Computational Ocean Acoustics

- Ray Tracing
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- Broadband Modeling

Broadband Modeling

- Fourier Synthesis
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  - Numerical Examples

Reference: Kuperman, W. A. and Henrick Schmidt. "Spectral and modal representations of the Doppler-shifted field in ocean waveguides." The Journal of the Acoustical Society of America 96, no.1 (July 1994): 386-395.

Doppler Shift in a Waveguide

Wave Equation - Moving Source

$$\nabla^2\psi(\mathbf{r},z,t) - \frac{1}{c^2}\,\frac{\partial^2\psi(\mathbf{r},z,t)}{\partial t^2} = -\delta(\mathbf{r}-\mathbf{v}_s t)\,\delta(z-z_s)\,e^{-i\Omega t}\,.$$

Inhomogeneous Helmholtz Equation

$$\left[\nabla^2 + k_\omega^2\right]\psi(\mathbf{r},z,\omega) = -\delta(z-z_s)\int\delta(\mathbf{r}-\mathbf{v}_s t)\,e^{i(\omega-\Omega)t}dt\,,$$

Wavenumber Integral Representation

2D Fourier Transform

$$\psi(\mathbf{r}, z; \omega) = \int \psi(\mathbf{k_r}, z; \omega)\, e^{i\mathbf{k_r}\cdot\mathbf{r}}\, d^2\mathbf{k_r}\,,$$

$$\psi(\mathbf{k_r}, z; \omega) = \frac{1}{(2\pi)^2} \int \psi(\mathbf{r}, z; \omega)\, e^{-i\mathbf{k_r}\cdot\mathbf{r}}\, d^2\mathbf{r}\,,$$

Depth-separated Wave Equation

$$\begin{aligned}\frac{d^2\psi(\mathbf{k_r}, z; \omega)}{dz} + \left[k_\omega^2 - k_r^2\right] \psi(\mathbf{k_r}, z; \omega) &= -\frac{\delta(z - z_s)}{(2\pi)^2} \int e^{i(\omega - \Omega - \mathbf{k_r}\cdot\mathbf{v}_s)t}\, dt \\ &= -\frac{\delta(z - z_s)}{2\pi}\, \delta(\omega - \Omega - \mathbf{k_r}\cdot\mathbf{v}_s)\,,\end{aligned}$$

Integral Identities

$$\int \delta(\mathbf{r} - \mathbf{v}_s t)\, e^{-i\mathbf{k_r}\cdot\mathbf{r}}\, d^2\mathbf{r} = e^{-\mathbf{k_r}\cdot\mathbf{v}_s t}\,,$$

$$\frac{1}{2\pi} \int e^{i(\omega - \Omega - \mathbf{k_r}\cdot\mathbf{v}_s)\,t}\, dt = \delta(\omega - \Omega - \mathbf{k_r}\cdot\mathbf{v}_s)\,.$$

Frequency-Wavenumber Solution

$$\psi(\mathbf{k_r}, z; \omega) = \delta(\omega - \Omega - \mathbf{k_r}\cdot\mathbf{v}_s)\, g(k_r, z; \omega)\,,$$

Time-domain Solution

$$\psi(\mathbf{r}, z, t) = \frac{1}{2\pi} \int e^{-i\omega t}\, d\omega \int \psi(\mathbf{k_r}, z, \omega)\, e^{i\mathbf{k_r}\cdot\mathbf{r}}\, d^2\mathbf{k_r}\ ,$$

$$\psi(\mathbf{r}, z, t) = \frac{1}{2\pi} \int g(k_r, z; \Omega + \mathbf{k_r}\cdot\mathbf{v}_s)\, e^{-i[(\Omega+\mathbf{k_r}\cdot\mathbf{v}_s)t - \mathbf{k_r}\cdot\mathbf{r}]}\, d^2\mathbf{k_r}\ .$$

Doppler Frequency Shift

Time-wavenumber coupling

$$\omega = \Omega + \mathbf{k_r}\cdot\mathbf{v}_s\ .$$

Moving Receiver

$$\psi(\mathbf{r}_0+\mathbf{v}_r t, z, t) = \frac{1}{2\pi}\int g(k_r, z; \Omega+\mathbf{k_r}\cdot\mathbf{v}_s)\, e^{-i\left[[\Omega+\mathbf{k_r}\cdot(\mathbf{v}_s-\mathbf{v}_r)]t-\mathbf{k_r}\cdot\mathbf{r}_0\right]}\, d^2\mathbf{k_r}\,.$$

Broadband Source

$$\psi(\mathbf{r}_0+\mathbf{v}_r t, z, t) = \frac{1}{4\pi^2}\int d\Omega S(\Omega)\int d^2\mathbf{k_r}$$
$$\times\, G(k_r, z; \Omega+\mathbf{k_r}\cdot\mathbf{v}_s) e^{-i[(\Omega+\mathbf{k_r}\cdot(\mathbf{v}_s-\mathbf{v}_r))t-\mathbf{k_r}\cdot\mathbf{r}_0]}\,.$$

