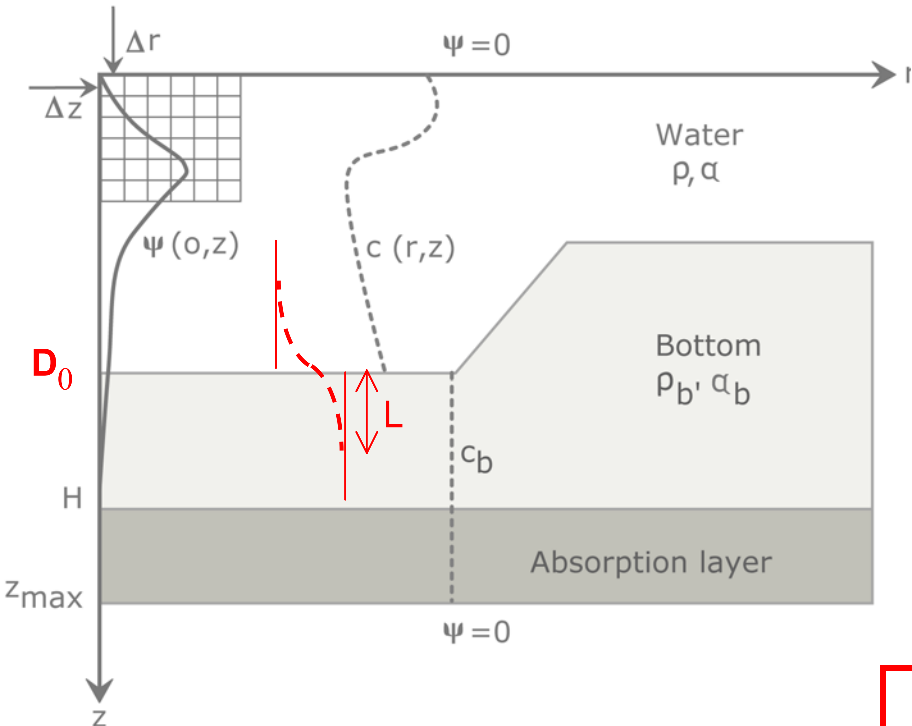
Artificial Absorbtion Layer

$$n^2 = n_b^2 + i\alpha \, \exp\left[- \left(\frac{z - z_{\max}}{D} \right)^2 \right],$$

Range-Depth Discretization

$$\Delta z \;\leq\; \lambda/4$$
$$\Delta r \;\simeq\; 2 - 5\,\Delta z$$

Convergence Analysis !!



Variable Density

$$\rho \nabla \cdot \left(\frac{1}{\rho} \nabla p \right) + k_0^2 \, n^2 \, p = 0 \, ,$$

$$\tilde{p} = \frac{p}{\sqrt{\rho}} \, ,$$

$$\nabla^2 \tilde{p} + k_0^2 \, \tilde{n}^2 \, \tilde{p} = 0 \, ,$$

Effective Index of Refraction

$$\tilde{n}^2 = n^2 + \frac{1}{2k_0^2} \left[\frac{1}{\rho} \nabla^2 \rho - \frac{3}{2\rho^2} (\nabla \rho)^2 \right] .$$

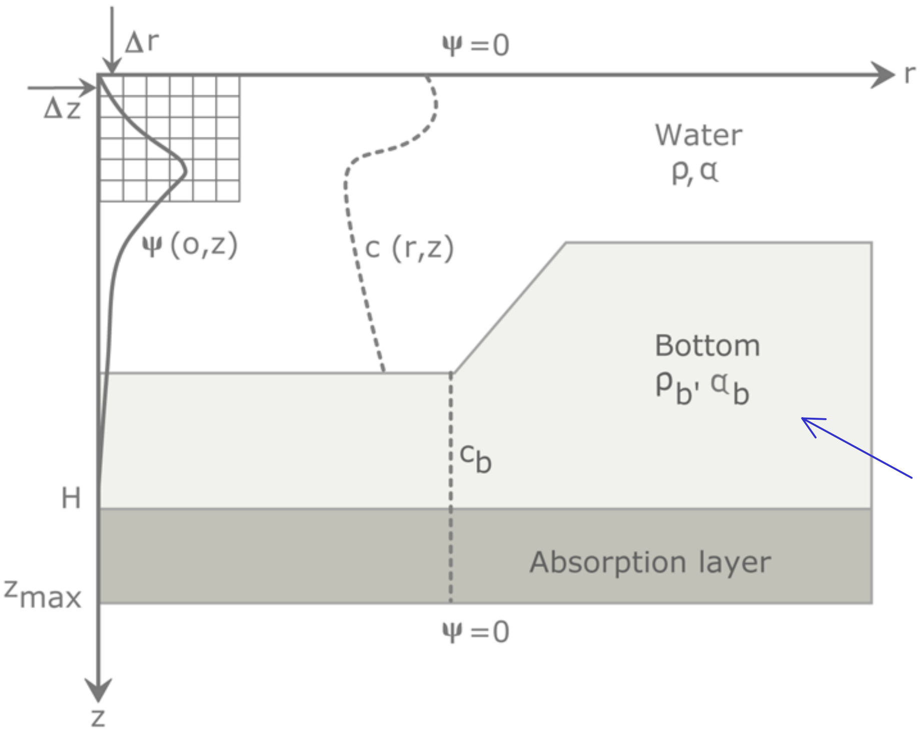
Depth Derivatives

$$\tilde{n}^2 = n^2 + \frac{1}{2k_0^2} \left[\frac{1}{\rho} \frac{\partial^2 \rho}{\partial z^2} - \frac{3}{2\rho^2} \left(\frac{\partial \rho}{\partial z} \right)^2 \right] .$$

Smoothing

$$\boxed{\rho(z) = \frac{1}{2}(\rho_2 + \rho_1) + \frac{1}{2}(\rho_2 - \rho_1) \tanh\left(\frac{z - D_0}{L} \right) ,}$$

$$k_0 L \simeq 2 \, .$$



Attenuation

Complex Wavenumber

$$k = \frac{\omega}{c} + i\alpha\,, \qquad \alpha > 0\,.$$

$$\alpha^{(\lambda)} = -20\log\left(\frac{e^{-\alpha(r+\lambda)}}{e^{-\alpha r}}\right) = \alpha\lambda\; 20\log e\,,$$

Complex Index of Refraction

$$n^2 = \left(\frac{k}{k_0}\right)^2 \simeq \left(\frac{c_0}{c}\right)^2 \left[1 + i\,\frac{2\alpha c}{\omega}\right].$$

$$\simeq \left(\frac{c_0}{c}\right)^2 \left[1 + i\,\frac{\alpha^{(\lambda)}}{27.29}\right].$$

Student Demos
Wavenumber Integration Models