Frequency-domain solution

$$\boxed{\psi(\mathbf{r}_0+\mathbf{v}_r t, z, \omega)} = \int dt e^{i\omega t}\psi(\mathbf{r}_0+\mathbf{v}_r t, z, t)$$
$$= \frac{1}{4\pi^2}\int d\Omega S(\Omega)\int d^2\mathbf{k_r} e^{i\mathbf{k_r}\cdot\mathbf{r}_0} G(k_r, z; \Omega+\mathbf{k_r}\cdot\mathbf{v}_s)$$
$$\times \int dt e^{-i(\Omega-\omega+\mathbf{k_r}\cdot(\mathbf{v}_s-\mathbf{v}_r))t}$$
$$= \frac{1}{2\pi}\int d^2\mathbf{k_r} e^{i\mathbf{k_r}\cdot\mathbf{r}_0}\int d\Omega S(\Omega)$$
$$\times\, G(k_r, z; \Omega+\mathbf{k_r}\cdot\mathbf{v}_s)\delta(\Omega-\omega+\mathbf{k_r}\cdot(\mathbf{v}_s-\mathbf{v}_r))$$
$$\boxed{= \frac{1}{2\pi}\int d^2\mathbf{k_r} e^{i\mathbf{k_r}\cdot\mathbf{r}_0} S(\Omega_k) G(k_r, z; \omega+\mathbf{k_r}\cdot\mathbf{v}_r)\,,}$$

Doppler-shifted Source Frequency

$$\boxed{\Omega_k = \omega - \mathbf{k_r}\cdot(\mathbf{v}_s-\mathbf{v}_r)\,.}$$

Far-field Approximation

$$
\begin{aligned}
&\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) \\
&\quad \simeq \int_0^\infty dk_r k_r J_0(k_r r_0) \\
&\qquad\qquad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos\theta_r) \\
&\quad = \frac{1}{2} \int_{-\infty}^{\infty} dk_r k_r H_0^{(1)}(k_r r_0) \\
&\qquad\qquad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos\theta_r)\,,
\end{aligned}
$$

$$\Omega_k = \omega - k_r(v_s \cos\theta_s - v_r \cos\theta_r)\,.$$

Temporal doppler

Spatial doppler

Normal Mode Representation

Depth-dependent Grean's Function

$$G(k_r, z; \omega) \simeq \frac{1}{2\pi\rho(z_s)} \sum \frac{\Psi_n(z)\Psi_n(z_s)}{k_r^2 - k_n^2},$$

Doppler Shifted Modal Wavenumbers

$$k_n^* \simeq k_n \left(1 + v_r \cos\theta_r \frac{dk_n}{d\omega}\right) = k_n \left(1 + \frac{v_r}{v_{ng}} \cos\theta_r\right),$$

Modal Field

$$\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) \simeq \frac{i}{4\rho(z_s)} \sum_n S(\Omega_n)$$

$$\times \Psi_n(z)\Psi_n(z_s) H_0^{(1)}\left(k_n r_0 (1 + \tfrac{v_r}{v_{ng}} \cos\theta_r)\right),$$

$$\begin{aligned} \Omega_n &= \omega - k_n(v_s \cos\theta_s - v_r \cos\theta_r) \\ &= \omega \left(1 - \frac{v_s}{v_{np}} \cos\theta_s + \frac{v_r}{v_{np}} \cos\theta_r\right), \end{aligned}$$

Temporal doppler

Spatial doppler

Adiabatic Approximation

$$\psi(r,z,\omega) \simeq$$

$$\frac{iS(\omega)e^{-i\pi/4}}{\rho(z_s)\sqrt{8\pi}} \sum_n \Psi_n(z)\Psi_n(z_s)\frac{e^{i\int_0^r k_n(r')dr'}}{\sqrt{\int_0^r k_n(r')dr'}} \ .$$

$$\psi(\mathbf{r}_0+\mathbf{v}_r t, z, \omega) \simeq$$

$$\frac{ie^{-i\pi/4}}{\rho(z_s)\sqrt{8\pi}} \sum_n S(\Omega_n^*)\Psi_n(z)\Psi_n(z_s)\frac{e^{i\int_0^{r_n^*} k_n(r')dr'}}{\sqrt{\int_0^{r_n^*} k_n(r')dr'}} \ .$$

Temporal doppler

$$\Omega_n^* = \omega\left(1 - \frac{v_s}{v_{np}(0)}\cos\theta_s + \frac{v_r}{v_{np}(r_0)}\cos\theta_r\right) \ ,$$

Doppler-perturbed Ranges

Spatial doppler

$$r_n^* = r_0\left(1 + \frac{v_r}{v_{ng}(r_0)}\cos\theta_r\right) \ .$$

Shallow Water Waveguide
Moving Source and Receiver

Far-field Approximation

$$
\begin{aligned}
&\psi(\mathbf{r}_0 + \mathbf{v}_r t, z, \omega) \\
&\quad \simeq \int_0^\infty dk_r k_r J_0(k_r r_0) \\
&\qquad\quad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos\theta_r) \\
&\quad = \frac{1}{2} \int_{-\infty}^{\infty} dk_r k_r H_0^{(1)}(k_r r_0) \\
&\qquad\quad \times S(\Omega_k) G(k_r, z; \omega + k_r v_r \cos\theta_r)\,,
\end{aligned}
$$

$$
\Omega_k = \omega - k_r \boxed{\underbrace{(v_s \cos\theta_s - v_r \cos\theta_r)}_{= 0}}\,.
$$